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The Leibniz algebras whose subalgebras are ideals

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Abstract: In this paper we obtain the description of the Leibniz algebras whose subalgebras are ideals.

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Introduction

An algebra L over a field F is said to be a ***Leibniz algebra*** (more precisely a ***left Leibniz algebra***) if it satisfies the Leibniz identity

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]] \quad \text{for all} \quad a,b,c \in L. \quad \textbf{(LI)}$$

Leibniz algebras are generalizations of Lie algebras. Indeed, a Leibniz algebra L is a Lie algebra if and only if $[a,a] = 0$ for every element $a \in L$. For this reason, we may consider Leibniz algebras as "non-anticommutative" analogs of Lie algebras.

Leibniz algebra appeared first in the papers of A.M. Bloh [1–3], in which he called them the D-algebras. However, that time, the real study of these algebras has not begun. Only two decades later, a real interest in these algebras raised. The impetus for this was the work of J.L. Loday [11], who introduced the term ***Leibniz algebras*** since it was Leibniz who discovered and proved the "Leibniz rule" for differentiation of functions. The Leibniz algebras appeared to be naturally related to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics (see, for example, [4, 7, 8]). Some papers concerning Leibniz algebras are devoted to the study of homological problems [5, 9, 12, 13]. The theory of Leibniz algebras has been developing quite intensively but very uneven. On one hand, some deep structural theorems were obtained as analogues of the corresponding results about Lie algebras. On the other hand, the study of Leibniz algebras does not look consistent and systematic. For any algebraic structure there are some natural questions that were not previously considered for Leibniz algebras. Thus, for example, a natural question on the structure of the cyclic subalgebras in Leibniz algebras has just recently been considered [6]. Another natural question is a question about the structure of Leibniz algebras, whose subalgebras are ideals. We note that it is not hard to prove that a Lie algebra, whose subalgebras are ideals, is abelian. But it is not true for Leibniz algebras. The following simple example justifies it. Let L be a vector space of dimension 2 over a field F, $\{a,b\}$ be a basis of L. Define the operation $[,]$ by the following rule:

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$[a,a] = b, [b,b] = [b,a] = [a,b] = 0$. A direct check shows that L becomes a Leibniz algebra. If $\lambda a + \mu b$ is an arbitrary element of L and $\lambda \neq 0$, then $[\lambda a + \mu b, \lambda a + \mu b] = \lambda^2 b$. By $\lambda^2 \neq 0$ we obtain that the subalgebra generated by $\lambda a + \mu b$ includes Fb. Since L/Fb is abelian, $\langle \lambda a + \mu b \rangle$ is an ideal. Hence every cyclic subalgebra of L is an ideal. It follows that every subalgebra of L is an ideal. As we shall see later, any non-abelian Leibniz algebra, whose subalgebras are ideals, is built of these algebras as the bricks. Now consider more details.

If L is a Leibniz algebra and M is a subset of L, then by $\langle M \rangle$ we denote the subalgebra generated by M.

As usual, a Leibniz algebra L is called ***abelian*** if $[x, y] = 0$ for all elements $x, \ y \in L$. In an abelian Leibniz algebra every subspace is a subalgebra and an ideal.

A Leibniz algebra L is called an ***extraspecial*** algebra, if it satisfies the following condition:

"(G) is non-trivial and has dimension 1,

$L/\zeta(L)$ is abelian.

As it turned out, not in any extraspecial Leibniz algebra every subalgebra is an ideal. Now, we give an example of an extraspecial Leibniz algebra showing it. Moreover, the presence of subalgebras, which are not ideals, depend on the choice of the field.

Let F be a field, put $L = Fa \oplus Fb \oplus Fc$. Define on L an operation $[,]$ by the following rule: $c = [a,a] = [b,b] = [a,b], [c,c] = [c,a] = [c,b] = [a,c] = [b,c] = [b,a] = 0$. From this definition we can see that $[L,L] \leq Fc, c \in \zeta(L), \langle c \rangle = Fc$. Since $[x, y], [y, z], [x, z] \in \zeta(L)$, the equality

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

occurs automatically. Thus, L is a Leibniz algebra. Let x be an arbitrary element of L. Then $x = \lambda a + \mu b + \nu c$ for some $\lambda, \mu, \nu \in F$. We have

$$\begin{gathered}[x,x] = [\lambda a + \mu b + \nu c, \lambda a + \mu b + \nu c] = \\ \lambda^2[a,a] + \lambda\mu[a,b] + \lambda\nu[a,c] + \lambda\mu[b,a] + \mu^2[b,b] + \mu\nu[b,c] + \lambda\nu[c,a] + \mu\nu[c,b] + \\ \nu 2[c,c] = \lambda^2 c + \lambda\mu c + \mu^2 c = (\lambda^2 + \lambda\mu + \mu^2)c.\end{gathered}$$

Let $F = \mathbb{F}_2$. If $(\lambda, \mu) \neq (0,0)$, then $\lambda^2 + \lambda\mu + \mu^2 = 1$, that is $[x,x] = c$ whenever $x \notin Fc$. It follows that $\zeta(L) = Fc$ and $\langle x \rangle = Fx \oplus Fc$. It follows that $\langle x \rangle$ is an ideal of L. Since Fc is an ideal, we obtain that every subalgebra of L is an ideal.

