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# Convex combination of analytic functions

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**Abstract:** Radii of convexity, starlikeness, lemniscate starlikeness and close-to-convexity are determined for the convex combination of the identity map and a normalized convex function  $F$  given by  $f(z) = \alpha z + (1 - \alpha)F(z)$ .

**Keywords:** Convex function, Starlike function, Close-to-convex function, Lemniscate of Bernoulli, Radius of starlikeness, Convolution, Convex combination

**MSC:** 30C80, 30C45

## 1 Introduction

Let  $\mathcal{A}$  be the class of analytic functions  $f$  defined on the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ , and normalized by  $f(0) = 0 = f'(0) - 1$ . Let  $\mathcal{S}$ ,  $\mathcal{ST}$ ,  $\mathcal{CV}$  and  $\mathcal{CCV}$  denote the subclasses of  $\mathcal{A}$  consisting of functions univalent, starlike, convex and close-to-convex respectively. Recall that a function  $f \in \mathcal{A}$  is *close-to-convex* if there exists a convex function  $g$  such that  $\operatorname{Re}(f'(z)/g'(z)) > 0$  for all  $z \in \mathbb{D}$ . The class  $\mathcal{ST}(\beta)$  of *starlike functions of order  $\beta$* ,  $0 \leq \beta < 1$ , consists of  $f \in \mathcal{A}$  satisfying  $\operatorname{Re}(zf'(z)/f(z)) > \beta$  for all  $z \in \mathbb{D}$  and  $\mathcal{ST} := \mathcal{ST}(0)$ . The class  $\mathcal{CV}(\beta)$  of *convex functions of order  $\beta$*  is defined by  $\mathcal{CV}(\beta) = \{f \in \mathcal{A} : zf'(z) \in \mathcal{ST}(\beta)\}$  and  $\mathcal{CV} := \mathcal{CV}(0)$ . The class  $\mathcal{SL}$  of *lemniscate starlike functions*, introduced by Sokól and Stankiewicz [1], consists of  $f \in \mathcal{A}$  satisfying  $|zf'(z)/f(z)|^2 - 1 < 1$  for all  $z \in \mathbb{D}$ , or, equivalently, if  $zf'(z)/f(z)$  lies in the region bounded by the right-half of the lemniscate of Bernoulli given by  $|w^2 - 1| < 1$ . For recent investigation on the class  $\mathcal{SL}$ , see [1–5]. Another class of our interest is the class  $\mathcal{M}_\beta$ ,  $\beta > 1$ , consisting of  $f \in \mathcal{A}$  satisfying  $\operatorname{Re}(zf'(z)/f(z)) < \beta$  for all  $z \in \mathbb{D}$ . The class  $\mathcal{M}_\beta$  was investigated by Uralegaddi *et al.* [6], while its subclass was investigated by Owa and Srivastava [7]. Related radius problem for this class can be found in [8] and [9].

Properties of linear combination, in particular, convex combination of functions belonging to various classes of functions were initially investigated by Rahmanov in 1952 and 1953 [10, 11]. The survey article of Campbell [12] provides several results concerning various combination of univalent functions as well as of locally univalent functions. Convex combination of univalent functions and the identity function were investigated by several authors (see Merkes [13] and references therein as well as [14]); in particular, Merkes [13] proved some results related to the present investigation. Obradovic and Nunokawa [15] investigated functions  $f \in \mathcal{A}$  satisfying the following condition

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z) - \alpha} \right) > 0 \quad (z \in \mathbb{D}) \quad (1)$$

for some  $\alpha \in [-1, 1)$  and obtained the following result.

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**Theorem 1.1** ([15, Theorem 2]). *If  $f \in \mathcal{A}$  satisfies the condition (1), then (i)  $f \in \mathcal{ST}$  for  $-\frac{1}{3} \leq \alpha \leq \frac{1}{3}$ , and (ii)  $f \in \mathcal{CCV}$  for  $-\frac{1}{3} \leq \alpha < 1$ .*

