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Research Article

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The algebraic size of the family of injective operators

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Abstract: In this paper, a criterion for the existence of large linear algebras consisting, except for zero, of one-to-one operators on an infinite dimensional Banach space is provided. As a consequence, it is shown that every separable infinite dimensional Banach space supports a commutative infinitely generated free linear algebra of operators all of whose nonzero members are one-to-one. In certain cases, the assertion holds for nonseparable Banach spaces.

Keywords: One-to-one operator, Point spectrum, Algebrability, Hypercyclic operator

MSC: 47L05, 15A03, 47A05, 47A16

1 Introduction

In the current millenium there has been a generalized interest in the search for large algebraic structures inside nonlinear sets. This area of research is called *lineability*. Concepts, results and references concerning this topic can be found in the survey [1] and the monograph [2]. In this note, we focus on the family of all *injective* (or one-to-one) operators defined on a Banach space. Obviously, this family is not a vector space. Only recently (see below) a number of assertions have been established within this context, in the real case. Our aim is to contribute to complete the existing knowledge on lineability of the mentioned family, mainly in the complex infinite dimensional setting.

Our notation will be rather usual. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will stand for the set of positive integers, the set $\mathbb{N} \cup \{0\}$, the set of rational numbers, the real line, and the field of complex numbers, respectively. The cardinality of the continuum, card $(\mathbb{R})$, is denoted by $\mathfrak{c}$. If X is a Banach space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ then X^* and $L(X)$ will represent, respectively, the topological dual space of X and the vector space of all operators on X, that is, the family of all continuous linear self-mappings $T : X \to X$. Recall that $L(X)$ becomes a linear algebra (in fact, a Banach algebra) if we endow this vector space with the internal law of composition of operators. If $T \in L(X)$ then the spectrum and the point spectrum of T will be denoted by $\sigma(T)$ and $\sigma_P(T)$, respectively. Recall that $\sigma(T) = \{\lambda \in \mathbb{K} : T - \lambda I \text{ is not invertible}\}$ ($I =$ the identity) and $\sigma_P(T)$ is the set of eigenvalues of T, that is, $\sigma_P(T) = \{\lambda \in \mathbb{K} : T - \lambda I \text{ is not injective}\}$ $(\subset \sigma(T))$. The adjoint of T is the operator $T^* \in L(X^*)$ given by $(T^*\varphi)(x) = \varphi(Tx)$ $(\varphi \in X^*, x \in X)$.

A few lineability concepts (see [2]) will be convenient in order to establish our findings appropriately. If X is a vector space, α is a cardinal number and $A \subset X$, then A is said to be: *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$, and α*-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$ (hence lineability means $\aleph_0$-lineability, where $\aleph_0 = \text{card}(\mathbb{N})$). If, in addition, X is contained in some (linear) algebra then A is called: *algebrable* if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite; α*-algebrable* if there is an α-generated algebra M with $M \setminus \{0\} \subset A$; *strongly algebrable* if $A \cup \{0\}$ contains an infinitely generated algebra

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that is isomorphic with a free algebra, and *strongly α-algebrable* if $A \cup \{0\}$ contains an α-generated algebra that is isomorphic with a free algebra. Note that if $B \subset X$ is a set whose elements commute respect to the algebra operation, then B is a generating set of some free algebra contained in $A \cup \{0\}$ if, and only if, for any $N \in \mathbb{N}$ and any nonzero polynomial P in N variables without a constant term and any distinct $y_1, \dots, y_N \in B$ we have $P(y_1, \dots, y_N) \in A \setminus \{0\}$. Of course, strong ($\alpha$-)algebrability implies (α-)algebrability, and this in turn implies (α-, resp.) lineability.

Even if no additional condition of linearity is imposed, there are not many injective self-mappings on $\mathbb{R}$ in the algebraic sense, as the following result (see [3]) shows.

Theorem 1.1. *The set of injective functions $\mathbb{R} \to \mathbb{R}$ is not 2-lineable.*

As for higher dimensions, we gather in the next theorem a number of assertions, which have been recently proved by Jiménez, Maghsoudi, Muñoz and Seoane [4] in the setting of *real* vector spaces.

