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An investigation on hyper S-posets over ordered semihypergroups

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Abstract: In this paper, we define and study the hyper S-posets over an ordered semihypergroup in detail. We introduce the hyper version of a pseudoorder in a hyper S-poset, and give some related properties. In particular, we characterize the structure of factor hyper S-posets by pseudoorders. Furthermore, we introduce the concepts of order-congruences and strong order-congruences on a hyper S-poset A, and obtain the relationship between strong order-congruences and pseudoorders on A. We also characterize the (strong) order-congruences by the ρ-chains, where ρ is a (strong) congruence on A. Moreover, we give a method of constructing order-congruences, and prove that every hyper S-subposet B of a hyper S-poset A is a congruence class of one order-congruence on A if and only if B is convex. In the sequel, we give some homomorphism theorems of hyper S-posets, which are generalizations of similar results in S-posets and ordered semigroups.

Keywords: Ordered semihypergroup, Hyper S-poset, (Strong) order-congruence, Pseudoorder, ρ-chain

MSC: 20N20, 06F05, 20M30

1 Introduction and preliminaries

It is well known that S-acts (also called S-systems) play an important role not only in studying properties of semigroups or monoids but also in other mathematical areas, such as graph theory and algebraic automata theory, for example, see [18, 22]. For a semigroup $(S, \cdot)$, a (*right*) S-*act* (or S-*system*) is a nonempty set A together with a mapping $A \times S \to A$ sending (a, s) to as such that $(as)t = a(st)$ for all $s, t \in S$ and $a \in A$. Further, for an ordered semigroup $(S, \cdot, \leq_S)$, a *right* S-*poset* A_S is a right S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq_S t$ then $as \leq_A at$, and if $a \leq_A b$ then $as \leq_A bs$. Left S-posets are defined analogously. During recent years a number of articles on S-posets theory have appeared, for example see [3, 19, 21, 26, 27, 33]. Also see [2] for an overview.

On the other hand, algebraic hyperstructures, particularly hypergroups, were introduced by Marty [23] in 1934. Later on, algebraic hyperstructures have been intensively studied, both from the theoretical point of view and especially for their applications in other fields (see [6, 7]). One of the main reason which attracts researches towards algebraic hyperstructures is its unique property that in algebraic hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are a suitable generalization of classical algebraic structures. The study on the theory of semihypergroups is one of the most active subjects in algebraic hyperstructure theory. Nowadays, many researchers studied different aspects of semihypergroups, for instance, Anvariyeh et al. [1], Chaopraknoi and Triphop [5], Davvaz [8], Hila et al. [14],

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Leoreanu [20] and Salvo et al. [25], also see [11, 24]. A theory of hyperstructures on ordered semigroups has been recently developed. In [13], Heidari and Davvaz applied the theory of hyperstructures to ordered semigroups and introduced the concept of ordered semihypergroups, which is a generalization of the concept of ordered semigroups. Since then many papers on ordered semihypergroups have been published, for instance, see [4, 9, 12, 28]. Our aim in this paper is to introduce a special type of hyperstructure, namely hyper S-posets, and study the properties of hyper S-posets over ordered semihypergroups. In particular, we define and discuss the order-congruences and strongly order-congruences of hyper S-posets, and give some homomorphism theorems of hyper S-posets by pseudoorders.

In the rest of this section, We recall the basic terms and definitions from the hyperstructure theory.

Definition 1.1. *A hypergroupoid $(S, \circ)$ is a nonempty set S together with a hyperoperation, that is a map $\circ : S \times S \to P^\star(S)$, where $P^\star(S)$ denotes the set of all the nonempty subsets of S. The image of the pair (x, y) is denoted by $x \circ y$.*

Definition 1.2. *A hypergroupoid $(S, \circ)$ is called a semihypergroup if the hyperoperation "$\circ$" is associative, that is, for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that*

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If $x \in S$ and A, B are nonempty subsets of S, then

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,\ A \circ x = A \circ \{x\}, \text{ and } x \circ B = \{x\} \circ B.$$

Generally, the singleton $\{x\}$ is identified by its element x.

Definition 1.3. *An algebraic hyperstructure $(S, \circ, \leq)$ is called an ordered semihypergroup (also called po-semihypergroup in [13]) if $(S, \circ)$ is a semihypergroup and $(S, \leq)$ is a partially ordered set such that: for any $x, y, a \in S$, $x \leq y$ implies $a \circ x \leq a \circ y$ and $x \circ a \leq y \circ a$. Here, if $A, B \in P^*(S)$, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.*

Clearly, every ordered semigroup can be regarded as an ordered semihypergroup, for instance, see [28].

Definition 1.4. *A nonempty subset A of an ordered semihypergroup S is called a left (resp. right) hyperideal of S if*
(1) $S \circ A \subseteq A$ (*resp.* $A \circ S \subseteq A$).
(2) *If $a \in A$ and $S \ni b \leq a$, then $b \in A$.*
If A is both a left and a right hyperideal of S, then it is called a (two-sided) hyperideal of S

For more information on hyperstructure theory, ordered semigroup theory and the properties of S-acts, the reader is referred to [7], [30] and [22], correspondingly.

2 Hyper S-acts over semihypergroups

In order to study the hyper S-posets over ordered semihypergroups in detail, in this section we first discuss the properties of hyper S-acts over semihypergroups. In particular, we investigate the congruences and strong congruences of hyper S-acts over semihypergroups.

We now recall the notion of hyper S-acts over semihypergroups from [10].

Definition 2.1. *Let $(S, \circ)$ be a semihypergroup and A a nonempty set. If we have a mapping $\mu : A \times S \to P^\star(A)$ | $(a, s) \mapsto \mu(a, s) := a * s \in P^\star(A)$, called the hyper action of S (or the S-hyperaction) on A, such that $a * (s \circ t) = (a * s) * t$, for all $a \in A, s, t \in S$, where*

$$(\forall T \subseteq S)\ a * T = \bigcup_{t \in T} a * t;\ (\forall B \subseteq A)\ B * t = \bigcup_{b \in B} b * t,$$

then we call A a right hyper S-act (also called right S-hypersystem in [10]), denoted by $A_{\mathcal{H}}$, or briefly A.

Left hyper S-acts are defined analogously, and in this paper we will often use the term hyper S-act to mean right hyper S-act.

Remark 2.2. *Every S-act over a semigroup can be regarded as a hyper S-act over a semihypergroup. In fact, if A is an S-act over a semigroup $(S, \cdot)$, the hyperoperation " $\circ$ " on S and the hyper S-action " $*$ " on A are defined respectively as $s \circ t := \{st\}$, $a * s := \{as\}$, for any $a \in A, s, t \in S$, then, clearly, A is a hyper S-act over a semihypergroup $(S, \circ)$.*

Definition 2.3. *Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and B a nonempty subset of A. B is called a hyper S-subact of A if B is closed under the hyper S-action on A, i.e., $b * s \subseteq B$ for any $b \in B, s \in S$.*

Clearly, for any $a \in A$, $a * S$ is a hyper S-subact of A, called *cyclic hyper S-subact*.

Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ an equivalence relation on A. If B and C are both nonempty subsets of A, then we write $B \,\overline{\rho}\, C$ to denote that for every $b \in B$, there exists $c \in C$ such that $b\rho c$ and for every $c \in C$, there exists $b \in B$ such that $b\rho c$. We write $B \,\overline{\overline{\rho}}\, C$ if for every $b \in B$ and for every $c \in C$, we have $b\rho c$. The equivalence relation ρ is called *congruence* if for every $(x, y) \in A \times A$, the implication $x\rho y \Rightarrow x * s \,\overline{\rho}\, y * s$, for all $s \in S$, is valid. ρ is called *strong congruence* if for every $(x, y) \in A \times A$, from $x\rho y$, it follows that $x * s \,\overline{\overline{\rho}}\, y * s$ for all $s \in S$. We denote by $C(A)$ (resp. $SC(A)$) the set of all congruences (resp. strong congruences) on a hyper S-act A.

Remark 2.4. *The set $C(A)$ of all congruences on a hyper S-act A is a complete lattice with respect to the intersection of set-theoretic and the union (also is called transitive product) defined as follows:*

$$(a, b) \in \prod_{\alpha \in \Gamma} \rho_\alpha \Leftrightarrow \exists c_\circ = a, c_1, \ldots, c_n = b \in A$$
$$\text{such that } (c_j, c_{j+1}) \in \rho_{\alpha_j} \text{ for some } \rho_{\alpha_j} \in \{\rho_\alpha\}_{\alpha \in \Gamma}.$$

It is worth pointing out that the equality relation 1_A and the universal relation $A \times A$ on A are the minimum element and greatest element of $C(A)$, respectively.

Theorem 2.5. *Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and ρ an equivalence relation on A. Then*
(1) *If ρ is a congruence, then A/ρ is a hyper S-act with respect to the following hyper S-action: $(a)_\rho \otimes s = \bigcup_{x \in a*s} (x)_\rho$, and it is called a factor hyper S-act.*
(2) *If ρ is a strong congruence, then A/ρ is a hyper S-act with respect to the following (hyper) S-action: $(a)_\rho \otimes s = (x)_\rho$ for all $x \in a * s$, and it is called a factor hyper S-act. In particular, if S is a semigroup and the operation on S is defined by $s \circ t := \{st\}$ for all $s, t \in S$, then A/ρ is an S-act.*

Proof. (1) Let ρ be a congruence on A. Then the hyper S-action " $\otimes$ " is well defined. Indeed, let $(a)_\rho, (b)_\rho \in A/\rho$ and $s, t \in S$ be such that $(a)_\rho = (b)_\rho, s = t$. Then $a\rho b$. Since ρ is a congruence on A, we have $a * s \,\overline{\rho}\, b * s$. Hence for any $x \in a * s$, there exists $y \in b * t$ such that $x\rho y$, i.e., $(x)_\rho = (y)_\rho$. Thus $(a)_\rho \otimes s = \bigcup_{x \in a*s} (x)_\rho \subseteq \bigcup_{y \in b*t} (y)_\rho = (b)_\rho \otimes t$. In a similar way ,it can be shown that $(b)_\rho \otimes t \subseteq (a)_\rho \otimes s$. Therefore, $(a)_\rho \otimes s = (b)_\rho \otimes t$.

Furthermore, let $s, t \in S$ and $(a)_\rho \in A/\rho$. Then we have

$$\begin{aligned}
(a)_\rho \otimes (s \circ t) &= \bigcup_{u \in s \circ t} ((a)_\rho \otimes u) = \bigcup_{u \in s \circ t} \bigcup_{x \in a*u} (x)_\rho \\
&= \bigcup_{x \in a*(s \circ t)} (x)_\rho = \bigcup_{x \in (a*s)*t} (x)_\rho \text{ (Since } A \text{ is a hyper } S\text{-act)} \\
&= \bigcup_{y \in a*s} \bigcup_{x \in y*t} (x)_\rho = \bigcup_{y \in a*s} ((y)_\rho \otimes t) = ((a)_\rho \otimes s) \otimes t.
\end{aligned}$$

Thus A/ρ is a hyper S-act over S.