If the function  $f$  is the convex combination  $f(z) = \alpha z + (1 - \alpha)F(z)$ , then the condition (1) is equivalent to the conditions that  $F \in \mathcal{CV}$ . If two subclasses  $\mathcal{G}$  and  $\mathcal{F}$  of  $\mathcal{A}$  are given, the  $\mathcal{G}$ -radius of  $\mathcal{F}$ , denoted by  $R_{\mathcal{G}}(\mathcal{F})$ , is the largest number  $R$  such that  $f(rz)/r \in \mathcal{G}$  for  $0 < r < R$ , and for all  $f \in \mathcal{F}$ . Whenever  $\mathcal{G}$  is characterized by possessing a geometric property  $P$ , the number  $R$  is also referred to as the radius of property  $P$  for the class  $\mathcal{F}$ . In this paper, we investigate radius problem for functions  $f$  satisfying the condition (1) to belong to one of the classes introduced above. We also prove the correct results corresponding to [15, Theorem 1(a) and Theorem 2(a), p. 100] that  $f \in \mathcal{CV}$  if  $f \in \mathcal{A}$  satisfies the condition (1) for some  $0 \leq \alpha \leq (12\sqrt{2} - 15)/9$ . Their result is correct only when  $\alpha = 0$ . Unlike the radii problems associated with starlikeness and convexity, where a central feature is the estimate for the real part of the expressions  $zf'(z)/f(z)$  or  $1 + zf''(z)/f'(z)$  respectively, the  $\mathcal{SL}$ -radius problems are tackled by first finding the disc that contains the values of  $zf'(z)/f(z)$  or  $1 + zf''(z)/f'(z)$ . The techniques used in this paper are earlier used for the class of uniformly convex functions investigated in [16–26].

For two analytic functions  $f, g \in \mathcal{A}$  with  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  and  $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ , their *convolution* or *Hadamard product*, denoted by  $f * g$ , is defined by  $(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n$ . We need the following results.

**Theorem 1.2** ([28, Theorem 2.1-2.2, p. 125]). *The classes  $\mathcal{ST}$ ,  $\mathcal{CV}$  and  $\mathcal{CCV}$  are closed under convolution with functions in  $\mathcal{CV}$ .*

**Theorem 1.3.** *The classes  $\mathcal{ST}(\beta)$ ,  $\mathcal{SL}$  and  $\mathcal{M}(\beta) \cap \mathcal{ST}$  are closed under convolution with functions in  $\mathcal{CV}$ .*

**Lemma 1.4** ([4, Lemma 2.2, p. 6559]). *Let  $0 < a < \sqrt{2}$ . If  $r_a$  is given by*

$$r_a = \begin{cases} \left( \sqrt{1-a^2} - (1-a^2) \right)^{1/2} & (0 < a \leq 2\sqrt{2}/3) \\ \sqrt{2}-a & (2\sqrt{2}/3 \leq a < \sqrt{2}), \end{cases}$$

*then  $\{w : |w - a| < r_a\} \subseteq \{w : |w^2 - 1| < 1\}$ .*

Theorem 1.3 is a special case of results of Shanmugam [27, Theorem 3.3, p. 336; Theorem 3.5, p. 337] (see also Ma and Minda [18, Theorem 5, p. 167]).

## 2 Radii problems associated with convex combinations

For functions satisfying the condition (1), we determine, in the first part of the following theorem, the range of  $\alpha$  so that the function is starlike of order  $\beta$  while the other parts of the theorem provide the radius of starlikeness of order  $\beta$ .

**Theorem 2.1.** *Let  $-1 \leq \alpha < 1$ ,  $0 \leq \beta < 1$  and  $f \in \mathcal{A}$  satisfy the condition (1).*

(a) *If  $0 \leq \beta \leq 1/2$  and  $|\alpha| \leq (1 - 2\beta)/(3 - 2\beta)$ , then  $f \in \mathcal{ST}(\beta)$ .*

(b) *If either*

(i)  *$0 \leq \beta \leq 1/2$ , and  $(1 - 2\beta)/(3 - 2\beta) < \alpha < 1$ , or*

(ii)  *$1/2 < \beta < 1$ , and  $(-\beta + 2\beta^2)/(6 - 7\beta + 2\beta^2) < \alpha < 1$ ,*

*then  $f(\rho_1 z)/(\rho_1) \in \mathcal{ST}(\beta)$  where  $\rho_1 = \rho_1(\alpha, \beta)$  is given by*

$$\rho_1(\alpha, \beta) = \left( \frac{\alpha(3 - 2\beta)^2 - (1 - 2\beta)^2}{4\sqrt{(1 - \alpha)\alpha^2(\alpha(\beta - 2) - \beta)(\beta - 1)} + \alpha(1 + 4\beta - 4\beta^2) + \alpha^2(7 - 12\beta + 4\beta^2)} \right)^{\frac{1}{2}}.$$