Theorem 1.2. *Let $n, m \in \mathbb{N}$. Then the following holds:*
(a) *The set $\{f : \mathbb{R}^n \to \mathbb{R}^m : f$ is injective$\}$ is m-lineable, but not $(m+1)$-lineable.*
(b) *The set $\{f : \mathbb{R}^{2^n m} \to \mathbb{R}^{2^n m} : f$ is linear and bijective$\}$ is 2^n-lineable.*
(c) *If $m \geq 3$ is odd then the set $\{f : \mathbb{R}^m \to \mathbb{R}^m : f$ is linear and bijective$\}$ is not m-lineable.*

Recall that if X is a finite dimensional Banach space then any linear mapping $X \to X$ is continuous, and any injective linear mapping $X \to X$ is bijective. Consistently, for a general Banach space, we consider the family

$$\begin{aligned} L_{1-1}(X) :&= \{\text{linear continuous injective mappings } X \to X\} \\ &= \{T \in L(X) : \operatorname{Ker}(T) = \{0\}\}. \end{aligned}$$

It is well known that the group of bijective operators on a Banach space (equivalently, by the Open Mapping Theorem, the group of invertible operators) is a nonempty open set (see e.g. [5, Chap. 7]). Hence $L_{1-1}(X)$ contains a nonempty open set, so it is not a too small set, in the topological sense. Thus, it is natural to raise the question of whether $L_{1-1}(X)$ is also large in the algebraic sense. Theorem 1.2 above gave us a partial answer in the realm of real finite dimensional spaces. In [4], the following theorem concerning an important class of infinite dimensional Banach spaces in the *real* case is also proved.

Theorem 1.3. *Let X be a real Banach space with a Schauder basis. Then $L_{1-1}(X)$ is lineable.*

In Section 2, we will briefly deal with the finite dimensional case in order to generalize Theorem 1.2. In Section 3, which is the main one, we will extend the aforementioned results to the *complex* setting when the Banach spaces have infinite dimension. Our main contribution consists of a criterion providing algebrability in this case. Through a modification of this criterion we will also improve Theorem 1.3 by showing that its conclusion still holds –in both real and complex cases– if X is just separable and infinite dimensional. Examples of nonseparable Banach spaces where the result remains valid are also provided.

2 Injective operators on finite dimensional Banach spaces

Theorem 1.1 and item (a) of Theorem 1.2 hold in the case $\mathbb{K} = \mathbb{C}$ with virtually the same proofs. (In the light of the original proof in [4, Theorem 2.4], we can even afford a slight generalization in the initial set $\mathbb{R}^n$.) Observe that, trivially, we have also 1-lineability for the set of bijective linear self-mappings of $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (any nonzero multiple of the identity is bijective). Item (c) of Theorem 1.2 also holds for $\mathbb{K} = \mathbb{C}$, not only for odd m but for all $m \in \mathbb{N}$ (hence Theorem 1.2(b) is false for $m = 1$ if we replace $\mathbb{R}$ by $\mathbb{C}$). Indeed, the proof of part (c) in [4] (see Remark 2.7, proof of Corollary 2.8 and Remark 2.9 of this reference) was based on properties of determinants and on the fact that any polynomial of odd degree with coefficients in $\mathbb{R}$ possesses at least one (real) zero. The restriction "m odd" is not needed in $\mathbb{C}$ by the Fundamental Theorem of Algebra.

As a consequence, we can state the following result.

Theorem 2.1. *Let $m \in \mathbb{N}$ and S be a set with* $\operatorname{card}(S) \le \mathfrak{c}$. *We have:*
(a) *For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the set $\{f : S \to \mathbb{K}^m : f \text{ is injective}\}$ is m-lineable, but not $(m+1)$-lineable.*
(b) *The set $\{f : \mathbb{C}^m \to \mathbb{C}^m : f \text{ is linear and bijective}\}$ is m-lineable if and only if $m = 1$.*

A natural problem remains open: for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $m \ge 2$, is the family $\{f : \mathbb{K}^m \to \mathbb{K}^m : f$ is linear and bijective$\}$ $(m-1)$-lineable?