Let $F = \mathbb{F}_5$. Suppose that $\lambda^2 + \lambda\mu + \mu^2 = 0$. It follows that $(\lambda + 1/2\mu)^2 = \mu^2(1/4 - 1)$. In field $\mathbb{F}_5$ a solution of an equation $4x = 1$ is 4, so that $1/4 - 1 = 3$. But the equation $x^2 = 3$ has no solutions in $\mathbb{F}_5$. This shows that the equality $\lambda^2 + \lambda\mu + \mu^2 = 0$ is possible only if $\lambda = \mu = 0$. Thus if $(\lambda, \mu) \neq (0,0)$, then $[x,x] \neq 0$ and $[x,x] \in Fc$. Hence in this case, every subalgebra of L is an ideal.

If $F = \mathbb{Q}$, then using the similar arguments we obtain again that every subalgebra of L is an ideal and the center of L is Fc.

Now consider the case when $F = \mathbb{F}_3$. For element $x = a + b$ we have $[a + b, a + b] = 3c = 0$. It follows that $\langle x \rangle = Fx$. But $[x, a] = [a + b, a] = c \notin Fx$, which shows that a cyclic subalgebra $\langle x \rangle$ is not an ideal.

The following theorem describes the structure of Leibniz algebras, in which every subalgebra is an ideal. But first, it will be useful to recall the following definitions.

A Leibniz algebra L has one specific ideal. Denote by $\mathbf{Leib}(L)$ the subspace generated by the elements $[a,a], a \in L$. It is not hard to prove that $\mathbf{Leib}(L)$ is an ideal of L. From its choice it follows that $L/\mathbf{Leib}(L)$ is a Lie algebra. And conversely, if H is an ideal of L such that L/H is a Lie algebra, then $\mathbf{Leib}(L) \leq H$.

The ideal $\mathbf{Leib}(L)$ is called the ***Leibniz kernel*** of algebra L.

We also note the following important property of the Leibniz kernel:

$$[[a,a], x] = 0 \text{ for arbitrary elements } a, x \in L.$$

The ***left*** (respectively ***right***) ***center*** $\zeta^{\mathbf{left}}(L)$ (respectively $\zeta^{\mathbf{right}}(L)$) of a Leibniz algebra L is defined by the rule

$$\zeta^{\mathbf{left}}(L) = \{x \in L | [x, y] = 0 \text{ for each element } y \in L\}.$$

(respectively

$$\zeta^{\mathbf{right}}(L) = \{x \in L | [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal. Moreover, $\mathbf{Leib}(L) \le \zeta^{\mathbf{left}}(L)$, so that $L/\zeta^{\mathbf{left}}(L)$ is a Lie algebra. In general, the left and the right centers are different, moreover, the left center is an ideal, but it is not true for the right center. One can find the corresponding counterexample in [10].

The ***center*** $\zeta(L)$ of L is defined by the rule

$$\zeta(L) = \{x \in L | [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L. In particular, we can consider the factor-algebra $L/\zeta(L)$.

Theorem A. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. If L is non-abelian, then $L = E \oplus Z$ where $Z \le \zeta(L)$ and E is an extraspecial subalgebra such that $[a, a] \ne 0$ for every element $a \notin \zeta(E)$.*

In Section 2 we establish a relation of extraspecial Leibniz algebras L such that $[a, a] \ne 0$ for every non-zero element a and the bilinear forms with no non-zero isotropic elements, which reduces the description of extraspecial Leibniz algebras to the description of these forms, and got some results on the structure of such forms. Here we present the main result of Section 2.

Let V be a vector space over a field F having countable dimension and Φ be a bilinear form on V. A basis $\{a_j | j \in \mathbb{N}\}$ is called ***left orthogonal***, if $\Phi(a_j, a_k) = 0$ whenever $j > k$.

Theorem B. *Let V be a vector space of countable dimension over a field F and Φ be a bilinear form on V. If $\Phi(a, a) \ne 0$ for each element $0 \ne a \in V$, then V has a left orthogonal basis.*

It should be noted that for the spaces of uncountable dimension a bilinear forms structure may be rather complicated. Even in the case of alternating forms there are quite exotic examples (see, for example, [14], Chapter 3).

1 On the structure of Leibniz algebras whose subalgebras are ideals

Let L be a Leibniz algebra over a field F. If B is a subspace of L and if $[L, B], [B, L] \le B$, then we will say that B is ***L-invariant***.

Lemma 1.1. *Let L be a Leibniz algebra over a field $F, a \in L$. Suppose that every subalgebra of L is an ideal of L. If A is an abelian subalgebra of L, then every subspace of A is L-invariant. Moreover, for every element $x \in L$ there are elements $a_x, \beta_x \in F$, which depend only of x, such that $[a, x] = \alpha_x a, [x, a] = \beta_x a$ for any element $a \in A$.*

Proof. Let B be a subspace of $A, b \in B$. Since A is abelian, $[b, b] = 0$, so that subspace Fb is a subalgebra. Then it is an ideal. It follows that $[x, b], [b, x] \in Fb \le B$, so that B is L-invariant.