(c) If either

- (i)  $0 \leq \beta \leq 1/2$ , and  $-1 \leq \alpha < (1-2\beta)/(-3+2\beta)$ , or
- (ii)  $1/2 < \beta < 1$ , and  $-1 \leq \alpha < (-\beta+2\beta^2)/(6-7\beta+2\beta^2)$ ,

then  $f(\rho_0 z)/\rho_0 \in \mathcal{ST}(\beta)$  where  $\rho_0 = \rho_0(\alpha, \beta)$  is given by

$$\rho_0(\alpha, \beta) = \frac{2(1-\beta)}{\alpha(\beta-2) + \beta + \sqrt{-4\alpha(1-\beta)^2 + (\alpha(\beta-2) + \beta)^2}}.$$

*Proof.* Define the function  $g : \mathbb{D} \rightarrow \mathbb{C}$  by

$$g(z) = \frac{z - \alpha z^2}{1 - z}. \quad (2)$$

For a fixed  $\alpha \in [-1, 1)$ , define the function  $f_1 : \mathbb{D} \rightarrow \mathbb{C}$  by

$$f_1(z) = \frac{f(z) - \alpha z}{1 - \alpha}. \quad (3)$$

If  $f$  satisfies the condition (1), then it follows that the function  $f_1 \in \mathcal{CV}$ . With the function  $g$  defined by (2), the equation (3) shows that

$$f(z) = \alpha z + (1 - \alpha) f_1(z) = f_1(z) * g(z).$$

If  $g$  is starlike in the disc  $\mathbb{D}_\rho$ , then  $g(\rho z)/\rho$  is starlike and hence, by Theorem 1.2,  $f_1(z) * (g(\rho z)/\rho)$  is starlike or equivalently  $f_1 * g$  is starlike in the disc  $\mathbb{D}_\rho$ . In view of this, it is enough to investigate the radius of starlikeness of the function  $g$  given by (2).

For the function  $g$  given by (2), we have

$$\frac{zg'(z)}{g(z)} = \frac{1}{1-z} - \frac{\alpha z}{1-\alpha z}.$$

With  $z = re^{it}$  and  $x = \cos t$ , a calculation shows that

$$\begin{aligned} \operatorname{Re} \left( \frac{zg'(z)}{g(z)} \right) &= \frac{1 - r \cos t}{1 + r^2 - 2r \cos t} + \frac{r\alpha(r\alpha - \cos t)}{1 + r^2\alpha^2 - 2r\alpha \cos t} \\ &= \frac{1 + 2r^2\alpha^2 + r^4\alpha^2 - r(1 + 3\alpha)(1 + r^2\alpha)x + 4r^2\alpha x^2}{(1 + r^2 - 2rx)(1 - 2rx\alpha + r^2\alpha^2)}. \end{aligned}$$

Therefore,  $g$  is starlike of order  $\beta$  in  $|z| < \rho_1$  if, for all  $0 \leq r < \rho_1$  and for all  $x \in [-1, 1]$ , we have

$$\begin{aligned} 1 + 2r^2\alpha^2 + r^4\alpha^2 - r(1 + 3\alpha)(1 + r^2\alpha)x + 4r^2\alpha x^2 \\ - \beta(1 + r^2 - 2rx)(1 - 2rx\alpha + r^2\alpha^2) \geq 0. \end{aligned}$$

This inequality is equivalent to

$$\begin{aligned} h(r, x) := 4r^2\alpha(1 - \beta)x^2 + r(1 + r^2\alpha)(-1 - 3\alpha + 2\beta + 2\alpha\beta)x \\ + 1 + 2r^2\alpha^2 + r^4\alpha^2 - \beta - r^2\beta - r^2\alpha^2\beta - r^4\alpha^2\beta \geq 0 \end{aligned}$$

for  $0 \leq r < \rho_1$  and for all  $x \in [-1, 1]$ . It follows that the derivative of function  $h(r, x)$  with respect to  $x$  vanishes for

$$x = x_0 = \frac{(1 + r^2\alpha)(1 - 2\beta + \alpha(3 - 2\beta))}{8r\alpha(1 - \beta)} \quad (\alpha \neq 0).$$

It should be noted for later use that, for  $\beta \leq 1/2$  and  $\alpha \geq 0$ ,

$$\begin{aligned} h(r, 1) - h(r, -1) &= (1 - r)(1 - r\alpha)(1 + r^2\alpha(1 - \beta) - \beta + r(\alpha(\beta - 2) + \beta)) \\ &\quad - (1 + r)(1 + r\alpha)(1 + r^2\alpha(1 - \beta) - \beta + r(\alpha(2 - \beta) - \beta)) \\ &= 2r(1 + r^2\alpha)(-1 - 3\alpha + 2\beta + 2\alpha\beta) \\ &\leq 2r(1 + r^2\alpha)(-2\alpha) \leq 0, \end{aligned} \quad (4)$$

and so we have  $h(r, 1) \leq h(r, -1)$ .

