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Research Article

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Rough semigroups and rough fuzzy semigroups based on fuzzy ideals

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Abstract: In this paper, we firstly introduce a special congruence relation $U(\mu, t)$ induced by a fuzzy ideal μ in a semigroup S. Then we define the lower and upper approximations based on a fuzzy ideal in semigroups. We can establish rough semigroups, rough ideals, rough prime ideals, rough fuzzy semigroups, rough fuzzy ideals and rough fuzzy prime ideals according to the definitions of rough sets and rough fuzzy sets. Furthermore, we shall consider the relationships among semigroups and rough semigroups, fuzzy semigroups and rough fuzzy semigroups, and some relative properties are also discussed.

Keywords: Fuzzy ideal, t-level set, Rough semigroup, Rough fuzzy semigroup, Rough (prime) ideal, Rough fuzzy (prime) ideal

MSC: 20N20, 16Y99

1 Introduction

In 1965, the notion of fuzzy sets was introduced by Zadeh [1] and he gave the way to study uncertainty problems. Thereafter, the theory has developed quickly and has been applied in many areas of our real world. Up to now, fuzzy mathematics has remained an important branch of mathematics. Meanwhile, in the early 20th century, some researchers began to research semigroups formally. Since 1950, the investigations of finite semigroups have become rather important in theoretical computer science because there are natural relations between finite semigroups and finite automatons. With the fuzzy mathematics being deeply studied, fuzzy algebras were also investigated. The study of fuzzy algebras is also one of the most important directions in fuzzy mathematics. In particular, a great number of studies on fuzzy groups and fuzzy semigroups were performed. It should be pointed out that the studies on fuzzy semigroups are deeply influencing the development of fuzzy theory. In recent years, a large number of fuzzy algebras have been developed by some scholars, for instances see [2–4]. Especially, Davvaz [5], Dudek [6] and Zhan [7] applied fuzzy theory to n-ary semigroups. Recently, Zhan [8, 9] studied some types of fuzzy k-ideals and fuzzy h-ideals of hemirings, respectively.

In 1982, Pawlak [10] firstly developed a rough set theory. It is rather important to deal with inexact, uncertain, or vague situations. More and more researchers have been attracted to study rough sets and their applications. While mentioning rough sets, we can not neglect the lower and upper approximations. In Pawlak rough sets, the equivalent relations are essential to construct the lower and upper approximations. To deal with imperfect data, rough set theory is a powerful theory [11]. In recent years, Bonikowaski [12], Iwinski [13], and Pomykala [14] have studied algebraic properties of rough sets. The notion of rough subgroups was introduced by Biswas and Nanda [15]. Then, Kuroki [16] put up the notion of rough ideals in semigroups. Also, in [17], Jun investigated the roughness of Γ-subsemigroups

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and ideals in Γ-semigroups. The notions of rough prime ideals and rough fuzzy prime ideals in semigroups were introduced in [18].

It is obvious that congruence relations in rough sets are rather important. In Pawlak rough sets, congruence relations are necessary in the constructions of the lower and upper approximations. Therefore, if we try to investigate the rough sets, we should define a congruence relation. In recent years, some researchers try to find something to serve as a congruence relation in some algebraic structures. For examples, Kuroki and Wang [19] made use of a normal subgroup of groups as a congruence relation to discuss the properties of the lower and upper approximations. In [20], Davvaz proved that an ideal of rings could be regarded as a congruence relation. Further, Davvaz studied rough subrings and rough ideals in rings. Especially, in [21] Davvaz constructed the t-level relation based on a fuzzy ideal μ and he proved that $U(\mu;t)$ is a congruence relation in rings.

Based on the above idea, in this paper we will study the properties of rough subsemigroups and rough fuzzy subsemigroups of semigroups. We construct the t-level relation $U(\mu;t)$ of a fuzzy ideal μ of semigroups and prove that $U(\mu;t)$ is a congruence relation. We divide this paper into four parts. In section 2, we recall some basic results. In section 3, based on this congruence relation, we construct rough subsemigroup and rough (prime) ideals of semigroups. Some examples are also presented. Finally, we investigate rough fuzzy semigroups of semigroups based on this congruence relation in section 4.

2 Preliminaries

In this section we will recall some basic definitions that can be used in this paper.

Definition 2.1. *A semigroup S is a nonempty set with a binary operation "$*$" such that*
(i) $a * b \in S$, *for all* $a, b \in S$.
(ii) $(a * b) * c = a * (b * c)$, *for all* $a, b, c \in S$.

