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**Research Article**

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# The hybrid power mean of quartic Gauss sums and Kloosterman sums

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**Abstract:** The main purpose of this paper is using the analytic method and the properties of the classical Gauss sums to study the computational problem of one kind fourth hybrid power mean of the quartic Gauss sums and Kloosterman sums, and give an exact computational formula for it.

**Keywords:** The quartic Gauss sums, Kloosterman sums, Classical Gauss sums, Fourth hybrid power mean, Computational formula

**MSC:** 11L05

## 1 Introduction

Let $q \geq 3$ be a positive integer. For any positive integer $k \geq 2$, integers $m$ and $n$, the $k$-th Gauss sums $G(m,k;q)$ and Kloosterman sums $K(m,n;q)$ are defined as

$$G(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k}{q}\right) \quad \text{and} \quad K(m,n;q) = \sideset{}{'}\sum_{a=1}^{q} e\left(\frac{ma+n\overline{a}}{q}\right),$$

where $\sum{}'^{q}_{a=1}$ denotes the summation over all $1 \leq a \leq q$ such that $(a,q)=1$, $e(y)=e^{2\pi iy}$, and $\overline{a}$ denotes the multiplicative inverse of $a$ mod $q$ ($a\overline{a} \equiv 1 \bmod q$).

Concerning the various properties of $G(m,k;q)$ and $K(m,n;q)$, many authors have studied them, and obtained several results, see [1-7]. For example, from the A.Weil's important work [1], one can get the upper bound estimate

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^k}{p}\right)\right| \ll_k \sqrt{p},$$

where $p$ is an odd prime, $\chi$ denotes any Dirichlet character mod $p$, and $\ll_k$ denotes the big-$O$ constant depending on $k$.

Zhang Wenpeng and Liu Huaning [2] studied the fourth power mean of $G(m,k,p)$, and obtained some sharp asymptotic formulae for it.

T. Estermann [3] proved the upper bound

$$|K(m,n;q)| \leq (m,n,q)^{\frac{1}{2}} \cdot d(q) \cdot q^{\frac{1}{2}},$$

where $(m,n,q)$ denotes the greatest common divisor of integers $m,n$ and $q$, $d(q)$ denotes the number of divisors of $q$.

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H. D. Kloosterman [4] studied the fourth power mean of $K(a,1;p)$, and proved the identity

$$\sum_{a=1}^{p-1} K^4(a,1;p) = 2p^3 - 3p^2 - 3p - 1.$$

For general odd integer $q \geq 3$ and $(n,q) = 1$, Zhang Wenpeng [8] proved the identity

$$\sum_{m=1}^{q} \mid K(m,1;q) \mid^4 = 3^{\omega(q)} q^2 \phi(q) \prod_{p\|q} \left( \frac{2}{3} - \frac{1}{3p} - \frac{4}{3p(p-1)} \right),$$

where $\phi(q)$ is Euler function, $\omega(q)$ denotes all distinct prime divisors of $q$, $\prod_{p\|q}$ denotes the product of all prime divisors of $q$ such that $p \mid q$ and $(p, q/p) = 1$.

Some related works can also be found in [8-10].

Now let $p$ be an odd prime with $p \equiv 1 \bmod 4$. For any positive integer $k$, we consider the fourth hybrid power mean

$$\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^{2h} \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2. \tag{1}$$

In this paper, we are concerned with the calculating problem of (1). Regarding this, as far as we knew, it seems that nobody has studied it yet, at least we are not aware of such work. The problem is interesting, because it can help us to understand more accurate information of the hybrid mean value of the quartic Gauss sums and the classical Kloosterman sums.

In this paper, we shall use the analytic methods and the properties of Gauss sums to study the calculating problem of (1), and give an interesting computational formula for (1) with $h = 1$. That is, we shall prove the following:

**Theorem.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 4$. Then we have the identity*

$$\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^{2} \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2$$
$$= \begin{cases} 3p^3 - 3p^2 - 3p + p\left(\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)\right), & \text{if } p \equiv 5 \bmod 8; \\ 3p^3 - 3p^2 - 3p - p\tau^2\left(\overline{\chi}_4\right) - p\tau^2(\chi_4) + 2\tau^5\left(\overline{\chi}_4\right) + 2\tau^5(\chi_4), & \text{if } p \equiv 1 \bmod 8, \end{cases}$$