(a) *Case (i)* Let  $0 \leq \beta \leq 1/2$  and  $0 \leq \alpha \leq (1-2\beta)/(3-2\beta)$ . If  $\beta = 1/2$ , then  $\alpha = 0$ ,  $h(r, x) = (1-r^2)/2$  and hence  $\min_{|x| \leq 1} h(r, x) > 0$  for  $0 \leq r < 1$ . If  $0 \leq \beta < 1/2$  and  $\alpha = 0$ , then

$$h(r, x) = 1 - \beta - r^2\beta + (2\beta - 1)rx$$

and so

$$\min_{|x| \leq 1} h(r, x) = h(r, 1) = (1-r)(1-\beta+\beta r) > 0 \quad \text{for } 0 \leq r < 1.$$

If  $0 \leq \beta < 1/2$  and  $0 < \alpha \leq (1-2\beta)/(3-2\beta)$ , then it can be verified that  $x_0 > 1$ . In view of this and from (4), it follows that  $\min_{|x| \leq 1} h(r, x) = h(r, 1)$ . Since  $h(r, 1)$  is a decreasing function of  $\alpha$ , we have, for  $0 \leq r < 1$ ,

$$\min_{|x| \leq 1} h(r, x) = h(r, 1) \geq \frac{(1-r)(3-2\beta+r(-1+2\beta))}{(3-2\beta)^2} > 0.$$

Therefore, for  $0 \leq r < 1$ ,  $\min_{|x| \leq 1} h(r, x) > 0$  and so  $g$  is starlike of order  $\beta$ .

*Case (ii)* Let  $0 \leq \beta < 1/2$  and  $(1-2\beta)/(-3+2\beta) < \alpha < 0$ . For fixed  $r$ , the second partial derivative of  $h(r, x)$  with respect to  $x$  is negative and so the minimum of  $h(r, x)$  in  $[-1, 1]$  is attained at the end points  $x = \pm 1$ . Using (4) and the fact that  $h(r, 1)$  is a decreasing function of  $\alpha$ ,  $\min_{|x| \leq 1} h(r, x) > 0$  for  $0 \leq r < 1$ , and  $\min_{|x| \leq 1} h(r, x) > 0$ . Therefore,  $g$  is starlike of order  $\beta$ .

(b) *Case (i)* Let  $0 \leq \beta \leq 1/2$  and  $(1-2\beta)/(3-2\beta) < \alpha < 1$ . It can be seen that  $-1 \leq x_0 \leq 1$  if

$$r \geq \frac{4\alpha(1-\beta) - \sqrt{(1-\alpha)\alpha(\alpha(3-2\beta)^2 - (1-2\beta)^2)}}{\alpha(1-2\beta + \alpha(3-2\beta))} := \gamma_0.$$

So, for  $\gamma_0 \leq r < \rho_1$ ,  $\min_{|x| \leq 1} h(r, x) = h(r, x_0) > 0$ , where

$$\begin{aligned} h(r, x_0) &= \frac{1-\alpha}{16\alpha(1-\beta)}(9\alpha - 1 + 4\beta - 12\alpha\beta - 4\beta^2 + 4\alpha\beta^2 - 2(1 + 7\alpha + 4\beta - 12\alpha\beta \\ &\quad - 4\beta^2 + 4\alpha\beta^2)\alpha r^2 + (9\alpha - 1 + 4\beta - 12\alpha\beta - 4\beta^2 + 4\alpha\beta^2)\alpha^2 r^4). \end{aligned}$$

Notice that the number  $\rho_1(\alpha, \beta)$  is the root of  $h(r, x_0) = 0$ . For  $0 \leq r < \gamma_0$ , since

$$s(\alpha) = 1 + r^2\alpha(1-\beta) - \beta + r(\alpha(\beta-2) + \beta)$$

is a decreasing function of  $\alpha$ ,

$$\begin{aligned} \min_{|x| \leq 1} h(r, x) &= h(r, 1) = (1-r)(1-r\alpha)(1+r^2\alpha(1-\beta) - \beta + r(\alpha(\beta-2) + \beta)) \\ &\geq (1-r)^4(1-\beta) > 0. \end{aligned} \tag{5}$$

Therefore,  $\min_{|x| \leq 1} h(r, x) \geq 0$  for  $0 \leq r < \rho_1$  and hence  $g$  is starlike of order  $\beta$  in  $|z| < \rho_1$ .