3 Injective operators on infinite dimensional Banach spaces

We turn to the infinite dimensional case when the field is $\mathbb{K} = \mathbb{C}$. Let us establish our criterion (Theorem 3.1), with which a high level of lineability is obtained. Observe that this criterion is not applicable to the finite dimensional setting, because any operator has always (complex) eigenvalues in this case. Note also that, despite the fact that the algebra $L(X)$ is not commutative, our approach furnishes a large commutative algebra.

Theorem 3.1. *Let X be a complex Banach space supporting an operator without eigenvalues. Then the family $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable.*

The proof will make use of some background about holomorphic functions of operators (see for instance [6, Chap. 1] or [7, Chap. 10]). Let X be a complex Banach space and $T \in L(X)$. If f is a complex function that is analytic on a neighborhood of $\sigma(T)$ then it is possible to define an operator $f(T) \in L(X)$ satisfying $f(T) = I$ if $f(z) \equiv 1$, $f(T) = T$ if $f(z) \equiv z$, $(f+g)(T) = f(T) + g(T)$ and $f(T)g(T) = (fg)(T)$ (where $f(T)g(T)$ denotes composition of $f(T)$ and $g(T)$, while fg denotes pointwise multiplication). Observe that, for fixed T, the operators $f(T)$ form a commutative linear algebra. In the special case of an entire function $f : \mathbb{C} \to \mathbb{C}$, we have that if f has Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$f(T) = \sum_{n=0}^{\infty} a_n T^n,$$

the series being convergent in the norm topology of $L(X)$. Here, $T^0 = I$ and $T^{n+1} = T^n \circ T$ $(n \ge 0)$. A special version of the spectral mapping theorem (see [7, Theorem 10.33]) reads as follows.

Theorem 3.2. *Assume that X is a complex Banach space. Let $T \in L(X)$ and $f : \Omega \to \mathbb{C}$ be holomorphic on an open set $\Omega \supset \sigma(T)$. Then $f(\sigma_P(T)) \subset \sigma_P(f(T))$. If, in addition, f is nonconstant on every connected component of Ω, then $f(\sigma_P(T)) = \sigma_P(f(T))$.*

The following auxiliary assertion provides a free algebra consisting of entire functions. Its proof is easy and is essentially contained in the proof of Lemma 2.4 in [8], which in turn is in the same spirit as [9, Proposition 7] (see also [10, Theorem 1.5] and [11]); so it will be omitted. Let $H_0(\mathbb{C})$ denote the set of all entire functions f with $f(0) = 0$.

Lemma 3.3. *Let $H \subset (0, +\infty)$ be a set with* $\operatorname{card}(H) = \mathfrak{c}$ *and which is linearly independent over the field $\mathbb{Q}$. For each $r > 0$, consider the function $E_r(z) := e^{rz} - 1$. Then $\{E_r : r \in H\}$ is a free system of generators of an algebra contained in $H_0(\mathbb{C})$.*

In the proof of our Theorem 3.1, Lemma 3.3 plays the role of generating the appropriate algebra by superposing a fixed operator belonging to the considered class with the representatives of a well chosen algebra of functions. This method was already used in [12].

Proof of Theorem 3.1. Consider the algebra $\mathcal{A}$ generated by the functions E_r of Lemma 3.3, that is, the collection all finite linear combinations of products $E_{r_1}^{m_1} \cdots E_{r_N}^{m_N}$ $(N \in \mathbb{N};\, r_1, \dots, r_N \in H;\, (m_1, \dots, m_N) \in \mathbb{N}_0^N \setminus$

$\{(0,\dots,0)\}$). Note that the cardinality of the free system $\{E_r\}_{r\in H}$ is $\mathfrak{c}$. By hypothesis, there exists $T \in L(X)$ such that $\sigma_P(T) = \varnothing$. Let $f \in H_0(\mathbb{C})$ such that f is not identically zero. Then f is not constant on $\Omega = \mathbb{C}$, which is connected. It follows from Theorem 3.2 that $\sigma_P(f(T)) = f(\sigma_P(T)) = f(\varnothing) = \varnothing$. In particular, $0 \notin \sigma_P(f(T))$. Thus, the operator $f(T)$ is injective, that is, $f(T) \in L_{1-1}(X)$. Now, consider the family