*where $\chi_4$ is any four order character* mod $p$, *$\tau(\chi) = \sum_{a=1}^{p-1} \chi(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums.*

Note that $|\tau(\chi_4)| = \sqrt{p}$, from our theorem we may immediately deduce the following two corollaries:

**Corollary 1.1.** *Let $p$ be an odd prime with $p \equiv 5 \bmod 8$. Then we have the asymptotic formula*

$$\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^{2} \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2 = 3p^3 + O\left(p^2\right).$$

**Corollary 1.2.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 8$. Then we have the asymptotic formula*

$$\sum_{b=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right) \right|^{2} \cdot \left| \sum_{c=1}^{p-1} e\left(\frac{bc + \bar{c}}{p}\right) \right|^2 = 3p^3 + O\left(p^{\frac{5}{2}}\right).$$

**Some notes.**

(A) *In our theorem, we only considered the case $p \equiv 1 \bmod 4$. If $p \equiv 3 \bmod 4$, then we have $|G(b,4;p)| = \sqrt{p}$ for any $(b,p) = 1$. So in this case, the conclusion is very simple.*

(B) *It is clear that in our theorem, there exist two terms $\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)$ and $\tau^5\left(\overline{\chi}_4\right) + \tau^5(\chi_4)$. These terms make our conclusions look slightly inelegant. Thus, how to compute the exact value of $\tau^2\left(\overline{\chi}_4\right) + \tau^2(\chi_4)$ and $\tau^5\left(\overline{\chi}_4\right) + \tau^5(\chi_4)$ will be the two meaningful problems.*

(C) *Whether there exits an exact calculating formula for (1) with $h \geq 2$ is also an open problem. This will be the focus of our further research.*

## 2 Several lemmas

In this section, we give several lemmas which are necessary for the proof of our theorem. Hereinafter, we shall use many properties of Gauss sums, all of them can be found in [11], so we will not be repeated here. First we have the following:

**Lemma 2.1.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 4$, $\chi$ be any character* mod $p$. *Then we have the identity*

$$\sum_{b=1}^{p-1}\chi(b)\left(\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)=\tau^2\left(\chi\chi_2\right)\sqrt{p}+\tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right)+\tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right),$$

*where $\chi_4$ is any four order character* mod $p$, $\left(\frac{*}{p}\right)=\chi_2$ *is the Legendre's symbol, and* $\tau(\chi)=\sum_{a=1}^{p-1}\chi(a)e\left(\frac{a}{p}\right)$ *denotes the classical Gauss sums.*

*Proof.* For any integer $b$ with $(b,p)=1$, note that $\chi_4^2=\chi_2$, $\tau(\chi_2)=\sqrt{p}$ and the trigonometric identity

$$\sum_{m=0}^{p-1}e\left(\frac{nm}{p}\right)=\begin{cases}p, & \text{if } (p,n)=p;\\ 0, & \text{if } (p,n)=1,\end{cases}$$

from the definition and properties of Gauss sums we have

$$\begin{aligned}&\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)=1+\sum_{a=1}^{p-1}e\left(\frac{ba^4}{p}\right)=1+\sum_{a=1}^{p-1}\left(1+\chi_4(a)+\chi_2(a)+\overline{\chi}_4(a)\right)e\left(\frac{ba}{p}\right)\\ &=\sum_{a=0}^{p-1}e\left(\frac{ba}{p}\right)+\overline{\chi}_4(b)\tau\left(\chi_4\right)+\chi_2(b)\tau\left(\chi_2\right)+\chi_4(b)\tau\left(\overline{\chi}_4\right)\\ &=\chi_2(b)\sqrt{p}+\overline{\chi}_4(b)\tau\left(\chi_4\right)+\chi_4(b)\tau\left(\overline{\chi}_4\right).\end{aligned}\tag{2}$$

where we have used the identity $\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma}{p}\right)=\overline{\chi}(m)\tau(\chi)$.