*Case (ii)* Let  $1/2 < \beta < 1$  and  $(1-2\beta)/(-3+2\beta) < \alpha < 1$ . Let  $r < \rho_1$ . It can be shown that

$$-1 \leq x_0 \leq 1 \text{ for } r \geq \frac{4\alpha(1-\beta) - \sqrt{(1-\alpha)\alpha(\alpha(3-2\beta)^2 - (1-2\beta)^2)}}{\alpha(1-2\beta + \alpha(3-2\beta))} := \gamma_1.$$

This shows that for  $\gamma_1 \leq r < \rho_1$ ,  $\min_{|x| \leq 1} h(r, x) = h(r, x_0) > 0$ .

On the other hand if  $0 < r < \gamma_1$ , then  $x_0 > 1$ . Now proceeding as in case (i) we get  $\min_{|x| \leq 1} h(r, x) = h(r, 1) > 0$ . Thus  $g$  is starlike of order  $\beta$  in  $|z| < \rho_1$ .

*Case (iii)* Let  $1/2 < \beta < 1$  and  $(-\beta+2\beta^2)/(6-7\beta+2\beta^2) < \alpha \leq (1-2\beta)/(-3+2\beta)$ . Let  $r_2$  be the root of the equation  $x_0 = -1$ . Then  $r_2 \leq \rho_1$ . Let  $r < \rho_1$ . If  $r \geq r_2$  then  $-1 \leq x_0 \leq 1$ . Thus  $\min_{|x| \leq 1} h(r, x) = h(r, x_0) > 0$ . On the other hand if  $r < r_2$ , then  $x_0 < -1$ . Also  $\rho_1 < \rho_0$ . Thus  $\min_{|x| \leq 1} h(r, x) = h(r, -1) > 0$  in  $|z| < \rho_1$  and hence  $g$  is starlike of order  $\beta$  in  $|z| < \rho_1$ .

(c) *Case (i)* Let  $0 \leq \beta \leq 1/2$  and  $-1 \leq \alpha < (1-2\beta)/(-3+2\beta)$ . For fixed  $r$ , the second derivative test shows that the minimum of  $h(r, x)$  in  $[-1, 1]$  is attained at the end points  $x = \pm 1$ . It follows from (4) that  $\min_{|x| \leq 1} h(r, x) = h(r, -1)$ . Notice that

$$h(r, -1) = (1+r)(1+r\alpha)(1-\beta - r(\beta + \alpha(-2+r(-1+\beta) + \beta)))$$

and the number  $\rho_0(\alpha, \beta)$  is the root of  $h(r, -1) = 0$ . Since  $h(r, -1) > 0$  for  $r < \rho_0$ ,  $\min_{|x| \leq 1} h(r, x) > 0$  in  $|z| < \rho_0$ . Thus  $g$  is starlike of order  $\beta$  in  $|z| < \rho_0$ .

*Case (ii)* Let  $1/2 < \beta < 1$  and  $0 < \alpha \leq (-\beta + 2\beta^2)/(6 - 7\beta + 2\beta^2)$ . Then  $r_2 \geq \rho_0$ . Let  $r < \rho_0$ . Since  $x_0 < -1$ , we have  $\min_{|x| \leq 1} h(r, x) = h(r, -1) > 0$  in  $|z| < \rho_0$ . Thus  $g$  is starlike of order  $\beta$  in  $|z| < \rho_0$ .

*Case (iii)* Let  $1/2 < \beta < 1$  and  $-1 \leq \alpha \leq 0$ . For fixed  $r$ , the minimum of  $h(r, x)$  in  $[-1, 1]$  is attained at the end points  $x = \pm 1$ . Also  $h(r, 1) - h(r, -1) > 0$ . Thus  $\min_{|x| \leq 1} h(r, x) = h(r, -1) > 0$  in  $|z| < \rho_0$ . Thus  $g$  is starlike of order  $\beta$  in  $|z| < \rho_0$ .

To prove the sharpness, consider the functions  $f$  and  $g$  given by

$$f(z) = g(z) = \frac{z - \alpha z^2}{1 - z}. \quad (6)$$

For the function  $f$  given in (6), we have

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z) - \alpha} \right) = \operatorname{Re} \left( \frac{1+z}{1-z} \right) > 0.$$

The function  $f$  satisfies the condition (1) and since radius is sharp for the function  $g$ , the sharpness follows.  $\square$

**Remark 2.2.** If  $-1/3 \leq \alpha \leq 1/3$  and  $f \in \mathcal{A}$  satisfy the condition (1), then  $f \in \mathcal{ST}(\beta)$  where

$$\beta = \frac{1 - 3|\alpha|}{2(1 - |\alpha|)} \geq 0.$$

In particular, we have the following corollary.