(2) The proof is similar to that of (1), and hence we omit the details. □

Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and B a hyper S-subact of A. The relation ρ_B on S is defined as follows:

$$\rho_B := \{(x, y) \in A\backslash B \times A\backslash B \mid x = y\} \cup (B \times B).$$

Clearly, ρ_B is an equivalence relation on A. Moreover, we have the following lemma.

Theorem 2.6. *Let A be a hyper S-act over a semihypergroup $(S, \circ)$ and B a hyper S-subact of A. Then ρ_B is a congruence on A and it is called Rees congruence induced by B.*

Proof. Let $x, y \in A$ and $x\rho_B y$. Then $x = y \in A\backslash B$ or $x, y \in B$. We consider the following cases:

Case 1. If $x = y \in A\backslash B$, then, for any $s \in S$, $x * s = y * s$. Hence $x * s\, \overline{\rho}_B\, y * s$.

Case 2. Let $x, y \in B$. Since B a hyper S-subact of A, we have $x * s \subseteq B, y * s \subseteq B$ for any $s \in S$. Thus, for any $a \in x * s, b \in y * s$, we have $(a, b) \in B \times B \subseteq \rho_B$. Thus $x * s\, \overline{\rho}_B\, y * s$.

Therefore, ρ_B is a congruence on A. □

Remark 2.7.

(1) *$A/\rho_B = \{\{x\} \mid x \in A\backslash B\} \cup \{B\}$, that is, for any $(a)_{\rho_B} \in A/\rho_B$, we have $(a)_{\rho_B} = \{a\}, a \in A\backslash B$ or $(a)_{\rho_B} = B$.*

(2) *By Theorems 2.5 and 2.6, $(A/\rho_B, \otimes_B)$ forms a factor hyper S-act, which is called Rees factor hyper S-act. Here the hyper S-action $\otimes_B$ on A/ρ_B is defined by $(a)_{\rho_B} \otimes_B s = \bigcup\limits_{x \in a*s} (x)_{\rho_B}, \forall (a)_{\rho_B} \in A/\rho_B, s \in S$.*

3 Hyper S-posets over ordered semihypergroups

In this section we shall introduce the concept of hyper S-posets over an ordered semihypergroup, and study the properties of hyper S-posets. In particular, we define and discuss the pseudoorders on hyper S-posets.

Definition 3.1. *Let $(S, \circ, \leq_S)$ be an ordered semihypergroup. A right hyper S-poset $(A, \leq_A)$, often denoted $A_{\mathcal{H}}$ (or briefly A), is a right hyper S-act A equipped with a partial order $\leq_A$ and, in addition, for all $s, t \in S$ and $a, b \in A$, if $s \leq_S t$ then $a * s \leq_A a * t$, and if $a \leq_A b$ then $a * s \leq_A b * s$. Here, $a * s$ stands for the result of the hyper action of s on a, and if $A_1, A_2 \in P^\star(A)$, then we say that $A_1 \leq_A A_2$ if for every $a_1 \in A_1$ there exists $a_2 \in A_2$ such that $a_1 \leq_A a_2$.*

Analogously, we can define a left hyper S-poset ${}_{\mathcal{H}}A$. Throughout this paper we shall use the term hyper S-poset to mean right hyper S-poset.

Remark 3.2.

(1) *Every S-poset over an ordered semigroup can be regarded as a hyper S-poset over an ordered semihypergroup.*

(2) *An ordered semihypergroup S is a hyper S-poset with respect to the hyperoperation of S.*

Let A be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq_S)$ and B a nonempty subset of A. B is called a *hyper S-subposet* of A if for any $b \in B, s \in S, b * s \subseteq B$, denoted by $B \leq A$. For any $a \in A$, $a * S$ is clearly a hyper S-subposet of A, called *cyclic hyper S-subposet*. It is easily seen that a hyperideal of an ordered semihypergroup S is a hyper S-subposet of S.

Definition 3.3. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, the "$*$" and "$\diamond$" are hyper S-actions on A and B, respectively, $f : A \to B$ a mapping from A to B. f is called isotone if $x \leq_A y$ implies $f(x) \leq_B f(y)$, for all $x, y \in A$. f is called reverse isotone if $x, y \in A$, $f(x) \leq_B f(y)$ implies $x \leq_A y$. f is called homomorphism (resp. strong homomorphism) if it is isotone and satisfies $f(a) \diamond s = \bigcup\limits_{x \in a*s} f(x)$ (resp. $f(a) \diamond s = f(x),\ \forall x \in a * s$), for all $a \in A, s \in S$. f is called isomorphism (resp. strong isomorphism) if it* *is homomorphism (resp. strong homomorphism), onto and reverse isotone. The hyper S-posets A and B are called strongly isomorphic, in symbol $A \cong B$, if there exists a strong isomorphism between them.*

Remark 3.4.

(1) *Suppose that $(A, \leq_A)$ and $(B, \leq_B)$ are two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$. If f is a strong homomorphism and reverse isotone mapping from A to B, then $A \cong Im(f)$.*

(2) *The class of right hyper S-posets and homomorphisms forms a category that we denote by $HPOS$-S. As usual, the monomorphisms of $HPOS$-S are exactly the injective homomorphisms.*

By Definition 3.1, we can see that the congruences and strong congruences on hyper S-posets can be defined exactly as in the case of hyper S-acts. Thus it is unnecessary to repeat the concepts of congruences and strong congruences on hyper S-posets.

Let A be a hyper S-act and ρ a (strong) congruence on A. Then, by Theorem 2.5, the set $A/\rho := \{(a)_\rho \mid a \in A\}$ is a hyper S-act and the hyper S-action on A/ρ is defined via the hyper S-action on A. The following question is natural: If $(A, \leq_A)$ is a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a (strong) congruence on A, then is the set A/ρ a hyper S-poset? A probable order on A/ρ could be the relation " $\preceq$ " on A/ρ defined by means of the order " $\leq_A$ " on A, that is

$$\preceq := \{((x)_\rho, (y)_\rho) \in A/\rho \times A/\rho \mid (x, y) \in \leq_A\}.$$

But this relation is not an order, in general. We illustrate it by the following example.

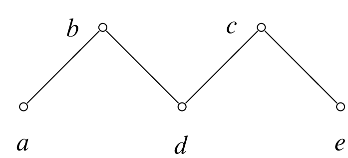
Example 3.5. *We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":*

$\circ$	a	b	c	d	e
a	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{d\}$	$\{d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{d\}$	$\{d\}$	$\{d\}$
c	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{d\}$	$\{d\}$	$\{c\}$	$\{d\}$	$\{c\}$

$$\leq := \{(a, a), (a, b), (b, b), (c, c), (d, b), (d, c), (d, d), (e, c), (e, e)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a, b), (d, b), (d, c), (e, c)\}.$$

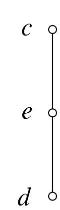


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{c, d, e\}$ defined by the order below:

$$\leq_A := \{(c, c), (d, d), (e, e), (d, e), (d, c), (e, c).$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(d, e), (e, c)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table.

Let ρ be a (strong) congruence on A defined as follows:

$$\rho := \{(c, c), (d, d), (e, e), (d, c), (c, d)\}.$$

Then $A/\rho = \{\{d, c\}, \{e\}\}$. Moreover, the relation on A/ρ defined by

$$\preceq := \{((x)_\rho, (y)_\rho) \in A/\rho \times A/\rho \mid (x, y) \in \leq_A\}$$

is not an order relation on A/ρ. In fact, since $d \leq_A e$, we have $(d)_\rho \preceq (e)_\rho$. Also, since $e \leq_A c$, we have $(e)_\rho \preceq (c)_\rho = (d)_\rho$. If "$\preceq$" is an order relation on A/ρ, then $(d)_\rho = (e)_\rho$, which is impossible. Thus $(A/\rho, \preceq)$ is not a hyper S-poset.

The following question arises: Is there a (strong) congruence ρ on A for which A/ρ is a hyper S-poset? To solve the above question, we first introduce the following definition.

Definition 3.6. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A relation ρ on A is called pseudoorder if it satisfies the following conditions:*
(1) $\leq_A \subseteq \rho$.
(2) *$a\rho b$ and $b\rho c$ imply $a\rho c$, i.e., $\rho \circ \rho \subseteq \rho$.*
(3) *$a\rho b$ implies $a * s \,\overline{\overline{\rho}}\, b * s$, for all $s \in S$.*

Note that an ordered semihypergroup S is a hyper S-poset with respect to the hyperoperation of S. Thus Definition 3.6 is a generalization of Definition 4.1 in [9]. For a similar definition about pseudoorders in ordered semigroups we refer the readers to Definition 1 in [16].

Theorem 3.7. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a pseudoorder on A. Then, there exists a strong congruence $\underline{\rho}$ on A such that $A/\underline{\rho}$ is a hyper S-poset over S.*

Proof. We denote by $\underline{\rho}$ the relation on A defined by

$$\underline{\rho} := \{(a, b) \in A \times A \mid a\rho b \text{ and } b\rho a\} \ (= \rho \cap \rho^{-1}).$$

First, we claim that $\underline{\rho}$ is a strong congruence on A. In fact, for any $a \in A$, clearly, $(a, a) \in \leq_A \subseteq \rho$, so $a\underline{\rho}a$. If $(a, b) \in \underline{\rho}$, then $a\rho b$ and $b\rho a$. Thus $(b, a) \in \underline{\rho}$. Let $(a, b) \in \underline{\rho}$ and $(b, c) \in \underline{\rho}$. Then $a\rho b, b\rho a, b\rho c$ and $c\rho b$. Hence $a\rho c$ and $c\rho a$, which imply that $(a, c) \in \underline{\rho}$. Thus $\underline{\rho}$ is an equivalence relation on A. Now, let $a\underline{\rho}b$ and $s \in S$. Then $a\rho b$ and $b\rho a$. Since ρ is a pseudoorder on A, by condition (3) of Definition 3.6, we have

$$a * s \,\overline{\overline{\rho}}\, b * s, b * s \,\overline{\overline{\rho}}\, a * s.$$

Thus, for every $x \in a * s$ and $y \in b * s$, we have $x\rho y$ and $y\rho x$. It implies that $x\underline{\rho}y$. Hence $a * s \,\overline{\overline{\underline{\rho}}}\, b * s$. Therefore, $\underline{\rho}$ is indeed a strong congruence on A. By Theorem 2.5, $A/\underline{\rho}$ is a hyper S-act over S.