Now for any character $\chi$ mod $p$, note that the identity

$$\sum_{b=1}^{p-1}\chi(b)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)=\sum_{c=1}^{p-1}e\left(\frac{\overline{c}}{p}\right)\sum_{b=1}^{p-1}\chi(b)e\left(\frac{bc}{p}\right)=\tau(\chi)\sum_{c=1}^{p-1}\overline{\chi}(c)e\left(\frac{\overline{c}}{p}\right)=\tau^2(\chi),$$

from (2) we have

$$\begin{aligned}&\sum_{b=1}^{p-1}\chi(b)\left(\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)\\ &=\sum_{b=1}^{p-1}\chi(b)\left(\chi_2(b)\sqrt{p}+\overline{\chi}_4(b)\tau\left(\chi_4\right)+\chi_4(b)\tau\left(\overline{\chi}_4\right)\right)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)\\ &=\tau^2\left(\chi\chi_2\right)\sqrt{p}+\tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right)+\tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right).\end{aligned}$$

This proves Lemma 2.1. □

**Lemma 2.2.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 4$, $\chi_4$ be any four order character* mod $p$. *Then we have the identity*

$$\sum_{\chi \bmod p}\tau^2\left(\chi\chi_4\right)\overline{\tau\left(\chi\overline{\chi}_4\right)}^2=-p(p-1).$$

*Proof.* It is clear that from the properties of the reduced residue system mod $p$ and that $\chi_2=\overline{\chi}_2$, $\chi_4^2=\chi_2$ we have

$$\tau\left(\chi\chi_4\right)\overline{\tau\left(\chi\overline{\chi}_4\right)}=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\chi_4(a)\overline{\chi}(b)\chi_4(b)e\left(\frac{a-b}{p}\right)$$

$$= \sum_{a=1}^{p-1} \chi(a)\chi_4(a) \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b(a-1)}{p}\right) = \tau(\chi_2) \sum_{a=1}^{p-1} \chi(a)\chi_4(a)\chi_2(a-1)$$
$$= \sqrt{p} \cdot \sum_{a=1}^{p-1} \chi(a)\chi_4(a)\chi_2(a-1). \tag{3}$$

Now from (3) and the orthogonality of characters mod $p$ we have

$$\sum_{\chi \bmod p} \tau^2(\chi\chi_4) \overline{\tau(\chi\overline{\chi}_4)}^2 = p \sum_{\chi \bmod p} \left( \sum_{a=1}^{p-1} \chi(a)\chi_4(a)\chi_2(a-1) \right)^2$$
$$= p \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\chi \bmod p} \chi(ab)\chi_4(ab)\chi_2((a-1)(b-1))$$
$$= p(p-1) \sum_{a=1}^{p-1} \chi_2((a-1)(\overline{a}-1)) = p(p-1) \sum_{a=1}^{p-1} \overline{\chi}_2(-a)\chi_2^2(a-1). \tag{4}$$

For $p \equiv 1 \bmod 4$, we know that $\chi_2(-1) = 1$, $\chi_2^2(a-1) = 1$ for all $2 \le a \le p-1$, and $\sum_{a=1}^{p-1} \chi_2(a) = 0$, from (4) we may immediately deduce that

$$\sum_{\chi \bmod p} \tau^2(\chi\chi_4) \overline{\tau(\chi\overline{\chi}_4)}^2 = p(p-1) \sum_{a=2}^{p-1} \chi_2(a) = -p(p-1).$$

This proves Lemma 2.2. □

**Lemma 2.3.** *Let $p$ be an odd prime with $p \equiv 1 \bmod 4$, $\chi_4$ be any four order character* mod $p$. *Then we have the identity*

$$\sum_{\chi \bmod p} \tau^2(\chi\chi_2) \overline{\tau(\chi\overline{\chi}_4)}^2 = \frac{p-1}{\sqrt{p}} \cdot \tau^4(\overline{\chi}_4).$$

*Proof.* It is clear that from the properties of the reduced residue system mod $p$ and that $\chi_2 = \overline{\chi}_2$, $\chi_4^3 = \overline{\chi}_4$ we have

$$\tau(\chi\chi_2) \overline{\tau(\chi\overline{\chi}_4)} = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi(a)\chi_2(a)\overline{\chi}(b)\chi_4(b) e\left(\frac{a-b}{p}\right)$$
$$= \sum_{a=1}^{p-1} \chi(a)\chi_2(a) \sum_{b=1}^{p-1} \overline{\chi}_4(b) e\left(\frac{b(a-1)}{p}\right) = \tau(\overline{\chi}_4) \sum_{a=1}^{p-1} \chi(a)\chi_2(a)\chi_4(a-1). \tag{5}$$