**Corollary 2.3** ([15, Theorem 2(b), p. 100]). If  $f$  satisfies the condition (1) for some  $\alpha$  with  $|\alpha| \leq \frac{1}{3}$ , then  $f \in \mathcal{ST}$ .

For other ranges of  $\alpha$ , we have the following corollary.

**Corollary 2.4.** The radius of starlikeness of the class of functions  $f \in \mathcal{A}$  satisfying the condition (1) for  $\alpha \in (-1, -1/3] \cup [1/3, 1)$  is given by

$$r = \begin{cases} \left( \frac{9\alpha - 1}{\alpha + 7\alpha^2 + 4\sqrt{2}\sqrt{(1-\alpha)\alpha^3}} \right)^{\frac{1}{2}} & \text{if } -\frac{1}{3} \leq \alpha < 1, \\ \frac{1}{\sqrt{\alpha(\alpha-1)-\alpha}} & \text{if } -1 < \alpha \leq -\frac{1}{3}. \end{cases}$$

**Theorem 2.5.** For  $-1 \leq \alpha < 1$ , the  $\mathcal{SL}$ -radius of the class of functions  $f \in \mathcal{A}$  satisfying the condition (1) is given by

$$\rho_2(\alpha) = \frac{2 - \sqrt{2}}{1 + \alpha - \sqrt{2}\alpha + \sqrt{(1-\alpha)(1-3\alpha+2\sqrt{2}\alpha)}}.$$

*Proof.* First we observe that

$$|(1 - \alpha z)(1 - z)| \geq (1 - \alpha r)(1 - r), \quad |z| = r < 1. \quad (7)$$

For  $0 \leq \alpha < 1$ , (7) is trivial. For  $-1 \leq \alpha < 0$ , the inequality (7) holds as the function

$$h(r, x) := |(1 - \alpha z)(1 - z)|^2 = (1 - 2r\alpha x + r^2\alpha^2)(1 - 2rx + r^2),$$

where  $x = \cos t$ , is a decreasing function. Since the function  $g$  given by (2) satisfies

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| = \left| \frac{(1-\alpha)z}{(1-z)(1-\alpha z)} \right| \leq \frac{r(1-\alpha)}{(1-r)(1-\alpha r)}, \quad (8)$$

Equation 8 and Lemma 1.4 yield

$$\left| \left( \frac{zg'(z)}{g(z)} \right)^2 - 1 \right| < 1 \quad (|z| < r)$$

provided

$$1 - \sqrt{2} + \sqrt{2} \left(1 - \sqrt{2}\alpha + \alpha\right) r + \alpha \left(1 - \sqrt{2}\right) r^2 \leq 0 \quad \text{or} \quad r \leq \rho_2.$$

Thus  $g(\rho_2 z)/\rho_2 \in \mathcal{SL}$ . As the function  $f$  satisfies (1), the function  $f_1$  defined by (3) is convex. Hence, by Theorem 1.3, we have

$$\frac{f(\rho_2 z)}{\rho_2} = f_1(z) * \frac{g(\rho_2 z)}{\rho_2} \in \mathcal{SL}$$

or, equivalently

$$\left| \left( \frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < 1 \quad (|z| < \rho_2).$$

For  $z = \rho_2$ , the function  $g$  given by (2) satisfies

$$\left| \left( \frac{zg'(z)}{g(z)} \right)^2 - 1 \right| = \left| \left( \frac{1}{1 - \rho_2} - \frac{\alpha \rho_2}{1 - \alpha \rho_2} \right)^2 - 1 \right| = \left| \left( \frac{\rho_2(1 - \alpha)}{(1 - \rho_2)(1 - \alpha \rho_2)} + 1 \right)^2 - 1 \right| = 1.$$

Thus, the result is sharp for the function

$$f(z) = g(z) = \frac{z - \alpha z^2}{1 - z}.$$

□

**Corollary 2.6.** *The  $\mathcal{SL}$ -radius of the class  $\mathcal{CV}$  of convex functions is  $2 - \sqrt{2}$ .*

**Theorem 2.7.** *For  $-1/3 \leq \alpha \leq 1/3$  and  $\beta > 1$ , the  $\mathcal{M}_\beta \cap \mathcal{ST}$  radius of the class of functions  $f \in \mathcal{A}$  satisfying the condition (1) is given by*

$$\rho_3(\alpha, \beta) = \frac{2(\beta - 1)}{\beta - 2\alpha + \alpha\beta + \sqrt{(\alpha - 1)(4\alpha - 4\alpha\beta - \beta^2 + \alpha\beta^2)}}.$$