$$\mathcal{B} := \{f(T) : f \in \mathcal{A}\},$$

which is clearly an algebra. Since $\mathcal{A} \subset H_0(\mathbb{C})$, we get $\mathcal{B} \setminus \{0\} \subset L_{1-1}(X)$. Clearly, the operators $E_r(T)$ $(r \in H)$ generate the algebra $\mathcal{B}$. It remains to show that they generate $\mathcal{B}$ in a free way. In other words, we should prove that, if $N \in \mathbb{N}$, Q is a complex polynomial in N variables without constant term, $r_1,\dots,r_N$ are different numbers in H and $Q(E_{r_1}(T),\dots,E_{r_N}(T)) = 0$, then $Q = 0$.

To this end, observe that, under the latter assumptions, we get from the properties of holomorphic functions of operators that $F(T) = 0$, where $F(z) := Q(E_{r_1}(z),\dots,E_{r_N}(z))$. Assume, by way of contradiction, that $Q \neq 0$. Since the E_r's generate a free algebra, we have $F \in H_0(\mathbb{C}) \setminus \{0\}$. From Theorem 3.2, it follows that

$$\{0\} = \sigma_P(0) = \sigma_P(F(T)) = F(\sigma_P(T)) = F(\varnothing) = \varnothing,$$

which is absurd. The proof is finished. □

Concerning applications of Theorem 3.1 (see Theorem 3.7 below), a tool that will be used is the following strong result due to Ovsepian and Pelczynski [13] about the structure of separable (real or complex) Banach spaces.

Theorem 3.4. *If X is an infinite-dimensional separable Banach space, then there are sequences $\{e_n\}_{n\ge1} \subset X$ and $\{\varphi_n\}_{n\ge1} \subset X^*$ with the following properties:*
(a) $\varphi_m(e_n) = \delta_{mn}$ *for all* $m,n \in \mathbb{N}$.
(b) *If* $\varphi_n(x) = 0$ *for all* $n \in \mathbb{N}$ *then* $x = 0$.
(c) $\|e_n\| = 1$ *for all* $n \in \mathbb{N}$ *and* $\sup_{n\in\mathbb{N}} \|\varphi_n\| < \infty$.

Remark 3.5. *It is proved in [13] that the sequence $\{e_n\}_{n\ge1}$ may satisfy, in addition, that $\overline{\operatorname{span}}\{e_n : n \in \mathbb{N}\} = X$. Nevertheless, we do not need this property at all.*

Since Theorem 3.1 is given in the complex setting, in order to apply it in the real one the technique of complexification will be needed. Recall that if X is a Banach space *on* $\mathbb{R}$ then its complexification $\widetilde{X}$ is the vector space on $\mathbb{C}$ given by X^2 endowed with the operations

$$(x,y) + (u,v) = (x+u, y+v) \quad \text{and} \quad c(x,y) = (ax - by, bx + ay),$$

where $c = a + ib$, $a,b \in \mathbb{R}$, $x,y,u,v \in X$. Then $\widetilde{X}$ becomes a Banach space on $\mathbb{C}$ under, for instance, the norm $\|(x,y)\| = \|x\| + \|y\|$. If $T \in L(X)$ then the complexification of T is the operator $\widetilde{T}$ on $\widetilde{X}$ defined as

$$\widetilde{T}(x,y) = (Tx, Ty).$$

Note that $\widetilde{T^n} = (\widetilde{T})^n$ for all $n \ge 0$. Observe also that if $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function with *real* Taylor coefficients a_n then the expression $f(T) = \sum_{n=0}^\infty a_n T^n$ also makes sense and defines an operator on X satisfying $\widetilde{f(T)} = f(\widetilde{T})$. The following is a variant of Theorem 3.1 when $\mathbb{K} = \mathbb{R}$.