Now, we define a relation $\preceq_\rho$ on $A/\underline{\rho}$ as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Then $(A/\underline{\rho}, \preceq_\rho)$ is a poset. Indeed, suppose that $(x)_{\underline{\rho}} \in A/\underline{\rho}$, where $x \in A$. Then $(x, x) \in \leq_A \subseteq \rho$. Hence, $(x)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Let $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (x)_{\underline{\rho}}$. Then $x\rho y$ and $y\rho x$. Thus $x\underline{\rho}y$, and we have $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Now, if $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $(y)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$, then $x\rho y$ and $y\rho z$. Hence $x\rho z$, and we conclude that $(x)_{\underline{\rho}} \preceq_\rho (z)_{\underline{\rho}}$.

Furthermore, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$, $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$ and $s \in S$. Then $x\rho y$. By hypothesis and Definition 3.6, $x * s \,\overline{\overline{\rho}}\, y * s$. Thus, for any $a \in x * s$ and $b \in y * s$, we have $a\rho b$. This implies that $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$. Hence we have

$$(x)_{\underline{\rho}} \otimes s = \bigcup_{a \in x * s} (a)_{\underline{\rho}} \preceq_\rho \bigcup_{b \in y * s} (b)_{\underline{\rho}} = (y)_{\underline{\rho}} \otimes s,$$

where the hyper S-action " $\otimes$ " on $A/\underline{\rho}$ is exactly that defined in Theorem 2.5. Moreover, let $s, t \in S, s \leq t$ and $(a)_{\underline{\rho}} \in A/\underline{\rho}$. Then, similarly as discussed above, we have $(a)_{\underline{\rho}} \otimes s \preceq_\rho (a)_{\underline{\rho}} \otimes t$.

Therefore, $A/\underline{\rho}$ is a hyper S-poset over S. ☐

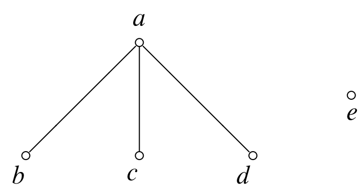
Example 3.8. *We consider a set $S := \{a, b, c, d, e\}$ with the following hyperoperation " $\circ$ " and the order " $\leq$ ":*

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$
c	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$
d	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a,d\}$
e	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$	$\{e\}$

$$\leq := \{(a,a),(b,b),(b,a),(c,c),(c,a),(d,a),(d,d),(e,e)\}.$$

The covering relation "$\prec$" and the figure of S are given by:

$$\prec = \{(b,a),(c,a),(d,a)\}.$$

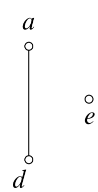


Then $(S, \circ, \leq)$ is an ordered semihypergroup (see [9]). We now consider the partially ordered set $A = \{a, d, e\}$ defined by the order below:

$$\leq_A := \{(a,a),(d,d),(e,e),(d,a).$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(d,a)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as above hyperoperation table.

Let ρ be a pseudoorder on A defined as follows:

$$\rho := \{(a,a),(d,d),(e,e),(a,d),(d,a),(e,a),(e,d)\}.$$

Applying Theorem 3.7, we get

$$\underline{\rho} := \{(a,a),(d,d),(e,e),(a,d),(d,a)\}.$$

Then $A/\underline{\rho} = \{\{a,d\},\{e\}\}$. Moreover, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S, where the order relation $\preceq_\rho$ on $A/\underline{\rho}$ is defined by

$$\preceq_\rho := \{(\{a,d\},\{a,d\}),(\{e\},\{e\}),(\{e\},\{a,d\})\}.$$

Theorem 3.9. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a pseudoorder on A. Let*

$$\mathcal{A} := \{\theta \mid \theta \text{ is a pseudoorder on } A \text{ such that } \rho \subseteq \theta\}.$$

Let $\mathcal{B}$ be the set of all pseudoorders on $A/\underline{\rho}$. Then, card($\mathcal{A}$) = card($\mathcal{B}$).

Proof. For $\theta \in \mathcal{A}$, we define a relation θ' on $A/\underline{\rho}$ as follows:

$$\theta' := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \theta\}.$$

First, we claim that θ' is a pseudoorder on $A/\underline{\rho}$. To prove our claim, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. Then, by Theorem 3.7, $(x, y) \in \rho \subseteq \theta$, which implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$. Thus, $\preceq_\rho \subseteq \theta'$. Now, assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and

$((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Then, $(x, y) \in \theta$ and $(y, z) \in \theta$. It implies that $(x, z) \in \theta$. Hence, $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \theta'$. Also, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta'$ and $s \in S$. Then, $(x, y) \in \theta$ and $s \in S$. Since θ is a pseudoorder on A, we have $x * s \;\overline{\overline{\theta}}\; y * s$. Thus, for every $a \in x * s, b \in y * s$, we have $(a, b) \in \theta$. This implies that $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta'$, and thus $(x)_{\underline{\rho}} \otimes s \;\overline{\overline{\theta'}}\; (y)_{\underline{\rho}} \otimes s$. Therefore, θ' is indeed a pseudoorder on $A/\underline{\rho}$.

Now, we define the mapping $f : \mathcal{A} \to \mathcal{B}$ by $f(\theta) = \theta', \forall \theta \in \mathcal{A}$. Then, f is a bijection from $\mathcal{A}$ onto $\mathcal{B}$. In fact,

(1) f is well defined. Indeed, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_1 = \theta_2$. Then, for any $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta_1'$, we have $(x, y) \in \theta_1 = \theta_2$. It implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta_2'$. Hence $\theta_1' \subseteq \theta_2'$. By symmetry, it can be obtained that $\theta_2' \subseteq \theta_1'$.

(2) f is one to one. In fact, let $\theta_1, \theta_2 \in \mathcal{A}$ and $\theta_1' = \theta_2'$. Assume that $(x, y) \in \theta_1$. Then, $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta_1'$ and thus $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \theta_2'$. This implies that $(x, y) \in \theta_2$. Thus, $\theta_1 \subseteq \theta_2$. Similarly, we obtain $\theta_2 \subseteq \theta_1$.

(3) f is onto. In fact, let $\delta \in \mathcal{B}$. We define a relation θ on A as follows:

$$\theta := \{(x, y) \in A \times A \mid ((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \delta\}.$$

We show that θ is a pseudoorder on A and $\rho \subseteq \theta$. Assume that $(x, y) \in \rho$. Then, by Theorem 3.7, $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho \subseteq \delta$, and thus $(x, y) \in \theta$. This implies that $\rho \subseteq \theta$. If $(x, y) \in \leq_A$, then $(x, y) \in \rho \subseteq \theta$. Hence, $\leq_A \subseteq \theta$. Let now $(x, y) \in \theta$ and $(y, z) \in \theta$. Then $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \delta$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \delta$. Hence $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \delta$, which implies that $(x, z) \in \theta$. Furthermore, let $(x, y) \in \theta$ and $s \in S$. Then $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \delta$ and $s \in S$. Since δ is a pseudoorder on $A/\underline{\rho}$, we have $(x)_{\underline{\rho}} \otimes s \;\overline{\overline{\delta}}\; (y)_{\underline{\rho}} \otimes s$, i.e., $\bigcup\limits_{a \in x*s} (a)_{\underline{\rho}} \;\overline{\overline{\delta}}\; \bigcup\limits_{b \in y*s} (b)_{\underline{\rho}}$. Thus, for every $a \in x * s$ and $b \in y * s$, $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \delta$. It means that $(a, b) \in \theta$. Hence we conclude that $x * s \;\overline{\overline{\theta}}\; y * s$. Moreover, clearly, $\theta' = \delta$. □

By the proof of Theorem 3.9, we immediately obtain the following corollary:

Corollary 3.10. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be pseudoorders on A such that $\rho \subseteq \theta$. We define a relation θ/ρ on $A/\underline{\rho}$ as follows:*

$$\theta/\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \theta\}.$$

Then θ/ρ is a pseudoorder on $A/\underline{\rho}$.

4 (Strong) order-congruences on hyper S-posets

In the above section, we have illustrated that for a (strong) congruence ρ on a hyper S-poset A the factor hyper S-act A/ρ is not necessarily a hyper S-poset, in general. To characterize the structure of hyper S-posets in detail, in this section we shall define and study the order-congruences and strong order-congruences on a hyper S-poset.

Definition 4.1. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$. A congruence (resp. strong congruence) ρ is called an order-congruence (resp. a strong order-congruence) if there exists an order relation "$\preceq$" on A/ρ such that:*

(1) $(A/\rho, \preceq)$ is a hyper S-poset, where the S-hyperaction "$\otimes$" on A/ρ is defined as one in Theorem 2.5.

(2) The canonical epimorphism $\varphi : A \to A/\rho, x \mapsto (x)_\rho$ is isotone, that is, φ is a homomorphism (resp. strong homomorphism) from A onto A/ρ.

It is clear that the equality relation 1_A and the universal relation $A \times A$ on A are both order-congruences, but 1_A is not a strong order-congruence on A. In general, a strong order-congruence example is given as follows:

Example 4.2. *We consider the ordered semihypergroup $(S, \circ, \leq)$ and the hyper S-poset $(A, \leq_A)$ over S in Example 3.5. Let ρ be a strong congruence on A defined as follows:*

$$\rho := \{((c, c), (d, d), (e, e), (c, e), (e, c)\}.$$

Then $S/\rho = \{\{c, e\}, \{d\}\}$. Moreover, ρ is a strong order-congruence on A. In fact, we define an order $\preceq_\rho$ on A/ρ as follows:

$$\preceq_\rho := \{(\{d\}, \{d\}), (\{c, e\}, \{c, e\}), (\{d\}, \{c, e\})\}.$$

Then $(A/\rho, \preceq_\rho)$ is a hyper S-poset and the mapping $\varphi : A \to A/\rho, x \mapsto (x)_\rho$ is isotone. Hence ρ is a strong order-congruence on A.

Proposition 4.3. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a pseudoorder on A. Then $\underline{\rho}$ is a strong order-congruence on A, where $\underline{\rho} = \rho \cap \rho^{-1}$.*

Proof. By Theorem 3.7, $(A/\underline{\rho}, \preceq_\rho)$ is a hyper S-poset over S, where the order relation $\preceq_\rho$ is defined as follows:

$$\preceq_\rho := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (x, y) \in \rho\}.$$

Also, let $x, y \in A$ and $x \leq_A y$. Then, since ρ is a pseudoorder on A, $(x, y) \in \leq_A \subseteq \rho$. Thus $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$, i.e., $(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}}$. Therefore, $\underline{\rho}$ is a strong order-congruence on A. □

In order to establish the relationship between strong order-congruences and pseudoorders on a hyper S-poset, the following lemma is essential.