Let $a, c \in A, x \in L$, then by above proved $[a, x] = \alpha a, [c, x] = \gamma c$ for some elements $\alpha, \gamma \in F$. We have $[a - c, x] = [a, x] - [c, x] = \alpha a - \gamma c$. On the other hand, $a - c \in A$, so that $F(a - c)$ must be L-invariant, i.e. $[a-c, x] = \eta(a-c) = \eta a - \eta c$. It follows that $\alpha a - \gamma c = \eta a - \eta c$, and hence $\alpha = \eta = \gamma$. For $[x, a]$ the arguments are similar. □

Lemma 1.2. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. If a is an element of L such that $[a, a] \ne 0$, then $\langle a \rangle = Fa \oplus F[a, a]$ and $[a, [a, a]] = [[a, a], a] = 0$.*

Proof. Put $b = [a, a]$. If $b \in Fa$, then $[a, a] = \gamma a$ for some $0 \ne \gamma \in F$. In this case we have $0 = [[a, a], a] = [\gamma a, a] = \gamma[a, a]$, which implies that $[a, a] = 0$, and we obtain a contradiction. This contradiction shows that the elements a, b are linearly independent. We have $b = [a, a] \in \mathbf{Leib}(L) \le \zeta^{\mathbf{left}}(L)$. Since the left center is an abelian ideal, Lemma 1.2 shows that $[a, b] = \alpha b$ for some element $\alpha \in F$. Suppose that $\alpha \in 0$ and consider an element $d = a - \alpha^{-1} b$. By such a choice $d \notin Fb$, so that $Fd \cap Fb = \langle 0 \rangle$. We have

$$[d, d] = [a - \alpha^{-1} b, a - \alpha^{-1} b] = [a, a] - [a, \alpha^{-1} b] = b - \alpha^{-1}[a, b] = b - b = 0.$$

It follows that $\langle d \rangle = Fd$. On the other hand, $[d,a] = [a,a] = b \notin \langle d \rangle$. It shows that a subalgebra $\langle d \rangle$ is not an ideal, and we obtain a contradiction. This contradiction shows that $\alpha = 0$, that is $[a,b] = 0$. □

Lemma 1.3. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. Then factor-algebra $L/\mathbf{Leib}(L)$ is abelian.*

Proof. Indeed, $L/\mathbf{Leib}(L)$ is a Lie algebra, and, moreover, it is a Lie algebra, whose subalgebras are ideals. So it is abelian. □

Corollary 1.4. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. Then a factor-algebra $L/\zeta^{\mathbf{left}}(L)$ is abelian.*

Lemma 1.5. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals, $C = \zeta^{\mathbf{left}}(L)$. If A is a maximal abelian ideal including C, then $A = C$.*

Proof. Let $a \in A$ and x be an arbitrary element of L. If c is a non-zero element of C, then $[c,x] = 0 = 0c$. Lemma 1.1 shows that $[a,x] = 0a = 0$. It follows that $a \in C$, that is $A = C$. □

Let L be a Leibniz algebra over a field F, M be a non-empty subset of L and H be a subalgebra of L. Put

$$\mathbf{Ann}_{H}\mathbf{left}(M) = \{a \in H \,|\, [a,M] = 0\}, \mathbf{Ann}_{H}\mathbf{right}(M) = \{a \in H \,|\, [M,a] = 0\}.$$

The subset $\mathbf{Ann}_{H}\mathbf{left}(M)$ is called the ***left annihilator*** or ***left centralizer*** of M in subalgebra H; the subset $\mathbf{Ann}_{H}\mathbf{right}(M)$ is called the ***right annihilator*** or ***right centralizer*** of M in subalgebra H. The intersection

$$\mathbf{Ann}_H(M) = \mathbf{Ann}_{H}\mathbf{left}(M) \cap \mathbf{Ann}_{H}\mathbf{right}(M) = \{a \in H \,|\, [a,M] = \langle 0 \rangle = [M,a]\}$$

is called ***annihilator*** or ***centralizer*** of M in subalgebra H.

It is not hard to see that all these subsets are subalgebras of L. Moreover, if M is a left ideal of L, then $\mathbf{Ann}_{L}\mathbf{left}(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathbf{Ann}_{H}\mathbf{left}(M)$, $b \in M$. Then

$$[[a,x],b] = [a,[x,b]] - [x,[a,b]] = 0 - [x,0] = 0 \text{ and}$$
$$[[x,a],b] = [x,[a,b]] - [a,[x,b]] = [x,0] - 0 = 0.$$

If M is an ideal of L, then $\mathrm{Ann}_L(M)$ is an ideal of L. Indeed, let x be an arbitrary element of L, $a \in \mathbf{Ann}_H(M)$, $b \in M$. Using the above arguments, we obtain that $[[a,x],b] = [[x,a],b] = 0$. Further

$$[b,[a,x]] = [[b,a],x]] + [a,[b,x]]] = [0,x] + 0 = 0 \text{ and}$$
$$[b,[x,a]] = [[b,x],a] + [x,[b,a]]] = 0 + [x,0] = 0.$$

Let L be a Lie algebra. Define the lower central series of L

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots \geq \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \ldots \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L, \gamma_2(L) = [L,L]$, and recursively $\gamma_{\alpha+1}(L) = [L,\gamma_\alpha(L)]$ for all ordinals α, and $\gamma_\lambda(L) = \cap_{\mu<\lambda}\, \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the ***lower hypocenter*** of L. We have $\gamma_\delta(L) = [L,\gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L,[L,[L,\ldots,L]\ldots L]$ is a left normed product of k copies of L.

Consider the factor $\gamma_k(L)/\gamma_{k+1}(L), k \in \mathbb{N}$. By definition, $[L,\gamma_k(L)] = \gamma_{k+1}(L)$, and also $[\gamma_k(L),L] = [\gamma_k(L),\gamma_1(L)] \leq \gamma_{k+1}(L)$ (see, for example, [10], Proposition 2.2).