Lemma 3.6. *Let X be a real Banach space and $T \in L(X)$ be an operator such that $\sigma_P(\widetilde{T}) = \varnothing$. Then the set $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Observe that Lemma 3.3 also works when the linear algebra $\mathcal{A}$ generated by the functions E_r $(r \in H)$ is considered over $\mathbb{K} = \mathbb{R}$. Then this algebra is also freely $\mathfrak{c}$-generated and, since $H \subset \mathbb{R}$, all its members are entire functions with real Taylor coefficients. As in the proof of Theorem 3.1 we get that the family $\mathcal{B} := \{f(\widetilde{T}) : f \in \mathcal{A}\}$ is a $\mathfrak{c}$-generated free algebra satisfying $\mathcal{B} \setminus \{0\} \subset L_{1-1}(\widetilde{X})$. Let us prove that the algebra

$$\mathcal{B}_1 = \{f(T) : f \in \mathcal{A}\}$$

is freely generated by the operators $E_r(T)$ $(r \in H)$. To this end, let us fix, as in the last part of the proof of Theorem 3.1, functions $E_{r_1}, \dots, E_{r_N}$ as well as a polynomial Q with degree N, but this time with real coefficients. Let $F := Q(E_{r_1}, \dots, E_{r_N})$ and assume that $F(T) = 0$. Then $F(\widetilde{T}) = \widetilde{F(T)} = \widetilde{0} = 0$, so $F = 0$ as in the mentioned proof. Hence $Q = 0$ because $\mathcal{A}$ was freely generated by the E_r's. Finally, each operator $f(T) \in \mathcal{B}_1 \setminus \{0\}$ is one-to-one because, otherwise, there would exist $x \in X \setminus \{0\}$ with $f(T)x = 0$. This would imply that the nonzero vector $(x, 0) \in \widetilde{X}$ satisfies

$$f(\widetilde{T})(x, 0) = \widetilde{f(T)}(x, 0) = (f(T)x, f(T)0) = (0, 0),$$

which contradicts the injectivity of $f(\widetilde{T})$. The proof is finished. □

We are now ready to show that separability is enough to guarantee algebrability for our family of one-to-one operators.

Theorem 3.7. *Assume that X is a separable infinite dimensional Banach space. Then $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Choose a pair of sequences $\{e_n\}_{n \geq 1} \subset X$ and $\{\varphi_n\}_{n \geq 1} \subset X^*$ with the properties given in Theorem 3.4. Define the mapping

$$T : x \in X \mapsto \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(x)\, e_{n+1} \in X. \tag{1}$$

From property (c) in Theorem 3.4, it follows that

$$\sum_{n=1}^{\infty} \left\| \frac{1}{2^n} \varphi_n(x)\, e_{n+1} \right\| \leq \sup_{n \in \mathbb{N}} \|\varphi_n\|\, \|x\| < \infty. \tag{2}$$

Since X is a complete space, (2) shows that the series in (1) converges to a vector of X, so T is well defined. Trivially, T is linear and, by (2), $\|Tx\| \leq C\|x\|$ $(x \in X)$ with $C = \sup_{n \in \mathbb{N}} \|\varphi_n\| < \infty$. In other words, $T \in L(X)$.

Let us show that T lacks eigenvalues. Assume, by way of contradiction, that there is $a \in \sigma_P(T)$. Then there exists $x \in X \setminus \{0\}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(x)\, e_{n+1} = a\, x. \tag{3}$$

If we let φ_1 act on both members of (3) then we get $0 = a\, \varphi_1(x)$, thanks to property (a) in Theorem 3.4. Suppose first that $a \neq 0$. This implies $\varphi_1(x) = 0$. By making φ_2 act on (3), we obtain $\frac{1}{2} \varphi_1(x) = a\, \varphi_2(x)$, hence $\varphi_2(x) = 0$. With this procedure, we successively derive $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$. It follows from property (b) in Theorem 3.4 that $x = 0$, which is absurd. Then $a = 0$ and $Tx = 0$. Letting φ_{m+1} $(m \geq 1)$ act on (3), we get $\frac{1}{2^m} \varphi_m(x) = 0$, so $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$, which again implies $x = 0$, a contradiction. Hence $\sigma_P(T) = \varnothing$. This is valid in both cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