*Proof.* From the inequality (8) for the function  $g$  given by (2), we have

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \leq 1 + \frac{r(1 - \alpha)}{(1 - r)(1 - \alpha r)} \leq \beta$$

provided

$$1 - \beta + (-2\alpha + \beta + \alpha\beta)r + \alpha(1 - \beta)r^2 \leq 0.$$

The last inequality holds if  $0 < r \leq \rho_3(\alpha, \beta)$ . Thus  $g(\rho_3 z)/\rho_3 \in \mathcal{M}_\beta$ . Since the function  $f$  satisfies (1), the function  $f_1$  defined by (3) is convex. Hence, by Theorem 1.3, we have

$$\frac{f(\rho_3 z)}{\rho_3} = f_1(z) * \frac{g(\rho_3 z)}{\rho_3} \in \mathcal{M}_\beta$$

or

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) < \beta \quad (|z| < \rho_3).$$

Also, by Corollary 2.3,

$$0 < \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) \quad (|z| < 1).$$

Thus  $f(\rho_3 z)/\rho_3 \in \mathcal{M}_\beta \cap \mathcal{ST}$ . For  $z = \rho_3$ , the function  $g$  given by (2) satisfies

$$\operatorname{Re} \frac{zg'(z)}{g(z)} = 1 + \frac{\rho_3(1 - \alpha)}{(1 - \rho_3)(1 - \alpha\rho_3)} = \beta.$$

Thus the result is sharp for the function

$$f(z) = g(z) = \frac{z - \alpha z^2}{1 - z}.$$

□

**Corollary 2.8.** *For  $\beta > 1$ , the  $\mathcal{M}_\beta \cap \mathcal{ST}$ -radius of the class  $\mathcal{CV}$  of convex functions is  $1 - \beta^{-1}$ .*

The results [15, Theorem 1(a) and Theorem 2(a), p. 100] of Obradovic and Nunokawa are incorrect. The correct version of these results are given in the following theorem.

**Theorem 2.9.** *For  $-1 \leq \alpha < 1$ , the radius of convexity of the class of functions  $f \in \mathcal{A}$  satisfying the condition (1) is*

$$\rho_4(\alpha) = \begin{cases} r_1, & \text{if } 0 \leq \alpha < 1, \\ r_2 & \text{if } -1 \leq \alpha \leq 0, \end{cases} \quad (9)$$

where  $r_1 \in (0, 1]$  is the root of the equation in  $r$ :

$$1 - r^2(1 + \alpha) - r^4\alpha(1 - 2\alpha) - 2r^3\alpha\sqrt{(1 - \alpha)(1 - r^2\alpha)} = 0$$

and  $r_2 \in (0, 1]$  is the root of the equation in  $r$ :

$$1 + 6\alpha r - (1 - 8\alpha - 8\alpha^2)r^2 + 2\alpha(1 + 9\alpha)r^3 + 15\alpha^2r^4 + 6\alpha^2r^5 + \alpha^2r^6 = 0.$$

*Proof.* If the function  $f$  satisfies (1) with  $\alpha = 0$ , then the function  $f$  is convex and so  $\rho_4(0) = 1$  as claimed. Now, assume that  $\alpha \neq 0$ . For the function  $g$  given by (2), a calculation shows that, with  $x = \cos t$ ,

$$\begin{aligned} \operatorname{Re} \left( 1 + \frac{zg''(z)}{g'(z)} \right) &= \frac{1 - r^2}{1 + r^2 - 2r \cos t} \\ &+ \frac{2r\alpha(-(1 + 3r^2\alpha)\cos t + r((2 + r^2)\alpha + \cos(2t)))}{1 + r^2(4 + r^2)\alpha^2 + 2r\alpha(-2(1 + r^2\alpha)\cos t + r\cos(2t))} \\ &= \frac{\phi(r, x)}{h(r, x)} \end{aligned}$$

where

$$h(r, x) = (1 + r^2 - 2rx)(1 + r^2(4 + r^2)\alpha^2 - 2r\alpha(r + 2x - 2rx^2 + 2r^2x\alpha))$$

and

$$\begin{aligned} \phi(r, x) &= 1 - r^2 - 4r^2\alpha + 8r^2\alpha^2 + 3r^4\alpha^2 + r^6\alpha^2 - 6r\alpha(1 - r^2 + 3r^2\alpha + r^4\alpha)x \\ &+ 12r^2\alpha(1 + r^2\alpha)x^2 - 8r^3\alpha x^3. \end{aligned} \quad (10)$$

Then

$$\frac{\partial}{\partial x} \phi(r, x) = -6r\alpha(1 - r^2 + 4r^2x^2 + 3r^2\alpha + r^4\alpha) + 24r^2x\alpha(1 + r^2\alpha).$$