As usual, we say that a Leibniz algebra L is ***nilpotent***, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be ***nilpotent of nilpotency class c*** if $\gamma_{\mathbf{c}+1}(L) = \langle 0 \rangle$, but $\gamma_{\mathbf{c}}(L) \neq \langle 0 \rangle$. We denote by $\mathbf{ncl}(L)$ the nilpotency class of L.

Lemma 1.6. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals, $C = \zeta^{\mathbf{left}}(L)$. Then $A = \mathbf{Ann}_{L}\mathbf{left}(C) = \mathbf{Ann}_L(C)$ is a nilpotent ideal,* $\mathbf{ncl}(A) \leq 2$, $\mathbf{codim}_F(A) \leq 1$.

Proof. As we noted above, C is an ideal of L, so that A is an ideal of L. Let a be an arbitrary element of A, then $[a, c] = 0$ for each element $c \in C$. On the other hand, $[c, a] = 0$ by the choice of c. It follows that $C \leq \zeta(A)$. By Corollary 1.4, a factor-algebra L/C is abelian, so that $[A, A] \leq C$ and A is nilpotent of nilpotency class at most 2.

Let $x \in L$. Lemma 1.1 shows that there exists an element $\alpha_x \in F$ such that $[x, c] = \alpha_x c$ for every element $c \in C$. Consider now the mapping $\mathbf{f} : L \longrightarrow F$, defined by the rule $\mathbf{f}(x) = \alpha x$ for each element $x \in L$. If $y \in L$, then

$$\alpha_{x+y} c = [x + y, c] = [x, c] + [y, c] = \alpha_x c + \alpha_y c = (\alpha_x + \alpha_y)c \text{ for every } c \in C.$$

This shows that $\mathbf{f}(x + y) = \alpha_{x+y} = \alpha_x + \alpha_y = \mathbf{f}(x) + \mathbf{f}(y)$. Furthermore, if $\beta \in F$, then

$$\alpha_{\beta x} c = [\beta x, c] = \beta[x, c] = \beta(\alpha_x c) = (\beta\alpha_x)c \text{ for every } c \in C,$$

which shows that $\mathbf{f}(\beta x) = \beta\alpha_x = \beta\mathbf{f}(x)$. It follows that mapping $\mathbf{f}$ is linear. Finally, $\mathbf{Ker(f)} = \{x \in L | \alpha_x = 0\}$. In other words, $x \in \mathbf{Ker(f)}$ is equivalent to $[x, c] = 0$ for each $c \in C$. This proves that $\mathbf{Ker(f)} = \mathbf{Ann}_{L}\,\mathbf{left}(C) = A$. Hence if $L \neq A$, then $\mathbf{dim}_F(L/A) = 1$. □

Define now the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq \ldots \leq \zeta_\alpha(L) \leq \zeta_{\alpha+1}(L) \leq \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \cup_{\mu<\lambda}\, \zeta_\mu(L)$ for the limit ordinals λ. By the definition each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the ***upper hypercenter*** of L. Denote by $\mathbf{zl}(L)$ the length of upper central series of L.

If $L = \zeta_\infty(L)$, then L is said to be ***hypercentral*** Leibniz algebra.

Lemma 1.7. *Let L be a Leibniz algebra over a field F and A be a non-zero ideal of L. If $A \leq \zeta_\infty(L)$, then $A \cap \zeta(L) \neq \langle 0 \rangle$.*

Proof. Suppose that $A \cap \zeta(L) = \langle 0 \rangle$. Since $A \leq \zeta_\infty(L)$, there is an ordinal α such that $A \cap \zeta_\alpha(L) \neq \langle 0 \rangle$. We choose the least ordinal λ with this property. Clearly λ is a non-limit ordinal, so that $C = A \cap \zeta_\lambda(L) \neq \langle 0 \rangle$, but $\langle 0 \rangle = A \cap \zeta_{\lambda-1}(L)$. The inclusion $C \leq \zeta_\lambda(L)$ implies that $[L, C], [C, L] \leq \zeta_{\lambda-1}(L)$. On the other hand, A is an ideal of L, and therefore $[L, C], [C, L] \leq A$. We obtain the inclusion $[L, C], [C, L] \leq A \cap \zeta_{\lambda-1}(L) = \langle 0 \rangle$. It follows that $\langle 0 \rangle \neq C \leq \zeta(L)$ and we obtain a contradiction. This contradiction proves a result. □

Corollary 1.8. *Let L be a hypercentral Leibniz algebra over a field F and A be a non-zero ideal of L. Then $A \cap \zeta(L) \neq \langle 0 \rangle$.*

Lemma 1.9. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. Suppose that L is nilpotent and* $\mathbf{ncl}(L) = 2$. *If a is an element of L such that $[a, a] = 0$, then $a \in \zeta(L)$.*

Proof. The equality $[a, a] = 0$ implies that $\langle a \rangle = Fa$, so that $\mathbf{dim}_F(\langle a \rangle) = 1$. Since $\langle a \rangle$ is an ideal, Corollary 1.8 implies that $\langle 0 \rangle \neq \langle a \rangle \cap \zeta(L)$. It follows that $\langle a \rangle = \langle a \rangle \cap \zeta(L)$, i.e. $a \in \zeta(L)$. □

Lemma 1.10. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. Suppose that L is nilpotent and* $\mathbf{ncl}(L) = 2$. *If $a \notin \zeta(L)$, then* $\mathbf{codim}_F(\mathbf{Ann}_{L}\,\mathbf{left}(\langle a \rangle)) = 1 = \mathbf{codim}_F(\mathbf{Ann}_{L}\,\mathbf{right}(\langle a \rangle))$.