Lemma 4.4. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and σ a relation on A. Then the following statements are equivalent:*
(1) *σ is a pseudoorder on A.*
(2) *There exist a hyper S-poset $(B, \leq_B)$ over S and a strong homomorphism $\varphi : A \to B$ such that*

$$\overrightarrow{ker}\varphi := \{(a, b) \in A \times A \mid \varphi(a) \leq_B \varphi(b)\} = \sigma,$$

where $\overrightarrow{ker}\varphi$ is called the directed kernel of φ.

Proof. (1) $\Rightarrow$ (2). Let σ be a pseudoorder on A. We denote by $\underline{\sigma}$ the strong congruence on A defined by

$$\underline{\sigma} := \{(a, b) \in A \times A \mid (a, b) \in \sigma, (b, a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

Then, by Theorem 3.7, the set $A/\underline{\sigma} := \{(a)_{\underline{\sigma}} \mid a \in A\}$ with the S-hyperaction $(a)_{\underline{\sigma}} \otimes s = (x)_{\underline{\sigma}}, \forall x \in a * s$, for all $a \in A, s \in S$ and the order

$$\preceq_\sigma := \{((x)_{\underline{\sigma}}, (y)_{\underline{\sigma}}) \in A/\underline{\sigma} \times A/\underline{\sigma} \mid (x, y) \in \sigma\}$$

is a hyper S-poset. Let $B = (A/\underline{\sigma}, \preceq_\sigma)$ and φ be the mapping of A onto $A/\underline{\sigma}$ defined by $\varphi : A \to A/\underline{\sigma} \mid a \mapsto (a)_{\underline{\sigma}}$. Then, by Proposition 4.3, φ is a strong homomorphism from A onto $A/\underline{\sigma}$ and clearly, $\overrightarrow{ker}\varphi = \sigma$.

(2) $\Rightarrow$ (1). If there exist a hyper S-poset $(B, \leq_B)$ over S and a strong homomorphism $\varphi : A \to B$ such that $\overrightarrow{ker}\varphi = \sigma$, then σ is a pseudoorder on A. Indeed, let $(a, b) \in \leq_A$. Then, by hypothesis, $\varphi(a) \leq_B \varphi(b)$. Thus $(a, b) \in \overrightarrow{ker}\varphi = \sigma$, and we have $\leq_A \subseteq \sigma$. Now, let $(a, b) \in \sigma$ and $(b, c) \in \sigma$. Then $\varphi(a) \leq_B \varphi(b) \leq_B \varphi(c)$. Hence $\varphi(a) \leq_B \varphi(c)$, i.e., $(a, c) \in \overrightarrow{ker}\varphi = \sigma$. Also, if $(a, b) \in \sigma$, then $\varphi(a) \leq_B \varphi(b)$. Since $(B, \leq_B)$ is a hyper S-poset over S, for any $s \in S$ we have $\varphi(a) \diamond s \leq_B \varphi(b) \diamond s$, where " $\diamond$ " is the S-hyperaction on B. Since φ is a strong homomorphism from A to B, for every $x \in a * s$ and $y \in b * s$, we have

$$\varphi(x) = \varphi(a) \diamond s \leq_B \varphi(b) \diamond s = \varphi(y).$$

Then $(x, y) \in \overrightarrow{ker}\varphi = \sigma$, and thus $a * s \,\overline{\overline{\sigma}}\, b * s$. Therefore, σ is a pseudoorder on A. □

Theorem 4.5. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in SC(A)$. Then the following statements are equivalent:*
(1) *ρ is a strong order-congruence on A.*
(2) *There exists a pseudoorder σ on A such that $\rho = \sigma \cap \sigma^{-1}$.*
(3) *There exist a hyper S-poset B over S and a strong homomorphism $\varphi : A \to B$ such that $\rho = ker(\varphi)$, where $ker\varphi = \{(a, b) \in A \times A \mid \varphi(a) = \varphi(b)\}$ is the kernel of φ.*

Proof. (1) $\Rightarrow$ (2). Let ρ be a strong order-congruence on A. Then there exist an order relation "$\preceq$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq)$ is a hyper S-poset over S, and $\varphi : A \to A/\rho$ is a strong homomorphism. Let $\sigma = \overrightarrow{ker}\varphi$. By Lemma 4.4, σ is a pseudoorder on A and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

(2) $\Rightarrow$ (3). For a pseudoorder σ on A, by Lemma 4.4, there exist a hyper S-poset B over S and a strong homomorphism $\varphi : A \to B$ such that $\sigma = \overrightarrow{ker\varphi}$. Then we have

$$ker\varphi = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

(3) $\Rightarrow$ (1). By hypothesis and Lemma 4.4, $\overrightarrow{ker\varphi}$ is a pseudoorder on A. Then, by Theorem 3.7, $\rho = \overrightarrow{ker\varphi} \cap (\overrightarrow{ker\varphi})^{-1}$ is a strong congruence on A. Thus, by the proof of Lemma 4.4, ρ is a strong order-congruence on A. □

Remark 4.6.
(1) *For a strong order-congruence ρ on A, since the order "$\preceq$" such that $(A/\rho, \preceq)$ is a hyper S-poset is not unique in general, we have the pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique.*
(2) *If σ is a pseudoorder on a hyper S-poset A, then $\rho = \sigma \cap \sigma^{-1}$ is the greatest strong order-congruence on A contained in σ. In fact, if ρ_1 is a strong order-congruence on A contained in σ, then $\rho_1 = \rho_1 \cap \rho_1^{-1} \subseteq \sigma \cap \sigma^{-1} = \rho$.*

Theorem 4.7. *Let ρ be a strong order-congruence on a hyper S-poset $(A, \leq_A)$. Then the least pseudoorder σ containing ρ is the transitive closure of relations $\leq_A \circ\rho$ (resp. $\rho\circ \leq_A$), that is,*

$$\sigma = \bigcup_{n=1}^{\infty} (\leq_A \circ\rho)^n = \bigcup_{n=1}^{\infty} (\rho\circ \leq_A)^n.$$

Proof. (1) Let $\sigma_1 = \bigcup_{n=1}^{\infty} (\leq_A \circ\rho)^n$. Clearly, $\rho \subseteq \leq_A \circ\rho \subseteq \sigma_1$. Similarly, since $\leq_A \subseteq \leq_A \circ \rho$, we have $\leq_A \subseteq \sigma_1$.

(2) If $(a,b) \in \sigma_1$, $(b,c) \in \sigma_1$, then there exist $m,n \in Z^+$ such that $(a,b) \in (\leq_A \circ\rho)^m$ and $(b,c) \in (\leq_A \circ\rho)^n$, where Z^+ denotes the set of positive integers. Thus $(a,c) \in (\leq_A \circ\rho)^{m+n} \subseteq \sigma_1$, i.e., σ_1 is transitive.

(3) Let $(a,b) \in \sigma_1$ and $s \in S$. Then there exists $n \in Z^+$ such that $(a,b) \in (\leq_A \circ\rho)^n$, that is, there exist $a_1, b_1, a_2, b_2, \ldots, a_n \in A$ such that

$$a \leq_A a_1\rho b_1 \leq_A a_2\rho b_2 \leq_A \cdots \leq_A a_n\rho b.$$

Since $(A, \leq_A)$ is a hyper S-poset and $\rho \in SC(A)$, we have

$$a * s \leq_A a_1 * s \,\overline{\overline{\rho}}\, b_1 * s \leq_A a_2 * s \,\overline{\overline{\rho}}\, b_2 * s \leq_A \cdots \leq_A a_n * s \,\overline{\overline{\rho}}\, b * s.$$

Then, for any $x \in a * s$, $y \in b * s$, there exist $x_i \in a_i * s$ $(i = 1, 2, \ldots, n)$, $y_j \in b_j * s$ $(j = 1, 2, \ldots, n-1)$ such that

$$x \leq_A x_1\rho y_1 \leq_A x_2\rho y_2 \leq_A \cdots \leq_A x_n\rho y.$$

It thus implies that $(x,y) \in (\leq_A \circ\rho)^n \subseteq \sigma_1$, and we obtain that $a * s \,\overline{\overline{\sigma}}_1\, b * s$. Thus $\bigcup_{n=1}^{\infty} (\leq_A \circ\rho)^n$ is a pseudoorder on A containing ρ.

Furthermore, since σ is transitive, and $\rho \subseteq \sigma, \leq_A \subseteq \sigma$, we have $\bigcup_{n=1}^{\infty} (\leq_A \circ\rho)^n \subseteq \sigma$. Thus, by hypothesis, $\sigma = \bigcup_{n=1}^{\infty} (\leq_A \circ\rho)^n$. In the same way, we can verify that $\sigma = \bigcup_{n=1}^{\infty} (\rho\circ \leq_A)^n$. □

In the following, we shall give some characterizations of (strong) order-congruences on hyper S-posets. In order to obtain the main results, we first introduce the following concept.

Definition 4.8. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ an equivalence relation on A. A finite sequence of the form $(x, a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y)$ of elements in A is called a ρ-chain if*
(1) $(a_1, b_1) \in \rho, (a_2, b_2) \in \rho, \ldots, (a_{n-1}, b_{n-1}) \in \rho, (a_n, y) \in \rho$;
(2) $x \leq_A a_1, b_1 \leq_A a_2, b_2 \leq_A a_3, \ldots, b_{n-2} \leq_A a_{n-1}, b_{n-1} \leq_A a_n$.
Briefly we write

$$x \leq_A a_1\rho b_1 \leq_A a_2\rho b_2 \leq_A \cdots \leq_A a_n\rho y.$$

The number n is called the length, x and y initial and terminal elements, respectively, of the ρ-chain. A ρ-chain is called close if its initial and terminal elements are equal, i.e. $x = y$.

We denote by $\rho^{C_{xy}}$ the set of all ρ-chains with x as the initial and y as the terminal elements in the sequel.

Lemma 4.9. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in C(A)$. Then the following statements are true:*
(1) $(x, y) \in (\leq_A \circ \rho)^n$ *if and only if there exists a ρ-chain with length n in $\rho^{C_{xy}}$, i.e., $\rho^{C_{xy}} \neq \emptyset$.*
(2) *For any $s \in S$, if $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in A$, then for every $u \in x * s$, there exists $v \in y * s$ such that $\rho^{C_{uv}} \neq \emptyset$.*

Proof. (1) The proof is straightforward by Definition 4.8, we omit it.