From (5) and the orthogonality of characters mod $p$, we have

$$\sum_{\chi \bmod p} \tau^2(\chi\chi_2) \overline{\tau(\chi\overline{\chi}_4)}^2 = \tau^2(\overline{\chi}_4) \sum_{\chi \bmod p} \left( \sum_{a=1}^{p-1} \chi(a)\chi_2(a)\chi_4(a-1) \right)^2$$
$$= \tau^2(\overline{\chi}_4) \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{\chi \bmod p} \chi(ab)\chi_2(ab)\chi_4((a-1)(b-1))$$
$$= \tau^2(\overline{\chi}_4)(p-1) \sum_{a=1}^{p-1} \chi_4((a-1)(\overline{a}-1)). \tag{6}$$

From the properties of Gauss sums we have

$$\sum_{a=1}^{p-1} \chi_4((a-1)(\overline{a}-1)) = \overline{\chi}_4(-1) \sum_{a=1}^{p-1} \overline{\chi}_4(a)\chi_4^2(a-1)$$

$$= \overline{\chi}_4(-1)\sum_{a=1}^{p-1}\overline{\chi}_4(a)\chi_2(a-1) = \frac{\overline{\chi}_4(-1)}{\tau(\chi_2)}\sum_{a=1}^{p-1}\overline{\chi}_4(a)\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b(a-1)}{p}\right)$$
$$= \frac{\overline{\chi}_4(-1)}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{-b}{p}\right)\sum_{a=1}^{p-1}\overline{\chi}_4(a)e\left(\frac{ab}{p}\right)$$
$$= \frac{\overline{\chi}_4(-1)\tau\left(\overline{\chi}_4\right)}{\tau(\chi_2)}\sum_{b=1}^{p-1}\chi_2(b)\chi_4(b)e\left(\frac{-b}{p}\right) = \frac{\tau^2\left(\overline{\chi}_4\right)}{\sqrt{p}}. \tag{7}$$

Combining (6) and (7) we deduce that

$$\sum_{\chi \bmod p}\tau^2\left(\chi\chi_2\right)\overline{\tau\left(\chi\overline{\chi}_4\right)}^2 = \frac{p-1}{\sqrt{p}}\cdot\tau^4\left(\overline{\chi}_4\right).$$

This proves Lemma 2.3. □

## 3 Proof of the theorem

In this section, we shall complete the proof of our theorem. First, from the orthogonality of characters mod $p$ we have

$$\sum_{\chi \bmod p}\left|\sum_{b=1}^{p-1}\chi(b)\left(\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)\right|^2 = (p-1)\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right|^2 \tag{8}$$

On the other hand, note that $|\tau(\chi_4)|^2 = p$, from Lemma 2.1 we have

$$\sum_{\chi \bmod p}\left|\sum_{b=1}^{p-1}\chi(b)\left(\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right)\left(\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right)\right|^2$$
$$= \sum_{\chi \bmod p}\left(\tau^2\left(\chi\chi_2\right)\sqrt{p} + \tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right) + \tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right)\right)$$
$$\times\left(\overline{\tau^2\left(\chi\chi_2\right)}\sqrt{p} + \overline{\tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right)} + \overline{\tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right)}\right)$$
$$= p\sum_{\chi \bmod p}\left(\left|\tau\left(\chi\chi_2\right)\right|^4 + \left|\tau\left(\chi\overline{\chi}_4\right)\right|^4 + \left|\tau\left(\chi\chi_4\right)\right|^4\right)$$
$$+\sqrt{p}\sum_{\chi \bmod p}\tau^2\left(\chi\chi_2\right)\left(\overline{\tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right)} + \overline{\tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right)}\right)$$
$$+\sqrt{p}\sum_{\chi \bmod p}\overline{\tau^2\left(\chi\chi_2\right)}\left(\tau^2\left(\chi\overline{\chi}_4\right)\tau\left(\chi_4\right) + \tau^2\left(\chi\chi_4\right)\tau\left(\overline{\chi}_4\right)\right)$$
$$+\chi_4(-1)\tau^2(\chi_4)\sum_{\chi \bmod p}\tau^2\left(\chi\overline{\chi}_4\right)\overline{\tau^2\left(\chi\chi_4\right)}$$
$$+\chi_4(-1)\tau^2(\overline{\chi}_4)\sum_{\chi \bmod p}\tau^2\left(\chi\chi_4\right)\overline{\tau^2\left(\chi\overline{\chi}_4\right)}. \tag{9}$$