Proof. Put $c = [a, a]$. Since $a \notin \zeta(L)$, Lemma 1.9 shows that $c \neq 0$. Then by Lemma 1.2 we obtain that $A = \langle a \rangle = Fa \oplus Fc$. Furthermore, $c \in \zeta(L)$, so that $[a, c] = [c, a] = 0$.

Let $x \in L$. Since $L/\zeta(L)$ is abelian, $[x, a] \in \zeta(L)$. On the other hand, A is an ideal, so that $[x, a] \in A$, that is $[x, a] \in A \cap \zeta(L) = Fc$. It follows that $[x, a] = \alpha_x c$ for some element $\alpha_x \in F$. Consider the mapping $\mathbf{f} : L \longrightarrow F$ defined by the rule $\mathbf{f}(x) = \alpha_x$ for each element $x \in L$. If $y \in L$, then

$$\alpha_{x+y} c = [x + y, a] = [x, a] + [y, a] = \alpha_x c + \alpha_y c = (\alpha_x + \alpha_y)c.$$

This shows that $\mathbf{f}(x+y) = \alpha_{x+y} = \alpha_x + \alpha_y = \mathbf{f}(x) + \mathbf{f}(y)$. Furthermore, if $\beta \in F$, then

$$\alpha_{\beta x} c = [\beta x, a] = \beta[x, a] = \beta(\alpha_x c) = (\beta\alpha_x)c,$$

which shows that $\mathbf{f}(\beta x) = \beta\alpha_x = \beta\mathbf{f}(x)$. It follows that mapping $\mathbf{f}$ is linear. Finally, $\mathbf{Ker(f)} = \{x \in L | \alpha_x = 0\}$. In other words, $x \in \mathbf{Ker(f)}$ is equivalent to $[x, a] = 0$. This proves that $\mathbf{Ker(f)} = \mathbf{Ann}_L^{\mathbf{left}}(a)$. It is not hard to see that $\mathbf{Ann}_L^{\mathbf{left}}(a) = \mathbf{Ann}_L^{\mathbf{left}}(A)$. The choice of a shows that $\mathbf{Ann}_L^{\mathbf{left}}(a) \neq L$. It follows that $\mathbf{dim}_F(L/\mathbf{Ann}_L^{\mathbf{left}}(A)) = 1$. □

Lemma 1.11. *Let L be an extraspecial Leibniz algebra over a field F. Then every subalgebra of L is an ideal if and only if $[a, a] \neq 0$ for every element $a \neq \zeta(L)$.*

Proof. Suppose that every subalgebra of L is an ideal and a is a non-zero element of L such that $[a, a] = 0$. Then Lemma 1.9 shows that $a \in \zeta(L)$.

Conversely, assume that L is an extraspecial algebra such that $[a, a] \neq 0$ for every element $a \notin \zeta(L)$. Let x be an arbitrary element of L. If $x \in \zeta(L)$, then $\langle x \rangle = Fx$, and a cyclic subalgebra $\langle x \rangle$ is an ideal. If $x \notin \zeta(L)$, then $y = [x, x] \in \zeta(L)$. It follows that $[x, [x, x]] = [[x, x], x] = 0$, and $\langle x \rangle = Fx \oplus F[x, x]$. Since $[x, x] \neq 0$, $F[x, x] = \zeta(L)$. Hence every cyclic subalgebra includes the center of L. Since $L/\zeta(L)$ is abelian, every cyclic subalgebra of L is an ideal. It follows that every subalgebra of L is an ideal. □

Lemma 1.12. *Let L be a Leibniz algebra over a field F and suppose that L is nilpotent and* $\mathbf{ncl}(L) = 2$. *If $a, b \notin \zeta(L)$ and $\langle a \rangle \cap \langle b \rangle = \langle 0 \rangle$, then L includes a subalgebra, which is not an ideal.*

Proof. Suppose the contrary. Let every subalgebra of L be an ideal. If we assume that $[a, a] = 0$, then as in Lemma 1.9 we obtain that $a \in \zeta(L)$. But this contradicts the choice of a. This contradiction proves that $[a, a] = c \neq 0$. Since A is nilpotent of class 2, $c \in \zeta(L)$, which implies that $[a, c] = [c, a] = 0$. In a similar way, $[b, b] = d \neq 0$ and $[b, d] = [d, b] = 0$. The subalgebras $\langle a \rangle$ and $\langle b \rangle$ are ideals, and therefore $[a, b], [b, a] \in \langle a \rangle \cap \langle b \rangle = \langle 0 \rangle$. It follows that $[a, b] = [b, a] = 0$. Let $x = a + b$, then

$$[x, x] = [a + b, a + b] = [a, a] + [b, b] = c + d.$$

As above $\langle x \rangle = Fx \oplus F[x, x]$ and $\langle x \rangle \cap \zeta(L) = F(c + d)$. On the other hand, $[x, a] = c, [x, b] = d$. It follows that an ideal, generated by x, must contain $Fc \oplus Fd$, and we obtain a contradiction. This contradiction proves a result. □

Proposition 1.13. *Let L be a Leibniz algebra over a field F, whose subalgebras are ideals. Suppose that L is nilpotent and* $\mathbf{ncl}(L) = 2$. *Then $L = E \oplus Z$ where E is an extraspecial subalgebra, $Z \leq \zeta(L)$.*

Proof. If we suppose that $[a, a] = 0$ for every element $a \in L$, then Lemma 1.9 shows that $L = \zeta(L)$, and we obtain a contradiction. Hence there exists an element a such that $[a, a] = c \neq 0$. Lemma 1.2 shows that $\langle a \rangle = Fa \oplus Fc$. Let B be a basis of L containing a, c. Then $B = \{a_\alpha | \alpha < \gamma\}$ for some ordinal α. We can put $a_1 = a, a_2 = c$. Consider the subalgebra $\langle a_1, a_2, a_3 \rangle$. If $[a_3, a_3] = 0$, then Lemma 1.9 shows that $a_3 \in \zeta(L)$. In this case, $\langle a_1, a_2, a_3 \rangle = \langle a_1 \rangle \oplus Fa_3$, where subalgebra $\langle a_1 \rangle$ is extraspecial, $Fa_3 \leq \zeta(L)$.