Definition 2.2 ([18]). *Let S be a semigroup, a subset A of S is called a subsemigroup of S if A is closed under "$*$"; A is called a left (right) ideal if A is closed under "$*$" and $SA \subseteq A(AS \subseteq A)$; If A is not only a left ideal but also is a right ideal, it is a bi-ideal; A is called a prime ideal if A is ideal, and if $a * b \in A$, we can get $a \in A$ or $b \in A$, for all $a, b \in A$.*

Definition 2.3 ([18]). *A fuzzy set μ of S is called a fuzzy subsemigroup of S if it satisfies $\mu(x * y) \geq \mu(x) \wedge \mu(y)$; μ is called a fuzzy left (right) ideal if it satisfies (1) $\mu(x * y) \geq \mu(x) \wedge \mu(y)$ (2) $\mu(x * y) \geq \mu(y)$ $(\mu(x * y) \geq \mu(x))$, for all $x, y \in S$; μ is called a fuzzy ideal if it is not only a fuzzy left ideal but also is a fuzzy right ideal; μ is called a fuzzy prime ideal if $\mu(x * y) = \mu(x) or \mu(x * y) = \mu(y)$, for all $x, y \in S$.*

Definition 2.4 ([3]). *Let R be an equivalence relation in a set A, if R is a binary relation and it satisfies*
(i) *Reflexive: for any* $a \in A, (a, a) \in R$.
(ii) *Symmetry: for all* $a, b \in A, (a, b) \in R, then (b, a) \in R$.
(iii) *Transitivity: for all* $a, b, c \in A, (a, b) \in R, (b, c) \in R, then (a, c) \in R$.
An equivalence relation R is called a congruence relation if $(a, b) \in R$, then $(ax, bx) \in R$ and $(xa, xb) \in R$, for all $x \in A$.

3 Rough semigroups based on fuzzy ideals

In the section, we introduce a new congruence relation of semigroups. Referring to [18], we study rough semigroups and, at the same time, we provide some examples. Throughout this paper, S is a semigroup.

Definition 3.1. *Let μ be a fuzzy ideal of S. For each $t \in [0, \mu(0)]$, the set $U(\mu, t) = \{(x, y) \in S \times S | (\mu(x) \wedge \mu(y)) \vee Id_S(x, y) \geq t\}$ is called a t-level relation of μ.*

For $Ids(x, y)$, we know $x = y$, then $Ids(x, y) = 1$; $x \neq y$, then $Ids(x, y) = 0$. Next, we prove $U(\mu, t)$ is a congruence relation.

Lemma 3.2. *Let μ be a fuzzy ideal of S and $t \in [0, \mu(0)]$. Then $U(\mu, t)$ is a congruence relation on S.*

Proof. First of all, we show $U(\mu, t)$ is an equivalence relation.

(i) Reflexive: For any element $x \in S$. $(\mu(x) \wedge \mu(x)) \vee Ids(x, x) = Ids(x, x) = 1 \geq t$.

(ii) Symmetry: Obviously, $U(\mu, t)$ is symmetric.

(iii) Transitivity: Let $(x, y) \in U(\mu, t)$ and $(y, z) \in U(\mu, t)$, and then we have $(\mu(x) \wedge \mu(y)) \vee Ids(x, y) \geq t, (\mu(y) \wedge \mu(z)) \vee Ids(y, z) \geq t$. If $x = y = z$, it is clear that $(x, z) \in U(\mu, t)$; If $x = y \neq z$, then $\mu(y, z) \geq t$ and $((\mu(x) \wedge \mu(z)) \vee Ids(x, z) = \mu(x) \wedge \mu(z) = \mu(y) \wedge \mu(z) \geq t$, therefore, $(x, z) \in U(\mu, t)$; if $x \neq y = z$, we have $\mu(x) \wedge \mu(y) \geq t$ and $(\mu(x) \wedge \mu(z)) \vee Ids(x, z) = \mu(x) \wedge \mu(z) = \mu(x) \wedge \mu(y) \geq t$, so $(x, z) \in U(\mu, t)$; if $x \neq y \neq z$, then we have $\mu(x) \wedge \mu(y) \geq t$, $\mu(y) \wedge \mu(z) \geq t$ and $(\mu(x) \wedge \mu(z)) \vee Ids(x, z) = \mu(x) \wedge \mu(z) \geq \mu(x) \wedge \mu(y) \wedge \mu(z) \geq t$, so $(x, z) \in U(\mu, t)$. Conclusion, $U(\mu, t)$ is an equivalence relation.