(2) Let $(x, a_1, b_1, a_2, b_2, \ldots, a_n, y) \in \rho^{C_{xy}}$ and $s \in S$. Then

$$x \leq_A a_1 \rho b_1 \leq_A a_2 \rho b_2 \leq_A \cdots \leq_A a_n \rho b.$$

Since $(A, \leq_A)$ is a hyper S-poset and $\rho \in C(A)$, we have

$$x * s \leq_A a_1 * s \,\overline{\rho}\, b_1 * s \leq_A a_2 * s \,\overline{\rho}\, b_2 * s \leq_A \cdots \leq_A a_n * s \,\overline{\rho}\, y * s.$$

Then, for any $u \in x \circ s$, there exist $x_i \in a_i * s$ $(i = 1, 2, \ldots, n)$, $y_j \in b_j * s$ $(j = 1, 2, \ldots, n-1)$, $v \in y * s$ such that

$$u \leq_A x_1 \rho y_1 \leq_A x_2 \rho y_2 \leq_A \cdots \leq_A x_n \rho v.$$

It thus implies that $(u, x_1, y_1, x_2, y_2, \ldots, x_n, v) \in \rho^{C_{uv}}$, i.e., $\rho^{C_{uv}} \neq \emptyset$. □

Lemma 4.10. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in SC(A)$. If $\rho^{C_{xy}} \neq \emptyset$ for some $x, y \in A$, then, for any $s \in S$, we have $\rho^{C_{uv}} \neq \emptyset$ for every $u \in x * s, v \in y * s$.*

Proof. The proof is similar to that of Lemma 4.9 with a slight modification. □

Lemma 4.11. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and ρ a (strong) congruence on A. If $(x, y) \in \rho$, $(k, z) \in \rho$, then $\rho^{C_{xk}} \neq \emptyset$ if and only if $\rho^{C_{yz}} \neq \emptyset$.*

Proof. ($\Longrightarrow$). If $\rho^{C_{xk}} \neq \emptyset$, by Lemma 4.9(1), there exists $n \in Z^+$ such that $(x, k) \in (\leq_A \circ \rho)^n$. Since $(x, y) \in \rho$, $(z, k) \in \rho$, we have

$$y \leq_A y \rho x (\leq_A \circ \rho)^n k \leq_A k \rho z,$$

which implies that $(y, z) \in (\leq_A \circ \rho)^{n+2}$. By Lemma 4.9(1), we have $\rho^{C_{yz}} \neq \emptyset$.

($\Longleftarrow$). Similar to the proof of necessity, we omit it. □

Now we shall give a characterization of order-congruences on a hyper S-poset.

Theorem 4.12. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in C(A)$. Then ρ is an order-congruence on A if and only if every close ρ-chain is contained in a single equivalent class of ρ.*

Proof. Let ρ be an order-congruence on A. Then there exists an order $\preceq$ on the factor hyper S-act A/ρ such that $(A/\rho, \preceq)$ is a hyper S-poset over S and $\varphi : A \to A/\rho$ is a homomorphism. For any $x \in A$, and every close ρ-chain $(x, a_1, b_1, \ldots, a_n, x)$ in $\rho^{C_{xx}}$, we have

$$x \leq_A a_1 \rho b_1 \leq_A a_2 \rho b_2 \leq_A \cdots \leq_A a_n \rho x.$$

Then,

$$\varphi(x) \preceq \varphi(a_1) = \varphi(b_1) \preceq \varphi(a_2) = \varphi(b_2) \preceq \cdots \preceq \varphi(a_n) = \varphi(x).$$

It implies that $\varphi(x) = \varphi(a_1) = \varphi(b_1) = \varphi(a_2) = \varphi(b_2) = \cdots = \varphi(a_n)$. Consequently, $(x, a_1, b_1, \ldots, a_n, x)$ is contained in a single ρ-class.

Conversely, since ρ is a congruence on A, by Theorem 2.5, A/ρ is a hyper S-act. We define a relation "$\preceq$" on the factor hyper S-act A/ρ as follows:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid \rho^{C_{xy}} \neq \emptyset\}.$$

(1) $\preceq$ is well-defined. In fact, let $x_1, y_1 \in A$ be such that $(x)_\rho = (x_1)_\rho$, $(y)_\rho = (y_1)_\rho$. If $(x)_\rho \preceq (y)_\rho$, then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.11, we have $\rho^{C_{x_1 y_1}} \neq \emptyset$, and $(x_1)_\rho \preceq (y_1)_\rho$.

(2) $\preceq$ is an ordered relation on A/ρ.

(α) $\preceq$ is reflexive. In fact, since for any $x \in A$, $x \leq_A x\rho x$, and we have $\rho^{C_{xx}} \neq \emptyset$, i.e., $((x)_\rho, (x)_\rho) \in \preceq$.

(β) $\preceq$ is transitive. Indeed, let $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (z)_\rho) \in \preceq$. Then we have $\rho^{C_{xy}} \neq \emptyset$, $\rho^{C_{yz}} \neq \emptyset$. By Lemma 4.9(1), there exist $m, n \in Z^+$ such that $(x, y) \in (\leq_A \circ \rho)^m$, $(y, z) \in (\leq_A \circ \rho)^n$. Then we have

$$(x, z) \in (\leq_A \circ \rho)^m \circ (\leq_A \circ \rho)^n = (\leq_A \circ \rho)^{m+n},$$

i.e., $\rho^{C_{xz}} \neq \emptyset$. Thus $((x)_\rho, (z)_\rho) \in \preceq$.

(γ) $\preceq$ is anti-symmetric. In fact, if $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (x)_\rho) \in \preceq$, then $\rho^{C_{xy}} \neq \emptyset$, $\rho^{C_{yx}} \neq \emptyset$. Similar to the above proof, it can be obtained that $\rho^{C_{xx}} \neq \emptyset$, i.e., there exists a close ρ-chain in $\rho^{C_{xx}}$ containing x and y. By hypothesis, $(x)_\rho = (y)_\rho$.

(3) $(A/\rho, \preceq)$ is a hyper S-poset over S. Indeed, let $(x)_\rho \preceq (y)_\rho$ and $s \in S$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.9(2), for every $u \in x * s$, there exists $v \in y * s$ such that $\rho^{C_{uv}} \neq \emptyset$, i.e., $(u)_\rho \preceq (v)_\rho$. Thus

$$(x)_\rho \otimes s = \bigcup_{u \in x*s} (u)_\rho \preceq \bigcup_{v \in y*s} (v)_\rho = (y)_\rho \otimes s.$$

Also, let $s, t \in S$ be such that $s \leq t$. Then $x * s \leq_A x * t$ for any $x \in A$. Thus, for every $u' \in x * s$, there exists $v' \in x * t$ such that $u' \leq_A v'$. It implies that $(u', v') \in \leq_A \circ \rho$, and we have $\rho^{C_{u'v'}} \neq \emptyset$. Hence $(u')_\rho \preceq (v')_\rho$, and we obtain

$$(x)_\rho \otimes s = \bigcup_{u' \in x*s} (u')_\rho \preceq \bigcup_{v' \in x*t} (v')_\rho = (x)_\rho \otimes t.$$

(4) The mapping $\varphi : A \to A/\rho \mid x \mapsto (x)_\rho$ is isotone. In fact, let $x, y \in A$ be such that $x \leq_A y$. Then $(x, y) \in \leq_A \circ \rho$, we have $\rho^{C_{xy}} \neq \emptyset$, i.e. $(x)_\rho \preceq (y)_\rho$.

Therefore, ρ is an order-congruence on A. □

Similarly, strong order-congruences on a hyper S-poset can be characterized as follows:

Theorem 4.13. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in SC(A)$. Then ρ is a strong order-congruence on A if and only if every close ρ-chain is contained in a single equivalent class of ρ.*

Proof. The proof is similar to that of Theorem 4.12 with suitable modification by using Lemma 4.10. □

Recall that a nonempty subset B of a poset $(A, \leq)$ is called *convex* if $a \leq b \leq c$ implies $b \in B$ for all $a, c \in B, b \in A$; B is called *strongly convex* if $a \in A, b \in B$ and $a \leq b$ imply $a \in B$. Any strongly convex subset of A is clearly convex, however, the converse does not hold in general.

Corollary 4.14. *If ρ is an order-congruence on a hyper S-poset A, then every ρ-class in A is convex.*

Proof. Let ρ be an order-congruence on A and B a congruence class of ρ. If $x \leq_A y \leq_A z$ and $x, z \in B$, then $(x)_\rho = (z)_\rho$. Thus we have $x \leq_A y\rho y \leq_A z\rho x$. Hence (x, y, y, z, x) is a close ρ-chain, by Theorem 4.12, we have $(x)_\rho = (y)_\rho = (z)_\rho$. It thus follows that $y \in B$, and B is convex. □

Furthermore, we have the following theorem.

Theorem 4.15. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and B a hyper S-subposet of A. Then B is a congruence class of one order-congruence on A if and only if B is convex.*

Proof. ($\Longrightarrow$). The proof is straightforward by Corollary 4.14.

($\Longleftarrow$). Let ρ_B be the Rees congruence induced by B on A. By Remark 2.7(1), B is a congruence class of ρ_B. Now we define a relation " $\preceq_B$ " on the factor hyper S-act A/ρ_B as follows:

$$(x)_{\rho_B} \preceq_B (y)_{\rho_B} \Leftrightarrow (x \leq_A y) \text{ or } (x \leq_A b, b' \leq_A y \text{ for some } b, b' \in B).$$

We claim that ρ_B is an order-congruence on A. To prove our claim, we first show that $\preceq_B$ is order relation on A/ρ_B, i.e., $\preceq_B$ is reflexive, anti-symmetric and transitive.

(1) Let $(x)_{\rho_B}$ be any element of A/ρ_B. Then, since $x \leq_A x$, we have $(x)_{\rho_B} \preceq_B (x)_{\rho_B}$.

(2) Let $(x)_{\rho_B} \preceq_B (y)_{\rho_B}$ and $(y)_{\rho_B} \preceq_B (x)_{\rho_B}$. Then $x \leq_A y$ or $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$, and $y \leq_A x$ or $y \leq_A b_1, b_1' \leq_A x$ for some $b_1, b_1' \in B$. We consider the following four cases:

Case 1. If $x \leq_A y$ and $y \leq_A x$, then $x = y$, and thus $(x)_{\rho_B} = (y)_{\rho_B}$.

Case 2. If $x \leq_A y$ and $y \leq_A b_1, b_1' \leq_A x$ for some $b_1, b_1' \in B$, then $b_1' \leq_A x \leq_A y \leq_A b_1$. Since B is convex and $b_1, b_1' \in B$, we have $x, y \in B$. Thus $(x)_{\rho_B} = (y)_{\rho_B} = B$.

Case 3. Let $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$ and $y \leq_A x$. Similar to the proof of Case 2, we have $(x)_{\rho_B} = (y)_{\rho_B}$.

Case 4. Let $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$ and $y \leq_A b_1, b_1' \leq_A x$ for some $b_1, b_1' \in B$. Then $b_1' \leq_A x \leq_A b$ and $b' \leq_A y \leq_A b_1$. Since B is convex, we have $x, y \in B$. Thus $(x)_{\rho_B} = (y)_{\rho_B}$.