Assume now that $[a_3, a_3] = b \neq 0$. Lemma 1.2 shows that $\langle a_3 \rangle = Fa_3 \oplus F_b$. In this case, $\mathbf{dim}_F(\langle a_1, a_3 \rangle) \leq \mathbf{dim}_F(\langle a_1 \rangle) + 2$. If $\mathbf{dim}_F(\langle a_1, a_3 \rangle) = \mathbf{dim}_F(\langle a_1 \rangle) + 2$, then $\langle a_1 \rangle \cap \langle a_3 \rangle =< 0 >$, and Lemma 1.12 implies that in this case, L includes a subalgebra, which is not an ideal. This contradiction proves that $\mathbf{dim}_F(\langle a_1, a_3 \rangle) = \mathbf{dim}_F(\langle a_1 \rangle) + 1$. If $\langle a_1, a_3 \rangle \cap \zeta(L) \neq Fc$, then $\langle a_1, a_3 \rangle \cap \zeta(L) = Fc \oplus Fd$ for some element $d \in \zeta(L)$. In this case, $\langle a_1, a_3 \rangle = \langle a_1 \rangle \oplus Fd$. If $\langle a_1, a_3 \rangle \cap \zeta(L) = Fc$, then $\langle a_1, a_3 \rangle$ is extraspecial.

Suppose that we have already constructed the subalgebras L_β for all $\beta < \alpha$ satisfying the following conditions: $a_\nu \in L_\beta$ for all $\nu \leq \beta$; $L_\beta = E_\beta \oplus Z_\beta$ where E_β is an extraspecial subalgebra, $\zeta(E_\beta) = Fc$, $Z_\beta \leq \zeta(L)$, $E_\nu \leq E_\beta$, $Z_\nu \leq Z_\beta$ whenever $\nu \leq \beta$. If α is a not limit ordinal, then $\alpha - 1$ exists. Let η be a least ordinal such that $a_\eta \notin L_{\alpha-1}$. Consider a subalgebra $L_\alpha = \langle L_{\alpha-1}, a_\eta \rangle$. If $[a_\eta, a_\eta] = 0$, then Lemma 1.9 shows that $a_\eta \in \zeta(L)$. In this case,

$$L_\alpha = L_{\alpha-1} \oplus Fa_\eta = E_{\alpha-1} \oplus Z_{\alpha-1} \oplus Fa_\eta = E_\alpha \oplus Z_\alpha$$

where $E_\alpha = E_{\alpha-1}, Z_\alpha = Z_{\alpha-1} \oplus Fa_\eta$.

If $[a_\eta, a_\eta] = u \neq 0$. Lemma 1.2 shows that $\langle a_\eta \rangle = Fa_\eta \oplus Fu$. In this case, $\mathbf{dim}_F(L_\alpha) \leq \mathbf{dim}_F(L_{\alpha-1}) + 2$. If $\mathbf{dim}_F(L_\alpha) = \mathbf{dim}_F(L_{\alpha-1}) + 2$, then $L_{\alpha-1} \cap \langle a_\eta \rangle = \langle 0 \rangle$. In particular, $\langle a_1 \rangle \cap \langle a_\eta \rangle = \langle 0 \rangle$, and Lemma 1.12 implies that in this case L includes a subalgebra, which is not an ideal. This contradiction proves that $\mathbf{dim}_F(L_\alpha) = \mathbf{dim}_F(L_{\alpha-1}) + 1$. If $L_\alpha \cap \zeta(L) \neq Fc$, then $L_\alpha \cap \zeta(L) = Fc \oplus Fv$ for some element $v \in \zeta(L)$. In this case,

$$L_\alpha = L_{\alpha-1} \oplus Fv = E_{\alpha-1} \oplus Z_{\alpha-1} \oplus Fv = E_\alpha \oplus Z_\alpha$$

where $E_\alpha = E_{\alpha-1}, Z_\alpha = Z_{\alpha-1} \oplus Fv$.

If $L_\alpha \cap \zeta(L) = Fc$, then $L_\alpha/Fc = L_{\alpha-1}/Fc \oplus \langle a_\eta \rangle/Fc$. In this case, $E_\alpha = \langle E_{\alpha-1}, a_\eta \rangle$ is extraspecial and $L_\alpha = E_\alpha \oplus Z_\alpha$ where $Z_\alpha = Z_{\alpha-1}$.

Assume that α is a limit ordinal. Put $E_\alpha = \cup_{\beta<\alpha} E_\beta, Z_\alpha = \cup_{\beta<\alpha} Z_\beta, L_\alpha = \cup_{\beta<\alpha} L_\beta$. Then clearly $L_\alpha = E_\alpha \oplus Z_\alpha$, E_α is extraspecial, $\zeta(E_\alpha) = Fc, Z_\alpha \leq \zeta(L)$.