A calculation shows that

$$\frac{\partial}{\partial x} \phi(r, x) = 0 \quad \text{if } x = x_0 = \frac{1 + r^2\alpha - r\sqrt{(1 - \alpha)(1 - r^2\alpha)}}{2r}$$

and

$$\frac{\partial^2}{\partial x^2} \phi(r, x_0) = 24r^3\alpha\sqrt{(1 - \alpha)(1 - r^2\alpha)} > 0 \quad \text{for } 0 < \alpha < 1.$$

So for  $0 < \alpha < 1$ ,

$$\begin{aligned} \min_{|x| \leq 1} \phi(r, x) &= \phi(r, x_0) = (1 - \alpha)(1 - r^2\alpha)(1 - r^2(1 + \alpha) - r^4\alpha(1 - 2\alpha) \\ &- 2r^3\alpha\sqrt{(1 - \alpha)(1 - r^2\alpha)}) > 0. \end{aligned}$$

On the other hand, if  $-1 < \alpha < 0$ , then, for  $r < r_2$ , it can be verified that

$$\begin{aligned} \min_{|x| \leq 1} \phi(r, x) &= \phi(r, -1) = 1 + 6\alpha r - (1 - 8\alpha - 8\alpha^2)r^2 + 2\alpha(1 + 9\alpha)r^3 \\ &+ 15\alpha^2r^4 + 6\alpha^2r^5 + \alpha^2r^6 > 0. \end{aligned}$$

Thus  $g(\rho_4 z)/\rho_4 \in \mathcal{CV}$ . Since the function  $f$  satisfies (1), the function  $f_1$  defined by (3) is convex. Hence

$$\frac{f(\rho_4 z)}{\rho_4} = f_1(z) * \frac{g(\rho_4 z)}{\rho_4} \in \mathcal{CV}$$

or

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (|z| < \rho_4).$$

The result is sharp for the function  $f$  given by (6).  $\square$

**Theorem 2.10.** *For  $-1 \leq \alpha < -1/3$ , the radius of close-to-convexity of the class of functions  $f \in \mathcal{A}$  satisfying the condition (1) is given by*

$$\rho_5(\alpha) = \frac{1}{\sqrt{\alpha(\alpha-1)} - \alpha}.$$

*Proof.* The function  $g_1 : \mathbb{D} \rightarrow \mathbb{C}$  by

$$g_1(z) = -\ln(1-z) \quad (z \in \mathbb{D})$$

is clearly convex in  $\mathbb{D}$ . For the functions  $g$  given by (2) and  $g_1$  above, we have

$$\operatorname{Re} \left( \frac{g'(re^{it})}{g_1'(re^{it})} \right) = \frac{1 + \alpha r^2 - (2\alpha + 1 + \alpha r^2)rx + 2\alpha r^2 x^2}{1 + r^2 - 2rx} \quad (11)$$

where  $x := \cos t$ . Let  $h : [-1, 1] \rightarrow \mathbb{R}$  be defined by

$$h(r, x) = 1 + \alpha r^2 - (2\alpha + 1 + \alpha r^2)rx + 2\alpha r^2 x^2.$$

Then

$$\frac{\partial}{\partial x} h(r, x) = 0 \quad \text{if} \quad x = x_0 = \frac{1 + 2\alpha + r^2 \alpha}{4r\alpha}$$

and

$$\frac{\partial^2}{\partial x^2} h(r, x) = 4r^2 \alpha < 0.$$

Therefore, for a fixed  $r$ , the minimum of  $h(r, x)$  is attained at  $x = \pm 1$ . Since

$$h(r, -1) - h(r, 1) = 2r(1 + 2\alpha + r^2 \alpha) < 0,$$

it follows that

$$\min_{|x| \leq 1} h(r, x) = h(r, -1) = (1 + r) \left( 1 + 2r\alpha + r^2\alpha \right) > 0 \quad \text{for} \quad r < \rho_5.$$

Thus  $g(\rho_5 z)/\rho_5 \in \mathcal{CCV}$ . Since the function  $f_1$  defined by (3) is convex as  $f$  satisfies (1), we have, by Theorem 1.2,

$$\frac{f(\rho_5 z)}{\rho_5} = f_1(z) * \frac{g(\rho_5 z)}{\rho_5} \in \mathcal{CCV}$$

or

$$\operatorname{Re} \left( \frac{g'(z)}{g_1'(z)} \right) > 0 \quad (|z| < \rho_5).$$

The result is sharp for the function

$$f(z) = g(z) = \frac{z - \alpha z^2}{1 - z}. \quad \square$$

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