(3) Let $(x)_{\rho_B} \preceq_B (y)_{\rho_B}$ and $(y)_{\rho_B} \preceq_B (z)_{\rho_B}$. Then $x \leq_A y$ or $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$, and $y \leq_A z$ or $y \leq_A b_1, b_1' \leq_A z$ for some $b_1, b_1' \in B$. There are four cases to be considered:

Case 1. If $x \leq_A y$ and $y \leq_A z$, then $x \leq_A z$, and thus $(x)_{\rho_B} \preceq_B (y)_{\rho_B}$.

Case 2. If $x \leq_A y$ and $y \leq_A b_1, b_1' \leq_A z$ for some $b_1, b_1' \in B$, then $x \leq_A y \leq_A b_1$ and $b_1' \leq_A z$. By the definition of $\preceq_B$, $(x)_{\rho_B} \preceq_B (z)_{\rho_B}$.

Case 3. Let $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$ and $y \leq_A z$. Analogous to the proof of Case 2, we have $(x)_{\rho_B} \preceq_B (z)_{\rho_B}$.

Case 4. Let $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$ and $y \leq_A b_1, b_1' \leq_A z$ for some $b, b' \in B$. Then $x \leq_A b$ and $b_1' \leq_A z$. Hence $(x)_{\rho_B} \preceq_B (z)_{\rho_B}$.

We now show that $(A/\rho_B, \preceq_B)$ is a hyper S-poset over S. Let $(x)_{\rho_B} \preceq_B (y)_{\rho_B}$ and $s \in S$. Then $x \leq_A y$ or $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$. We consider the following two cases:

Case 1. If $x \leq_A y$, then $x * s \leq_A y * s$. Thus for every $u \in x * s$, there exists $v \in y * s$ such that $u \leq_A v$, and we have $(u)_{\rho_B} \preceq_B (v)_{\rho_B}$. Thus

$$(x)_{\rho_B} \otimes_B s = \bigcup_{u \in x*s} (u)_{\rho_B} \preceq_B \bigcup_{v \in y*s} (v)_{\rho_B} = (y)_{\rho_B} \otimes_B s.$$

Case 2. Let $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$. Then $x*s \leq_A b*s, b'*s \leq_A y*s$. Thus for every $u \in x*s$, there exists $b_1 \in b * s$ such that $u \leq_A b_1$, and for some $b_1' \in b' * s$ there exists $v \in y * s$ such that $b_1' \leq_A v$. Since B is a hyper S-subposet of A and $b, b' \in B$, we have $b_1 \in b * s \subseteq B, b_1' \in b' * s \subseteq B$. On the other hand, $u \leq_A b_1, b_1' \leq_A v$ for some $b_1, b_1' \in B$. Hence $(u)_{\rho_B} \preceq_B (v)_{\rho_B}$, and thus $(x)_{\rho_B} \otimes_B s \preceq_B (y)_{\rho_B} \otimes_B s$.

Also, let $s, t \in S$ be such that $s \leq t$. Then $x * s \leq_A x * t$ for any $x \in A$. Thus, for every $u' \in x * s$, there exists $v' \in x * t$ such that $u' \leq_A v'$. It implies that $(u')_{\rho_B} \preceq_B (v')_{\rho_B}$, and we have

$$(x)_{\rho_B} \otimes_B s = \bigcup_{u' \in x*s} (u')_{\rho_B} \preceq_B \bigcup_{v' \in x*t} (v')_{\rho_B} = (x)_{\rho_B} \otimes_B t.$$

Therefore, $(A/\rho_B, \preceq_B)$ is a hyper S-poset over S.

Furthermore, by the definition of $\preceq_B$, it can be shown that the canonical epimorphism $\varphi : A \to A/\rho_B, x \mapsto (x)_{\rho_B}$ is isotone. Thus ρ_B is an order-congruence on A. This completes the proof. □

By the proof of the above theorem, we immediately obtain the following corollary:

Corollary 4.16. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and B a strongly convex hyper S-subposet of A. Then $(A/\rho_B, \preceq_B)$ forms a hyper S-poset over S and the Rees congruence ρ_B induced by B on A is an order-congruence, where the order relation "$\preceq_B$" on A/ρ_B is defined as follows:*

$$(x)_{\rho_B} \preceq_B (y)_{\rho_B} \Leftrightarrow (x \leq_A y) \text{ or } (x \leq_A b, b' \leq_A y \text{ for some } b, b' \in B).$$

Corollary 4.16 shows that the Rees congruence ρ_B induced by B on A is an order-congruence. But we state that ρ_B is not necessarily a strong order-congruence on A in general. We illustrate it by the following example.

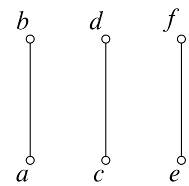
Example 4.17. *We consider a set $S := \{a, b, c, d, e, f\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":*

$\circ$	a	b	c	d	e	f
a	$\{a\}$	$\{a,b\}$	$\{c\}$	$\{c,d\}$	$\{e\}$	$\{e,f\}$
b	$\{b\}$	$\{b\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$
c	$\{c\}$	$\{c,d\}$	$\{c\}$	$\{c,d\}$	$\{c\}$	$\{c,d\}$
d	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
e	$\{e\}$	$\{e,f\}$	$\{c\}$	$\{c,d\}$	$\{e\}$	$\{e,f\}$
f	$\{f\}$	$\{f\}$	$\{d\}$	$\{d\}$	$\{f\}$	$\{f\}$

$$\leq := \{(a,a), (a,b), (b,b), (c,c), (c,d), (d,d), (e,e), (e,f), (f,f)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{(a,b), (c,d), (e,f)\}.$$

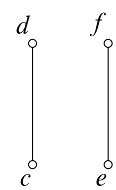


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{c, d, e, f\}$ defined by the order below:

$$\leq_A := \{(c,c), (d,d), (e,e), (f,f), (c,d), (e,f).$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(c,d), (e,f)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as in the above hyperoperation table. Let $B = \{c, d\}$. It is easy to check that B is a strongly convex hyper S-subposet of A. Then $\rho_B = \{(c,c), (d,d), (e,e), (f,f), (c,d), (d,c)\}$. One can easily verify that ρ_B is an order-congruence on A. But we claim that ρ_B is not a strong congruence on A. In fact, since $(e,e) \in \rho_B$, while $e * b\ \overline{\overline{\rho}}_B\ e * b$ doesn't hold. Thus ρ_B is not a strong order-congruence on A.

As a generalization of Theorem 2 in [32], we have the following theorem. The following theorem can be proved using similar techniques as in the proof of Theorem 4.15.

Theorem 4.18. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and B a strongly convex hyper S-subposet of A. We define an order relation "$\preceq_1$" on A/ρ_B $(= \{\{x\} \mid x \in A \backslash B\} \cup \{B\})$ as follows:*

$$\preceq_1 := \{(B, \{x\}) \mid x \in A \backslash B\} \cup \{(\{x\}, \{y\}) \mid x, y \in A \backslash B, x \leq_A y\} \cup \{(B, B)\}.$$

Then $(A/\rho_B, \preceq_1)$ is a hyper S-poset over S, and ρ_B is an order-congruence on A.

Proposition 4.19. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and B a strongly convex hyper S-subposet of A. Then the order relations defined in Corollary 4.16 and Theorem 4.18 are different. Moreover, $\preceq_B \subseteq \preceq_1$.*

Proof. Let $(x)_{\rho_B}, (y)_{\rho_B} \in A/\rho_B$ and $(x)_{\rho_B} \preceq_B (y)_{\rho_B}$. Then $x \leq_A y$ or $x \leq_A b, b' \leq_A y$ for some $b, b' \in B$. Since B is strongly convex, we have $x \leq_A y$ or $x \in B$ and $b' \leq_A y$ for some $b' \in B$. The first case implies $(x)_{\rho_B} \preceq_1 (y)_{\rho_B}$, and the second case implies $(x)_{\rho_B} \preceq_1 (b')_{\rho_B} \preceq_1 (y)_{\rho_B}$, i.e., $(x)_{\rho_B} \preceq_1 (y)_{\rho_B}$. Hence $\preceq_B \subseteq \preceq_1$. □

The following example shows that $\preceq_B \subsetneq \preceq_1$ in general.

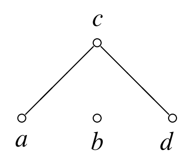
Example 4.20. *We consider a set $S := \{a, b, c, d\}$ with the following hyperoperation "$\circ$" and the order "$\leq$":*

$\circ$	a	b	c	d
a	$\{a, d\}$	$\{a, d\}$	$\{a, d\}$	$\{a\}$
b	$\{a, d\}$	$\{b\}$	$\{a, d\}$	$\{a, d\}$
c	$\{a, d\}$	$\{a, d\}$	$\{c\}$	$\{a, d\}$
d	$\{a\}$	$\{a, d\}$	$\{a, d\}$	$\{d\}$

$$\leq := \{(a, a), (a, c), (b, b), (c, c), (d, c), (d, d)\}.$$

We give the covering relation "$\prec$" and the figure of S as follows:

$$\prec = \{((a, c), (d, c)\}.$$

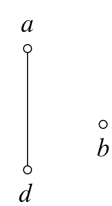


Then $(S, \circ, \leq)$ is an ordered semihypergroup. We now consider the partially ordered set $A = \{a, b, d\}$ defined by the order below:

$$\leq_A := \{(a, a), (b, b), (d, d), (d, a).$$

We give the covering relation "$\prec_A$" and the figure of A.

$$\prec_A = \{(d, a)\}.$$



Then $(A, \leq_A)$ is a hyper S-poset over S with respect to S-hyperaction on A as in the above hyperoperation table. Let $B = \{a, d\}$. We can easily verify that B is a strong convex hyper S-subposet of A. Since $a \nleq_A b$ and there does not exist $x \in B$ such that $x \leq_A b$, we have $(a)_{\rho_B} \npreceq_B (b)_{\rho_B}$. But, by the definition of $\preceq_1$, we have $(a)_{\rho_B} \preceq_1 (b)_{\rho_B}$.

In the following we shall define and study the strong order-congruence generated by a strong congruence on a hyper S-poset.

Definition 4.21. *Let ρ be a strong congruence on a hyper S-poset A. A strong order-congruence σ is called the strong order-congruence generated by ρ on A, if σ satisfies the following conditions:*
(1) *$\rho \subseteq \sigma$.*
(2) *If there exists a strong order-congruence η on A such that $\rho \subseteq \eta$, then $\sigma \subseteq \eta$.*

Theorem 4.22. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$ and $\rho \in SC(A)$. Then*
(1) *If we define a relation ρ^* on A as follows:*

$$(x, y \in A)\ (x, y) \in \rho^* \text{ if and only if } \rho^{C_{xy}} \neq \emptyset,$$

then ρ^* is a pseudoorder on A.
(2) *R_ρ is a relation on A defined as follows:*

$$(x, y \in A)\ (x, y) \in R_\rho \Longleftrightarrow (x, y) \in \rho^* \text{ and } (y, x) \in \rho^*.$$

Then R_ρ is the strong order-congruence generated by ρ on A.