For $\alpha = \gamma$ we proved the result. □

Proof of Theorem A. Let C be a left center of L. By Lemma 1.5 C is a maximal abelian ideal of L and Corollary 1.4 shows that L/C is abelian. Since $[C, L] = \langle 0 \rangle$, $\mathbf{Ann}_L^{\mathbf{left}}(C) = \mathbf{Ann}_L(C)$. Then Lemma 1.6 shows that A is a nilpotent ideal, $\mathbf{ncl}(A) \leq 2$, and $\mathbf{dim}_F(L/A) \leq 1$. Suppose that $L = A$. By Proposition 1.13, $L = E \oplus Z$ where E is an extraspecial subalgebra, $Z \leq \zeta(L)$. Lemma 1.11 shows that $[a, a] \neq 0$ for every element $a \notin \zeta(E) = E \cap C$.

Suppose now that $L \neq A$ and choose an element $d \notin A$. Put $e = [d, d]$. By Lemma 1.2 $\langle d \rangle = Fd \oplus Fe$ and $[d, e] = [e, d] = 0$. Since C is abelian, Lemma 1.1 shows that there are the elements $\lambda, \rho \in F$ such that $[d, c] = \lambda c, [c, d] = \rho c$ for each element $c \in C$. Since $e \in \mathbf{Leib}(L) \leq \zeta^{\mathbf{left}}(L), \lambda = \rho = 0$. In other words, $[d, c] = 0$ for each element $c \in C$. This means that $d \in \mathbf{Ann}_L^{\mathbf{left}}(C) = A$. But this contradicts the choice of d. This contradiction shows that $L = A$, which proves the result. □

2 Extraspecial Leibniz algebras and bilinear forms

We can connect a bilinear form to an extraspecial algebra in the following way. Let $Z = \zeta(L), V = L/Z$ and c be a fixed non-zero element of Z. Define the mapping $\Phi : V \times V \longrightarrow F$ by the following rule: If $x, y \in L$, then $[x, y] \in Z$, so that $[x, y] = \alpha c$ for some element $\alpha \in F$. Put $\Phi(x + Z, y + Z) = \alpha$. This mapping is definitely correct. Indeed, let x_1, y_1 be elements of L such that $x_1 + Z = x + Z, y_1 + Z = y + Z$. Then $x_1 = x + c_1, y_1 = y + c_2$ for some elements $c_1, c_2 \notin Z$. Then

$$[x_1, y_1] = [x + c_1, y + c_2] = [x, y] + [x, c_2] + [c_1, y] + [c_1, c_2] = [x, y].$$

The mapping Φ is bilinear. In fact, let $x, y, u \notin L, [x, u] = \lambda c, [y, u] = \mu c$. Then $[x + y, u] = [x, u] + [y, u] = \lambda c + \mu c = (\lambda + \mu)c$, so that

$$\Phi(x + Z + y + Z, u + Z) = \Phi(x + y + Z, u + Z) = (\lambda + \mu)c = \lambda c + \mu c = \\ \Phi(x + Z, u + Z) + \Phi(y + Z, u + Z).$$

Similarly we can show that

$$\Phi(x + Z, y + Z + u + Z) = \Phi(x + Z, u + Z) + \Phi(x + Z, y + Z).$$

Let $\beta \in F$, then $[\beta x, y] = \beta[x, y] = \beta(\alpha c) = (\beta\alpha)c$. It follows that

$$\Phi(\beta(x + Z), y + Z) = \Phi(\beta x + Z, y + Z) = (\beta\alpha)c = \beta(\alpha c) = \beta\Phi(x + Z, y + Z).$$

Similarly we can show that

$$\Phi(x + Z), \beta(y + Z) = \beta\Phi(x + Z, y + Z).$$

By the definition of an extraspecial algebra we obtain that a bilinear form Φ is non-degenerate. Moreover, Lemma 1.9 shows that $\Phi(x, x) \neq 0$ for every non-zero element x.

Conversely, let V be a vector space over a field F and Φ be a bilinear form on V such that $\Phi(x,x) \neq 0$ for every non-zero element x. Put $L = V \oplus F$. Define the operation [,] on L by the following rule: if $a, b \in V, \alpha, \beta \in F$, then $[(a,\alpha),(b,\beta)] = (0, \Phi(a,b))$. Put $C = \{(0,\alpha)|\alpha \in F\}$. Then $\mathbf{dim}_F(C) = 1$. By this definition $[L,L] = [L,C] =]C,L] = [C,C] = \langle 0 \rangle$. It follows that the constructed algebra is a Leibniz algebra. Furthermore, $C \leq \zeta(L)$. Moreover, $C = \zeta(L)$. Indeed, let $(z,\gamma) \in \zeta(L)$ and suppose that $z \neq 0$. Then $[(z,\gamma),(a,\alpha)] = [(a,\alpha),(z,\gamma)] = (0,0)$, in particular, $[(z,\gamma),(z,\gamma)] = (0,0)$. But $[(z,\gamma),(z,\gamma)] = (\Phi(z,z),0)$. Since $z \neq 0$, $(\Phi(z,z) \neq 0$, and we obtain a contradiction. This contradiction proves the equality $C = \zeta(L)$.

Let V be a vector space over a field F, U a subspace of V and Φ be a bilinear form on V. Put

$$^{\perp}U = \{x \in V|\Phi(x,u) = 0 \text{ for all elements } u \in U\},$$
$$U^{\perp} = \{x \in V|\Phi(u,x) = 0 \text{ for all elements } u \in U\}.$$

Clearly $^{\perp}U$ and $U^{\perp}$ are the subspaces of $V, ^{\perp}U$ is called ***left orthogonal complement*** of U in $V, U^{\perp}$ is called ***right orthogonal complement*** of U in V.