Next, we prove that $U(\mu, t)$ is a congruence relation. For $(x, y) \in U(\mu, t)$, we prove $(ax, ay) \in U(\mu, t)$ and $(xa, ya) \in U(\mu, t)$ is also right. In the following, we only prove the former, the latter is the same. In other words, we only prove $(\mu(ax) \wedge \mu(ay)) \vee Ids(ax, ay) \geq t$. If $ax = ay$, clearly, $1 \geq t$. If $ax \neq ay$, then $x \neq y$, then

$$\begin{aligned}(\mu(ax) \wedge \mu(ay)) \vee Ids(ax, ay) &= \mu(ax) \wedge \mu(ay) \\ &\geq [\mu(a) \vee \mu(x)] \wedge [\mu(a) \vee \mu(y)] \\ &= [\mu(a) \wedge \mu(a)] \vee [\mu(x) \wedge \mu(a)] \vee [\mu(a) \wedge \mu(y)] \vee [\mu(x) \wedge \mu(y)] \\ &\geq \mu(x) \wedge \mu(y) \\ &\geq t\end{aligned}$$

Thus, $U(\mu, t)$ is a congruence relation. □

For any fuzzy ideal μ of S, we know that $\mu(0) \geq \mu(x)$ and $\mu(0) \leq 1$, so when $t \in [0, \mu(0)]$, the above lemma is proper. We say x is congruent to y model μ, written $x \equiv_t y(mod \mu)$. If for elements $x, y \in S$, $t \in [0, 1]$, $(\mu(x) \wedge \mu(y)) \vee Ids(x, y) \geq t$, we use $[x]_{(\mu,t)}$ as the equivalence class of x. However, $U(\mu, t)$ is not a complete congruence relation. Through our research, we can obtain Lemma 3.3 as follows.

Lemma 3.3. *Let μ be a fuzzy ideal of S and $t \in [0, 1]$, then $[x]_{(\mu,t)}[y]_{(\mu,t)} \subseteq [xy]_{(\mu,t)}$.*

Proof. Let $m \in [x]_{(\mu,t)}[y]_{(\mu,t)}$, then $m = x'y'$, where $x' \in [x]_{(\mu,t)}$ and $y' \in [y]_{(\mu,t)}$. Hence $\mu(x, x') \geq t$ and $\mu(y, y') \geq t$. Since $U(\mu, t)$ is a congruence relation in S, we have $\mu(xy, x'y') \geq t$, and so $x'y' \in [xy]_{(\mu,t)}$. This means that $m \in [xy]_{(\mu,t)}$. Thus $[x]_{(\mu,t)}[y]_{(\mu,t)} \subseteq [xy]_{(\mu,t)}$. □

Now, we give an example to prove that we can not use "=" to replace "⊆" in Lemma 3.3.

Example 3.4. *Let $S = \{a, b, c, d\}$ be a semigroup with the following "$*$" table.*

*	a	b	c	d
a	a	b	b	d
b	b	b	b	d
c	b	b	b	d
d	d	d	d	d

Assume that $\mu = \frac{0.3}{a} + \frac{0.5}{b} + \frac{0.1}{c} + \frac{0.8}{d}$ is a fuzzy ideal of S. Here $t = 0.4$, then we have $U(\mu; 0.4) = \{(a, a), (b, b), (c, c), (d, d), (b, d)\}$, so we have $[a]_{(\mu,0.4)} = \{a\}$, $[b]_{(\mu,0.4)} = \{b, d\}$, $[c]_{(\mu,0.4)} = \{c\}$, $[d]_{(\mu,0.4)} = \{b, d\}$. $[a]_{(\mu,0.4)}[c]_{(\mu,0.4)} = \{b\}$. Here, $a * c = b$, so $[a * c]_{(\mu,0.4)} = \{(b, d)\}$. Obviously, $[a]_{(\mu,0.4)}[c]_{(\mu,0.4)} \subseteq [a * c]_{(\mu,0.4)}$.

From the above example, we see we can not write $[x]_{(\mu,t)}[y]_{(\mu,t)} = [xy]_{(\mu,t)}$, it is only a containment relation. However there exists special $U(\mu,t)$ such that $[x]_{(\mu,t)}[y]_{(\mu,t)} = [xy]_{(\mu,t)}$. So we give the definition to make $U(\mu,t)$ a complete congruence relation as follows.