Proof. (1) Let $x, y \in A$ be such that $x \leq_A y$. Then there is a ρ-chain from x to y: (x, y, y), i.e., $\rho^{C_{xy}} \neq \emptyset$. Thus $x \leq_A y$ implies $x\rho^* y$, and we have $\leq_A \subseteq \rho^*$. Assume that $(x, y) \in \rho^*$ and $(y, z) \in \rho^*$. Then there exist

$a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_m \in A$ such that

$$x \leq_A a_1 \rho b_1 \leq_A a_2 \rho b_2 \leq_A \cdots \leq_A a_{n-1} \rho b_{n-1} \leq_A a_n \rho y,$$
$$y \leq_A c_1 \rho d_1 \leq_A c_2 \rho d_2 \leq_A \cdots \leq_A c_{m-1} \rho d_{m-1} \leq_A d_m \rho z.$$

Thus, $x \leq_A a_1 \rho b_1 \leq_A a_2 \rho b_2 \leq_A \cdots \leq_A a_{n-1} \rho b_{n-1} \leq_A a_n \rho y \leq_A c_1 \rho d_1 \leq_A c_2 \rho d_2 \leq_A \cdots \leq_A c_{m-1} \rho d_{m-1} \leq_A d_m \rho z$, which is a ρ-chain from x to z. Hence $(x, z) \in \rho^*$ and ρ^* is transitive. Furthermore, let $(x, y) \in \rho^*$ and $s \in S$. Then $\rho^{C_{xy}} \neq \emptyset$. By Lemma 4.10, for every $u \in x * s, v \in y * s$, we have $\rho^{C_{uv}} \neq \emptyset$, which implies that $(u, v) \in \rho^*$. It thus follows that $x * s \ \overline{\overline{\rho^*}}\ y * s$. Therefore, ρ^* is a pseudoorder on A.

(2) By (1), ρ^* is a pseudoorder on A. Clearly, $R_\rho = \rho^* \cap (\rho^*)^{-1}$. By Proposition 4.3, R_ρ is a strong order-congruence on A. We claim that R_ρ is the strong order-congruence generated by ρ on A. To prove our claim, let $(x, y) \in \rho$. Since ρ is a strong congruence on A, we have $(y, x) \in \rho$. Consequently, $(x, y) \in R_\rho$. Hence $\rho \subseteq R_\rho$. Furthermore, suppose that η is a strong order-congruence on A and $\rho \subseteq \eta$. Then $R_\rho \subseteq \eta$. Indeed, let $(x, y) \in R_\rho$. Then $(x, y) \in \rho^*$ and $(y, x) \in \rho^*$. By definition of ρ^*, there exist $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{n-1}, c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_{m-1} \in A$ such that

$$x \leq_A a_1 \rho b_1 \leq_A a_2 \rho b_2 \leq_A \cdots \leq_A a_{n-1} \rho b_{n-1} \leq_A a_n \rho y,$$
$$y \leq_A c_1 \rho d_1 \leq_A c_2 \rho d_2 \leq_A \cdots \leq_A c_{m-1} \rho d_{m-1} \leq_A d_m \rho x.$$

Thus, by $\rho \subseteq \eta$, we have $x \leq_A a_1 \eta b_1 \leq_A a_2 \eta b_2 \leq_A \cdots \leq_A a_{n-1} \eta b_{n-1} \leq_A a_n \eta y \leq_A c_1 \eta d_1 \leq_A c_2 \eta d_2 \leq_A \cdots \leq_A c_{m-1} \eta d_{m-1} \leq_A d_m \eta x$. Since η is a strong order-congruence on A, by Theorem 4.13 we can conclude that the closed η-chain $(x, a_1, b_1, a_2, b_2, \ldots, a_{n-1}, b_{n-1}, a_n, y, c_1, d_1, c_2, d_2, \ldots, c_{m-1}, d_{m-1}, d_m, x)$ is contained in a single equivalence class of η. In particular, we have $(x, y) \in \eta$. Therefore, R_ρ is the strong order-congruence generated by ρ on A. □

By Theorem 4.22, we immediately obtain the following corollary:

Corollary 4.23. *Every strong congruence on a hyper S-poset A is contained in a strong order-congruence on A.*

5 Homomorphism theorems of hyper S-posets

Homomorphism theorems of semigroups and S-acts based on congruences have been given in [15] and [22], respectively. In cases of ordered semigroups and S-posets, pseudoorders play the role congruences which are "bigger" than the congruences, for example, see [17, 31, 32]. In the current section, we discuss homomorphism theorems of hyper S-posets by pseudoorders defined in Section 3.

Let σ be a pseudoorder on a hyper S-poset $(A, \leq_A)$. Then, by Theorem 4.5, $\rho = \sigma \cap \sigma^{-1}$ is a strong order-congruence on A. We denote by $\rho^\sharp$ the canonical epimorphism from A onto A/ρ, i.e., $\rho^\sharp : A \to A/\rho \mid x \mapsto (x)_\rho$, which is a strong homomorphism. In the following, we give a homomorphism theorem of hyper S-posets by pseudoorders, which is a generalization of Theorem 12 in [32]. For a similar result about ordered semigroups we refer the readers to Theorem 1 in [17].

Theorem 5.1. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, $\varphi : A \to B$ a strong homomorphism. Then: If σ is a pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, then there exists the unique strong homomorphism $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\rho^\sharp"'] & B \\ A/\rho \arrow[ur, "f"'] & \end{tikzcd}$$

commutes, where $\rho = \sigma \cap \sigma^{-1}$. Moreover, $Im(\varphi) = Im(f)$. Conversely, if σ is a pseudoorder on A for which there exists a strong homomorphism $f : (A/\rho, \preceq_\sigma) \to (B, \leq_B)$ $(\rho = \sigma \cap \sigma^{-1})$ such that the above diagram commutes, then $\sigma \subseteq \overrightarrow{ker}\varphi$.

Proof. Let σ be a pseudoorder on A such that $\sigma \subseteq \overrightarrow{ker}\varphi$, $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$. Then

(1) f is well defined. Indeed, if $(a)_\rho = (b)_\rho$, then $(a, b) \in \rho \subseteq \sigma$. Since $\sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(a), \varphi(b)) \in \leq_B$. Furthermore, since $(b, a) \in \sigma \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(b), \varphi(a)) \in \leq_B$. Therefore, $\varphi(a) = \varphi(b)$.

(2) f is a strong homomorphism and $\varphi = f \circ \rho^\sharp$. In fact: By Lemma 4.4, there exist an order relation "$\preceq_\sigma$" on the factor hyper S-act A/ρ such that $(A/\rho, \preceq_\sigma)$ is a hyper S-poset and the canonical epimorphism $\rho^\sharp$ is a strong homomorphism. Moreover, we have

$$\begin{aligned}(a)_\rho \preceq_\sigma (b)_\rho &\Rightarrow (a, b) \in \sigma \subseteq \overrightarrow{ker}\varphi\\ &\Rightarrow \varphi(a) \leq_B \varphi(b)\\ &\Rightarrow f((a)_\rho) \leq_B f((b)_\rho).\end{aligned}$$

Also, let $(a)_\rho \in A/\rho$ and $s \in S$. For any $(x)_\rho \in (a)_\rho \otimes s$, we have $x \in a * s$. Since φ is a strong homomorphism from A to B, we have

$$f((a)_\rho) \diamond s = \varphi(a) \diamond s = \varphi(x) = f((x)_\rho),$$

where " $\diamond$ " is the S-hyperaction on B. Furthermore, for any $a \in A$, $(f \circ \rho^\sharp)(a) = f((a)_\rho) = \varphi(a)$, and thus $\varphi = f \circ \rho^\sharp$.

We claim that f is a unique strong homomorphism from A/ρ to B. To prove our claim, let g be a strong homomorphism from A/ρ to B such that $\varphi = g \circ \rho^\sharp$. Then, for any $(a)_\rho \in A/\rho$, we have

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^\sharp)(a) = g((a)_\rho).$$

Moreover, $Im(f) = \{f((a)_\rho) \mid a \in A\} = \{\varphi(a) \mid a \in A\} = Im(\varphi)$.

Conversely, let σ be a pseudoorder on A, $f : A/\rho \to B$ is a strong homomorphism and $\varphi = f \circ \rho^\sharp$. Then $\sigma \subseteq \overrightarrow{ker}\varphi$. Indeed, by hypothesis, we have

$$\begin{aligned}(a, b) \in \sigma \Leftrightarrow (a)_\rho \preceq_\sigma (b)_\rho &\Rightarrow f((a)_\rho) \leq_B f((b)_\rho)\\ &\Rightarrow (f \circ \rho^\sharp)(a) \leq_B (f \circ \rho^\sharp)(b)\\ &\Rightarrow \varphi(a) \leq_B \varphi(b) \Rightarrow (a, b) \in \overrightarrow{ker}\varphi,\end{aligned}$$

where the order $\preceq_\sigma$ on A/ρ is defined as in the proof of Lemma 4.4, that is

$$\preceq_\sigma := \{((x)_\rho, (y)_\rho) \in A/\rho \times A/\rho \mid (x, y) \in \sigma\}. \qquad \square$$

Corollary 5.2. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$ and $\varphi : A \to B$ a strong homomorphism. Then $A/ker\varphi \cong Im(\varphi)$, where $ker\varphi$ is the kernel of φ.*

Proof. Let $\sigma = \overrightarrow{ker}\varphi$ and $\rho = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1}$. Then, by Theorems 4.5 and 5.1, ρ is a strong order-congruence on A and $f : A/\rho \to B \mid (a)_\rho \mapsto \varphi(a)$ is a strong homomorphism. Moreover, f is inverse isotone. In fact, let $(a)_\rho, (b)_\rho$ be two elements of A/ρ such that $f((a)_\rho) \leq_B f((b)_\rho)$. Then $\varphi(a) \leq_B \varphi(b)$, and we have $(a, b) \in \overrightarrow{ker}\varphi$. Thus, by Lemma 4.4, $((a)_\rho, (b)_\rho) \in \preceq_\sigma$, i.e., $(a)_\rho \preceq_\sigma (b)_\rho$. Clearly, $\rho = ker\varphi$. By Remark 3.4(1), $A/ker\varphi \cong Im(f)$. Also, by Theorem 5.1, $Im(f) = Im(\varphi)$. Therefore, $A/ker\varphi \cong Im(\varphi)$. $\square$

Remark 5.3. *Note that if $(A, \leq_A)$ and $(B, \leq_B)$ are both S-posets, then Corollary 5.2 coincides with Corollary 13 in [32].*

Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ pseudoorders on A and $\rho \subseteq \theta$. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by θ/ρ as follows:

$$\theta/\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (a, b) \in \theta\},$$

where $\preceq_\rho := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid (a, b) \in \rho\}$, $\underline{\rho} = \rho \cap \rho^{-1}$. By Corollary 3.10, θ/ρ is a pseudoorder on $(A/\underline{\rho}, \preceq_\rho)$.