According to Theorem 3.1, $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable if $\mathbb{K} = \mathbb{C}$. Finally, we will prove by using Lemma 3.6 that the conclusion also holds if $\mathbb{K} = \mathbb{R}$. All that we have to show is $\sigma_P(\widetilde{T}) = \varnothing$. Assume, contrariwise, that there is $c = a + ib \in \mathbb{C}$ as well as a vector $z = (x, y) \in \widetilde{X} \setminus \{(0, 0)\}$ such that $\widetilde{T} z = c\, z$. Then $Tx = ax - by$ and $Ty = bx + ay$. Note that if $a = 0 = b$ then $Tx = 0 = Ty$, in which case $0 \in \sigma_P(T) = \varnothing$, which is absurd. Consequently, $c \neq 0$ or, that is the same, $a^2 + b^2 \neq 0$. Therefore we have that for some $(a, b) \neq (0, 0)$ and some $(x, y) \neq (0, 0)$ the following holds:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(x)\, e_{n+1} = ax - by \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi_n(y)\, e_{n+1} = bx + ay. \tag{4}$$

Letting φ_1 and then φ_{m+1} $(m \in \mathbb{N})$ act on both equalities of (4) we obtain

$$a\, \varphi_1(x) - b\, \varphi_1(y) = 0, \;\; b\, \varphi_1(x) + a\, \varphi_1(y) = 0, \text{ and} \tag{5}$$

$$a\, \varphi_{m+1}(x) - b\, \varphi_{m+1}(y) = 2^{-m} \varphi_m(x),$$

$$b\,\varphi_{m+1}(x) + a\,\varphi_{m+1}(y) = 2^{-m}\varphi_m(y). \tag{6}$$

Since (5) is a homogeneous linear system in the unknowns $\varphi_1(x)$, $\varphi_1(y)$ whose determinant is $a^2 + b^2 \neq 0$, one derives $\varphi_1(x) = 0 = \varphi_1(y)$. By proceeding recursively and assuming $\varphi_m(x) = 0 = \varphi_m(y)$ for an $m \in \mathbb{N}$, one finds that (6) is, again, a homogeneous linear system with determinant $a^2 + b^2 \neq 0$. Hence its unique solution is $\varphi_{m+1}(x) = 0 = \varphi_{m+1}(y)$. To summarize, $\varphi_m(x) = 0 = \varphi_m(y)$ for all $m \in \mathbb{N}$, from which it follows that $(x, y) = (0, 0)$ because of Theorem 3.4(b). This contradiction concludes the proof. □

Remarks 3.8.

1. *In particular* $L_{1-1}(X)$ *is lineable under the assumptions of Theorem 3.7. Notice that our approach is radically different from that of Theorem 1.3 given in [4, Theorem 2.12]. Observe also that this theorem is strengthened in a double direction: the conclusion is reinforced and extended, and our assumptions are weaker because every Banach space with a Schauder basis is separable but, as Enflo proved in [14], the reverse is not true.*
2. *The conclusion of Theorem 3.7 is optimal, in terms of the cardinality of the generating family of operators. Indeed, since* X *is separable, it follows that* $\mathrm{card}(X) = \mathfrak{c}$. *Again by separability,* $\mathrm{card}(C(X)) = \mathfrak{c}$. *Hence* $\mathrm{card}(L(X)) = \mathfrak{c}$ *because this cardinality lies between* $\mathrm{card}(X)$ *and* $\mathrm{card}(C(X))$. *Then the cardinality of the generating set of the algebra founded in Theorem 3.7 is optimal.*

Let us give an example of an application of Theorem 3.1 that cannot be derived from Theorem 3.7. Assume that Ω is a topological space and that X is a Banach space of continuous functions $\Omega \to \mathbb{K}$. Assume that $m : \Omega \to \mathbb{R}$ is a function such that $m \cdot f \in X$ for all $f \in X$. Then the mapping