Using usual methods of linear algebra the following proposition can be proved.

Proposition 2.1. *Let V be a finite dimensional vector space over a field F, U a subspace of V and Φ be a non-degenerate bilinear form on V. If the restriction of Φ on U is non-degenerate, then* $\mathbf{dim}_F(^{\perp}U) = \mathbf{dim}_F(U^{\perp}) = \mathbf{dim}_F(V) - \mathbf{dim}_F(U)$.

Corollary 2.2. *Let V be a finite dimensional vector space over a field F and Φ be a bilinear form on V. If $\Phi(a,a) \neq 0$ for each element $0 \neq a \in V$, then V has a left orthogonal basis.*

Indeed, let U be an arbitrary non-zero subspace of V. If we suppose that a restriction of Φ on U is degenerate, then $^{\perp}U \cap U \neq \langle 0 \rangle$. Let $0 \neq a \in {}^{\perp}U \cap U$, then $\Phi(a,u) = 0$ for all elements $u \in U$. In particular, $\Phi(a,a) = 0$, and we obtain a contradiction. Hence the restriction of Φ on every non-zero subspace is non-degenerate, and we can apply Proposition 2.1.

We note that if V is a finite dimensional vector space and Φ is a bilinear form on V, such that V has a left orthogonal basis, then the matrix of a form Φ in this basis is triangular.

Proof of Theorem B.

Theorem B. *Let V be a vector space over a field F, having countable dimension, and Φ be a bilinear form on V. If $\phi(a,a) \neq 0$ for each element $0 \neq a \in V$, then V has a left orthogonal basis.*

Choose in space V an arbitrary basis $\{a_j | j \in \mathbb{N}\}$. Put $V_j = Fa_1 + \ldots + Fa_j, j \in N$. Since $\mathbf{dim}_F(V_2) = 2$ is finite, Corollary 2.2 shows that V_2 has a basis $\{v_1, v_2\}$ such that $[v_2, v_1] = 0$. Since $\mathbf{dim}_F(V_3)$ is finite, Proposition 2.1 implies that $\mathbf{dim}_F(V_3 \cap^{\perp} V_2) = \mathbf{dim}_F(V_3) - \mathbf{dim}_F(V_2) = 1$. By the above remarks, the restriction of Φ on every non-zero subspace is non-degenerate, which implies that $(V_3 \cap^{\perp} V_2) \cap V_2 = \langle 0 \rangle$. Let $0 \neq v_3 \in V_3 \cap^{\perp} V_2$, then $\{v_1, v_2, v_3\}$ is a basis of V_3 such that $[v_2, v_1] = [v_3, v_1] = [v_3, v_2] = 0$. Using similar arguments, ordinary induction and the fact that $V = \cup_{n \in \mathbb{N}} V_n$, we proved the above statement. □

Corollary 2.3. *Let L be an extraspecial Leibniz algebra over a field F, having countable dimension. If $[a,a] \neq 0$ for every element $a \notin \zeta(L)$, then L has a basis $\{e_n | n \in \mathbb{N}\}$ such that $[e_1, e_n] = [e_n, e_1] = 0, [e_j, e_n] \in Fe_1$ for all $j, n \in \mathbb{N}, [e_j, e_n] = 0$ whenever $j > n$.*

References

[1] Bloh A.M. On a generalization of the concept of Lie algebra. Doklady AN USSR-165 (1965), 471-473.

[2] Bloh A.M. Cartan-Eilenberg homology theory for a generalized class of Lie algebras. Doklady AN USSR-175 (1967), 824-826.

[3] Bloh A.M. A certain generalization of the concept of Lie algebra. Algebra and number theoy. Moskov. Gos. Pedagogical Inst., Uchenye Zapiski-375 (1971), 9-20.

[4] Butterfield J., Pagonis C. From Physics to Philosophy. Cambridge Univ. Press: Cambridge, 1999.
[5] Casas J.M., Pirashvili T. Ten-term exact sequence of Leibniz homology. J. Algebra-231(2000), 258-264.
[6] Chupordya V.A., Kurdachenko L.A., Subbotin I.Ya. On some "minimal" Leibniz algebras. Journal of Algebra and Applications, 2016, in press.
[7] Lie Theory and its applications in physic. IX International workshop, Editor V. Dobrev-Springer: Tokyo 2013.
[8] Noncommutative Structures in Mathematics and Physics. Proceedings of the NATO advanced research workshop. Editors S. Duplij, J. Wess. Springer Science + Business media Dordrecht 2001.
[9] Frabetti A. Leibniz homology of dialgebras of matrices. Journal Pure Appl. Algebra-129 (1998), 123-141.
[10] Kurdachenko L.A., Otal J., Pypka A.A. On some relationships between the factors of the canonical central series in Leibniz algebras. European Journal of Mathematics-2(2016), 565-577.
[11] Loday J.L. Une version non commutative des algebres de Lie; les algebras de Leibniz. Enseign. Math. 39 (1993), 269-293.
[12] Loday J.L., Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology. Math. Ann.-296(1993), 139-158.
[13] Pirashvili T. On Leibniz homology. Ann. Inst. Fourier (Grenoble)-44(2)(1994), 401-411.
[14] Tomkinson M.J. FC-groups. PITMAN: Boston-1984.