Theorem 5.4. *Let $(A, \leq_A)$ be a hyper S-poset over an ordered semihypergroup $(S, \circ, \leq)$, ρ and θ pseudoorders on A and $\rho \subseteq \theta$. Then $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$.*

Proof. Since θ/ρ is a pseudoorder on $A/\underline{\rho}$, we have the mapping $\varphi : A/\underline{\rho} \to A/\underline{\theta} \mid (a)_{\underline{\rho}} \mapsto (a)_{\underline{\theta}}$ is a strong homomorphism. In fact:

(1) φ is well-defined. Indeed, let $(a)_{\underline{\rho}} = (b)_{\underline{\rho}}$. Then $(a, b) \in \underline{\rho}$. Thus, by the definition of $\underline{\rho}$, $(a, b) \in \rho \subseteq \theta$ and $(b, a) \in \rho \subseteq \theta$. This implies that $(a, b) \in \underline{\theta}$, and thus $(a)_{\underline{\theta}} = (b)_{\underline{\theta}}$.

(2) φ is a strong homomorphism. In fact, let $(a)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. Then, since $\underline{\rho}, \underline{\theta} \in SC(A)$, for any $x \in a * s$, we have

$$(a)_{\underline{\rho}} \otimes_\rho s = (x)_{\underline{\rho}}, (a)_{\underline{\theta}} \otimes_\theta s = (x)_{\underline{\theta}},$$

where " $\otimes_\rho$ " and " $\otimes_\theta$ " are the S-hyperaction on $A/\underline{\rho}$ and $A/\underline{\theta}$, respectively. Thus

$$\varphi((a)_{\underline{\rho}}) \otimes_\theta s = (a)_{\underline{\theta}} \otimes_\theta s = (x)_{\underline{\theta}} = \varphi((x)_{\underline{\rho}}).$$

Also, if $(a)_{\underline{\rho}} \preceq_\rho (b)_{\underline{\rho}}$, then $(a, b) \in \rho \subseteq \theta$. It implies that $(a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}}$, and thus φ is isotone.

On the other hand, it is easy to see that φ is onto, since

$$Im(\varphi) = \{\varphi((a)_{\underline{\rho}}) \mid a \in A\} = \{(a)_{\underline{\theta}} \mid a \in A\} = A/\underline{\theta}.$$

It thus follows from Corollary 5.2 that $A/\underline{\rho}/Ker\varphi \cong Im(\varphi) = A/\underline{\theta}$.

Furthermore, let $\overrightarrow{ker}\varphi := \{((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \mid \varphi((a)_{\underline{\rho}}) \preceq_\theta \varphi((b)_{\underline{\rho}})\}$. Then

$$\begin{aligned}((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \overrightarrow{ker}\varphi &\Longleftrightarrow (a)_{\underline{\theta}} \preceq_\theta (b)_{\underline{\theta}} \Longleftrightarrow (a, b) \in \theta \\ &\Longleftrightarrow ((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \theta/\rho.\end{aligned}$$

Therefore, $Ker\varphi = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1} = (\theta/\rho) \cap (\theta/\rho)^{-1} = \underline{\theta/\rho}$. We have thus shown that $(A/\underline{\rho})/\underline{\theta/\rho} \cong A/\underline{\theta}$. □

Definition 5.5. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a homomorphism. Then, f is called a (ρ, θ)-homomorphism if $(x, y) \in \rho$ implies $(f(x), f(y)) \in \theta$, for all $x, y \in A$.*

Lemma 5.6. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the mapping $\overline{f} : (A/\underline{\rho}, \preceq_\rho) \to (B/\underline{\theta}, \preceq_\theta)$ defined by*

$$(\forall x \in A)\ \overline{f}((x)_{\underline{\rho}}) := (f(x))_{\underline{\theta}}$$

is a strong homomorphism of hyper S-posets, where the orders $\preceq_\rho$, $\preceq_\theta$ on $A/\underline{\rho}$ and $B/\underline{\theta}$, respectively, are both defined as in the proof of Lemma 4.4.

Proof. Let $f : A \to B$ be a (ρ, θ)-homomorphism and $\overline{f} : A/\underline{\rho} \to B/\underline{\theta} \mid (x)_{\underline{\rho}} \mapsto (f(x))_{\underline{\theta}}$. Then

(1) $\overline{f}$ is well defined. In fact, let $(x)_{\underline{\rho}}, (y)_{\underline{\rho}} \in A/\underline{\rho}$ be such that $(x)_{\underline{\rho}} = (y)_{\underline{\rho}}$. Then $(x, y) \in \underline{\rho} \subseteq \rho$. Since f is a (ρ, θ)-homomorphism, we have $(f(x), f(y)) \in \theta$. It implies that $((f(x))_{\underline{\theta}}, (f(y))_{\underline{\theta}}) \in \preceq_\theta$. Similarly, since $(y, x) \in \rho$, we have $((f(y))_{\underline{\theta}}, (f(x))_{\underline{\theta}}) \in \preceq_\theta$. Therefore, $((f(x))_{\underline{\theta}} = (f(y))_{\underline{\theta}}$, i.e., $\overline{f}((x)_{\underline{\rho}}) = \overline{f}((y)_{\underline{\rho}})$.

(2) $\overline{f}$ is a strong homomorphism. Indeed, let $(x)_{\underline{\rho}} \in A/\underline{\rho}$ and $s \in S$. Since f is a homomorphism, for any $a \in x * s$, we have $f(a) \in f(x) \diamond s$, where "$\diamond$" is the S-hyperaction on B. By Theorem 3.7, $\underline{\rho} \in SC(A)$, $\underline{\theta} \in SC(B)$. Thus, by Theorem 2.5 we have

$$\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s = (f(x))_{\underline{\theta}} \otimes_\theta s = (f(a))_{\underline{\theta}} = \overline{f}((a)_{\underline{\rho}}).$$

Also, since f is a (ρ, θ)-homomorphism, we have

$$(x)_{\underline{\rho}} \preceq_\rho (y)_{\underline{\rho}} \Rightarrow (x, y) \in \rho \Rightarrow (f(x), f(y)) \in \theta$$

$$\Rightarrow (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}} \Rightarrow \overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}}).$$

Hence $\overline{f}$ is isotone. Therefore, $\overline{f}$ is a strong homomorphism. □

Lemma 5.7. *Let $(A, \leq_A)$ and $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. We define a relation on the hyper S-poset $(A/\underline{\rho}, \preceq_\rho)$ denoted by ρ_f as follows:*

$$\rho_f := \{((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in A/\underline{\rho} \times A/\underline{\rho} \mid (f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}\}.$$

Then ρ_f is a pseudoorder on $A/\underline{\rho}$.

Proof. Assume that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \preceq_\rho$. By Lemma 5.6, $\overline{f}$ is a strong homomorphism. Then $\overline{f}((x)_{\underline{\rho}}) \preceq_\theta \overline{f}((y)_{\underline{\rho}})$, i.e., $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. It implies that $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$, and thus $\preceq_\rho \subseteq \rho_f$. Now, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $((y)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$ and $(f(y))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. Thus, by the transitivity of $\preceq_\theta$, $(f(x))_{\underline{\theta}} \preceq_\theta (f(z))_{\underline{\theta}}$. This implies that $((x)_{\underline{\rho}}, (z)_{\underline{\rho}}) \in \rho_f$. Moreover, let $((x)_{\underline{\rho}}, (y)_{\underline{\rho}}) \in \rho_f$ and $s \in S$. Then $(f(x))_{\underline{\theta}} \preceq_\theta (f(y))_{\underline{\theta}}$. Since $(B/\underline{\theta}, \preceq_\theta)$ is a hyper S-poset over S, it can be obtained that $(f(x))_{\underline{\theta}} \otimes_\theta s \preceq_\theta (f(y))_{\underline{\theta}} \otimes_\theta s$, that is, $\overline{f}((x)_{\underline{\rho}}) \otimes_\theta s \preceq_\theta \overline{f}((y)_{\underline{\rho}}) \otimes_\theta s$. Then, since $\overline{f}$ is a strong homomorphism, for every $a \in x * s$ and for every $b \in y * s$, we have $\overline{f}((a)_{\underline{\rho}}) \preceq_\theta \overline{f}((b)_{\underline{\rho}})$, which means that $(f(a))_{\underline{\theta}} \preceq_\theta (f(b))_{\underline{\theta}}$. Hence $((a)_{\underline{\rho}}, (b)_{\underline{\rho}}) \in \rho_f$. It thus implies that $(x)_{\underline{\rho}} \otimes_\rho s\, \overline{\overline{\rho}}_f\, (y)_{\underline{\rho}} \otimes_\rho s$. Therefore, ρ_f is a pseudoorder on $A/\underline{\rho}$. □

By Lemmas 5.6 and 5.7, we immediately obtain the following two corollaries.

Corollary 5.8. *$Ker\,\overline{f} = \underline{\rho_f}$.*

Corollary 5.9. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. Then, the following diagram*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\underline{\rho}^\sharp"'] & B \arrow[d, "\underline{\theta}^\sharp"] \\
A/\underline{\rho} \arrow[r, "\overline{f}"'] & B/\underline{\theta}
\end{tikzcd}$$

commutates.

Theorem 5.10. *Let $(A, \leq_A)$, $(B, \leq_B)$ be two hyper S-posets over an ordered semihypergroup $(S, \circ, \leq)$, ρ, θ be two pseudoorders on A and B, respectively, and the mapping $f : A \to B$ a (ρ, θ)-homomorphism. If σ is a pseudoorder on $A/\underline{\rho}$ such that $\sigma \subseteq \rho_f$, then there exists the unique strong homomorphism $\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta} \mid ((a)_{\underline{\rho}})_{\underline{\sigma}} \mapsto \underline{f}((a)_{\underline{\rho}})$ such that the diagram*

$$\begin{tikzcd}
A/\underline{\rho} \arrow[r, "\overline{f}"] \arrow[d, "\underline{\sigma}^\sharp"'] & B/\underline{\theta} \\
(A/\underline{\rho})/\underline{\sigma} \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

commutes.

Conversely, if σ is a pseudoorder on $A/\underline{\rho}$ for which there exists a strong homomorphism $\varphi : (A/\underline{\rho})/\underline{\sigma} \to B/\underline{\theta}$ such that the above diagram commutes, then $\sigma \subseteq \rho_f$.

Proof. The proof is straightforward by Lemmas 5.6, 5.7 and Theorem 5.1, and we omit the details. □

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