$$f \in X \mapsto m \cdot f \in X$$

is well defined and linear, so it defines an operator $M_m \in L(X)$ (the *multiplication operator* by m) due to the closed graph theorem. Instances of such spaces are the space X_1 of all continuous functions $[0, 1] \to \mathbb{K}$ (endowed with the supremum norm $\|\cdot\|_\infty$) and the space X_2 of functions $[0, 1] \to \mathbb{K}$ that are continuous and of bounded variation (endowed with the total variation norm $\|f\| = |f(0)| + \mathrm{Var}_{[0,1]}(f)$). Notice that X_1 is separable, while X_2 is not. Since each of these spaces X_i is in fact a Banach algebra, one can choose as m any member of X_i. We impose, in addition, that every a-point set $m^{-1}(\{a\})$ $(a \in \mathbb{K})$ has empty interior in Ω (for instance, take $m(x) = x$ in the above examples $X = X_1, X_2$). Then

$$\sigma_P(M_m) = \varnothing \text{ and, if } K = \mathbb{R},\ \sigma_P(\widetilde{M_m}) = \varnothing. \tag{9}$$

Theorem 3.1 yields that $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable. It is enough to prove (9). With this aim, assume that there exists $\lambda \in \sigma_P(M_m)$, so that there is $f \in X \setminus \{0\}$ with $m\,f = \lambda f$. By continuity, there is a nonempty open set $G \subset \Omega$ such that $f(x) \neq 0$ for all $x \in G$. Then $m = \lambda$ on G, hence $m^{-1}(\{\lambda\})$ has nonempty interior, a contradiction. Finally, suppose that $\mathbb{K} = \mathbb{R}$ and that there exists $\lambda = a + bi \in \sigma_P(\widetilde{M_m})$, so that there are functions $f, g \in X$ with $(f, g) \neq (0, 0)$ satisfying $m\,f = af - bg$ and $m\,g = bf + ag$. Fix $x \in \Omega \setminus m^{-1}(\{a\})$. It follows that

$$(m(x) - a) f(x) + bg(x) = 0 \quad \text{and} \quad -bf(x) + (m(x) - a)\,g(x) = 0.$$

The determinant of this homogeneous linear system with unknowns $f(x), g(x)$ is $(m(x) - a)^2 + b^2$, which is nonzero because $m(x) \neq a$. Then its unique solution is $(f(x), g(x)) = (0, 0)$. Therefore $f = 0$ and $g = 0$ on the dense set $\Omega \setminus m^{-1}(\{a\})$. By continuity, $f = 0 = g$, that again is a contradiction.

In order to furnish another class of Banach spaces to which Theorem 3.1 applies, we need to recall a concept and some properties coming from hypercyclicity, for whose general theory and results (updated up to 2011) we refer the reader to the excellent books [15] and [16]. If X is a (Hausdorff) topological vector space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) then an operator $T \in L(X)$ is said to be *hypercyclic* if it possesses a dense orbit, that is, if there is a vector $x_0 \in X$, called hypercyclic for T, such that the set

$$\{T^n x_0 : n \geq 1\} \text{ is dense in } X.$$

It is evident that if X supports some hypercyclic operator then X must be separable. In addition, X cannot be finite dimensional. Conversely, if X is an infinite dimensional separable Fréchet (in particular, Banach) space then there is a hypercyclic operator $T \in L(X)$; see, e.g., [16, Chap. 8].

Theorem 3.9. *Assume that X is a complex Banach space that is the dual space of some separable infinite dimensional Banach space. Then $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. By hypothesis, there is a separable infinite dimensional Banach space Y such that $X = Y^*$. Choose a hypercyclic operator $S \in L(Y)$ and define $T := S^*$. But the adjoint of any hypercyclic operator has no eigenvalues: see, e.g., [16, Lemma 2.53(a)]. Thus, it suffices to apply Theorem 3.1. □

Remarks 3.10.