It is clear that if $\chi\chi_2 \neq \chi_0$, the principal character mod $p$, then $\left|\tau\left(\chi\chi_2\right)\right|^4 = p^2$; If $\chi\chi_2 = \chi_0$, then $\left|\tau\left(\chi\chi_2\right)\right|^4 = 1$ (The same reason for $\left|\tau\left(\chi\overline{\chi}_4\right)\right|^4$ and $\left|\tau\left(\chi\chi_4\right)\right|^4$). So we have

$$\sum_{\chi \bmod p}\left(\left|\tau\left(\chi\chi_2\right)\right|^4 + \left|\tau\left(\chi\overline{\chi}_4\right)\right|^4 + \left|\tau\left(\chi\chi_4\right)\right|^4\right)$$
$$= 3p^2(p-2) + 3 = 3(p-1)(p^2-p-1). \tag{10}$$

From Lemma 2.2 we have

$$\chi_4(-1)\left(\tau^2(\chi_4)\sum_{\chi \bmod p}\tau^2\left(\chi\overline{\chi}_4\right)\overline{\tau^2\left(\chi\chi_4\right)} + \tau^2(\overline{\chi}_4)\sum_{\chi \bmod p}\tau^2\left(\chi\chi_4\right)\overline{\tau^2\left(\chi\overline{\chi}_4\right)}\right)$$

$$= -\chi_4(-1)\left(\tau^2(\chi_4)+\tau^2(\overline{\chi}_4)\right)p(p-1). \tag{11}$$

Note that $\overline{\tau(\chi_4)} = \chi_4(-1)\tau(\overline{\chi}_4)$, from Lemma 2.3 we have

$$\sqrt{p}\sum_{\chi \bmod p}\tau^2(\chi\chi_2)\left(\overline{\tau^2(\chi\overline{\chi}_4)\tau(\chi_4)}+\overline{\tau^2(\chi\chi_4)\tau(\overline{\chi}_4)}\right) = \chi_4(-1)(p-1)\left(\tau^5(\overline{\chi}_4)+\tau^5(\chi_4)\right). \tag{12}$$

From (12) we also have

$$\sqrt{p}\sum_{\chi \bmod p}\overline{\tau^2(\chi\chi_2)}\left(\tau^2(\chi\overline{\chi}_4)\tau(\chi_4)+\tau^2(\chi\chi_4)\tau(\overline{\chi}_4)\right)$$
$$= \chi_4(-1)(p-1)\left(\overline{\tau^5(\overline{\chi}_4)}+\overline{\tau^5(\chi_4)}\right) = (p-1)\left(\tau^5(\overline{\chi}_4)+\tau^5(\chi_4)\right), \tag{13}$$

where we have used the identity $\overline{\tau(\chi_4)} = \overline{\chi}_4(-1)\tau(\overline{\chi}_4) = \chi_4(-1)\tau(\overline{\chi}_4)$.

It is clear that if $p = 8k+5$, then $\chi_4(-1) = -1$; if $p = 8k+1$, then $\chi_4(-1) = 1$. So if $p = 8k+5$, then from (8)-(13) we may immediately deduce the identity

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right|^2 = 3p^3-3p^2-3p+p\left(\tau^2(\overline{\chi}_4)+\tau^2(\chi_4)\right). \tag{14}$$

If $p = 8k+1$, then from (8)-(13) we can also deduce the identity

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ba^4}{p}\right)\right|^2\cdot\left|\sum_{c=1}^{p-1}e\left(\frac{bc+\overline{c}}{p}\right)\right|^2$$
$$= 3p^3-3p^2-3p-p\left(\tau^2(\overline{\chi}_4)+\tau^2(\chi_4)\right)+2\left(\tau^5(\overline{\chi}_4)+\tau^5(\chi_4)\right). \tag{15}$$

Now our theorem follows from (14) and (15).

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