1. *Theorem 3.9 covers important examples of nonseparable Banach spaces, such as $\ell_\infty(\mathbb{C}) := \{x = (x_n)_{n\geq 1} \in \mathbb{C}^{\mathbb{N}} : (x_n)_{n\geq 1}$ is bounded$\}$ (under the norm $\|x\| = \sup_{n\geq 1} |x_n|$) and $BV([0,1],\mathbb{C}) := \{f : [0,1] \to \mathbb{C} : f$ is of bounded variation$\}$ (under the total variation norm) because the former is the dual space of the space $\ell_1(\mathbb{C})$ of all absolutely summable complex sequences, while the latter is the dual space of the space $C([0,1],\mathbb{C})$ of continuous complex functions on $[0,1]$ (Riesz's theorem).*
2. *We conjecture that Theorem 3.9 also holds for $\mathbb{K} = \mathbb{R}$. This is supported by the fact that any separable infinite dimensional Banach space Y supports a mixing operator S, that is, an operator satisfying the following property: for any pair U, V of nonempty open subsets of Y, there exists some $N \in \mathbb{N}$ such that $S^n(U) \cap V \neq \varnothing$ for all $n \geq N$ [16, Chap. 8]. This easily implies that $\widetilde{S}$ is hypercyclic on $\widetilde{Y}$. Then $\sigma_P((\widetilde{S})^*) = \varnothing$. The handicap in order to apply Lemma 3.6 lies in the fact that we need $\sigma_P(\widetilde{S^*}) = \varnothing$, but $(\widetilde{S})^*$ acts on $(Y \times Y)^*$, while $\widetilde{S^*}$ does so on $Y^* \times Y^*$.*

Despite the last remark, the big algebraic size of the family of injective operators on $\ell_\infty(\mathbb{R})$ happens to be true. In fact, an approach similar to that used in the proof of Theorem 3.7 shows the following.

Theorem 3.11. *Let X be a Banach space that is a subset of the sequence space $\mathbb{K}^{\mathbb{N}}$. Assume that every member of the canonical unit sequence $(e_n)_{n\geq 1}$ belongs to X and that the projections*

$$\varphi_m : x = (x_n)_{n\geq 1} \in X \mapsto x_m \in \mathbb{K} \quad (m \in \mathbb{N})$$

are continuous. Then $L_{1-1}(X)$ is strongly $\mathfrak{c}$-algebrable.

Proof. Define T by

$$T : x = (x_n)_{n\geq 1} \in X \mapsto \sum_{n=1}^{\infty} 2^{-n} \|\varphi_n\|^{-1} \|e_{n+1}\|^{-1} x_n e_{n+1} \in X$$

and mimic the proof of Theorem 3.7. The details are left as an exercise. □

If our Banach space is reflexive, a result due to H. Salas provides us with a dual pair of large algebras of one-to-one operators.

Theorem 3.12. *Let X be a complex infinite dimensional separable reflexive Banach space. Then there are families $\mathcal{F} \subset L(X)$, $\mathcal{G} \subset L(X^*)$ satisfying the following properties:*

(a) *$\mathcal{F}$ and $\mathcal{G}$ are commutative linear algebras.*
(b) *$\mathcal{F}$ and $\mathcal{G}$ are freely $\mathfrak{c}$-generated.*
(c) *Every member of $\mathcal{F}$ or $\mathcal{G}$ is injective.*
(d) *$\mathcal{G} = \{S^* : S \in \mathcal{F}\}$.*

Proof. In 2007, Salas [17] proved that if X is an infinite dimensional Banach space whose dual X^* is separable, then there exists a hypercyclic operator T on X such that its adjoint T^* is also hypercyclic. Under our assumptions, X is, in addition, reflexive, so $X = X^{**} = (X^*)^*$ is separable. Hence X^* is separable (because if the dual Y^* of a Banach space Y is separable then Y is itself separable). Therefore we can find a hypercyclic operator $T \in L(X)$ such that $T^* \in L(X^*)$ is hypercyclic. From [16, Lemma 2.53(a)] we have

$$\sigma_P(T^*) = \varnothing \text{ and } \sigma_P(T) = \sigma_P(T^{**}) = \sigma_P((T^*)^*) = \varnothing.$$

Then Theorem 3.1 furnishes families $\mathcal{F} \subset L(X)$, $\mathcal{G} \subset L(X^*)$ satisfying properties (a), (b) and (c). But it is known (see, e.g., [6, Chap. 1] or [7, Chap. 10]) that $f(T^*) = (f(T))^*$ for every entire function f, hence by the construction given in the proof of Theorem 3.1 one obtains that (d) is also fulfilled. □

We want to finish this paper by posing the following problem, which is in the same spirit as [4, Question 2.14].

Problem. *Is $L_{1-1}(X)$ large –in any algebraic sense– for all infinite dimensional Banach spaces?*

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