Definition 3.5. *$U(\mu,t)$ is called a complete congruence relation if it satisfies: for any $x, y \in S$, $[x]_{(\mu,t)}[y]_{(\mu,t)} = [xy]_{(\mu,t)}$.*

Example 3.6. *Let $S = \{0, a, b, c\}$ be a semigroup with the following "$*$" table.*

*	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

Assume that $\mu = \frac{0.1}{0} + \frac{0.4}{a} + \frac{0.7}{b} + \frac{0.7}{c}$ is a fuzzy ideal in S. Here $t = 0.7$, then we have $U(\mu; 0.7) = \{(0,0), (a,a), (b,b), (c,c), (b,c)\}$, so we have $[0]_{(\mu,0.7)} = \{0\}$, $[a]_{(\mu,0.7)} = \{a\}$, $[b]_{(\mu,0.7)} = \{b,c\}$, $[c]_{(\mu,0.7)} = \{b,c\}$. Obviously, we can easily check $U(\mu,t)$ is a complete congruence relation.

Let μ be a fuzzy ideal of S and $t \in [0,1]$. We see that $U(\mu,t)$ is a congruence relation. According to Pawlak rough sets, we can obtain the approximation space (S,μ,t), just like (U,θ) in Pawlak rough sets.

Definition 3.7. *Let μ be a fuzzy ideal of S and $U(\mu,t)$ be a t-level set. For $X \subseteq S$ and $X \neq \emptyset$, we define the upper and lower approximations over (S,μ,t) as follows:*

$$\underline{U}(\mu,t,X) = \{x \in S \mid [x]_{(\mu,t)} \subseteq X\} \text{ and } \overline{U}(\mu,t,X) = \{x \in S \mid [x]_{(\mu,t)} \cap X \neq \emptyset\}.$$

(1) *If $\overline{U}(\mu,t,X)(\underline{U}(\mu,t,X))$ is a subsemigroup of S, then we call X an upper (lower) rough subsemigroup of S;*
(2) *If $\overline{U}(\mu,t,X)(\underline{U}(\mu,t,X))$ is a (prime) ideal of S, then we call X an upper (lower) rough (prime) ideal of S.*

Proposition 3.8. *Let μ be any fuzzy ideal of S, A and B be any nonempty subsets of S, then the following hold:*
(1) $\underline{U}(\mu,t,A) \subseteq A \subseteq \overline{U}(\mu,t,A)$,
(2) $\overline{U}(\mu,t,A \cap B) \subseteq \overline{U}(\mu,t,A) \cap \overline{U}(\mu,t,B)$,
(3) $\underline{U}(\mu,t,A \cap B) = \underline{U}(\mu,t,A) \cap \underline{U}(\mu,t,B)$,
(4) $\overline{U}(\mu,t,A \cup B) = \overline{U}(\mu,t,A) \cup \overline{U}(\mu,t,B)$,
(5) $\underline{U}(\mu,t,A \cup B) \supseteq \underline{U}(\mu,t,A) \cup \underline{U}(\mu,t,B)$,
(6) $A \subseteq B$, then $\overline{U}(\mu,t,A) \subseteq \overline{U}(\mu,t,B)$ and $\underline{U}(\mu,t,A) \subseteq \underline{U}(\mu,t,B)$,
(7) $\mu \subseteq \nu$, then $\overline{U}(\mu,t,A) \subseteq \overline{U}(\nu,t,A)$ and $\underline{U}(\mu,t,A) \supseteq \underline{U}(\nu,t,A)$,
(8) $\overline{U}(\mu,t,\overline{U}(\mu,t,A)) = \overline{U}(\mu,t,A)$,
(9) $\underline{U}(\mu,t,\underline{U}(\mu,t,A)) = \underline{U}(\mu,t,A)$,
(10) $\overline{U}(\mu,t,\underline{U}(\mu,t,A)) = \overline{U}(\mu,t,A)$,
(11) $\underline{U}(\mu,t,\overline{U}(\mu,t,A)) = \underline{U}(\mu,t,A)$.

Proof. The proofs are trivial. □

Proposition 3.9. *Let μ be a fuzzy ideal of S and $t \in [0,1]$. If A_1 and A_2 are nonempty subsets of S, then*

$$\overline{U}(\mu,t,A_1) \cdot \overline{U}(\mu,t,A_2) \subseteq \overline{U}(\mu,t,A_1A_2).$$

Proof. Suppose that $x \in \overline{U}(\mu,t,A_1) \cdot \overline{U}(\mu,t,A_2)$, then there exists $a_i \in \overline{U}(\mu,t,A_i)(i = 1,2)$, and $x = a_1a_2$, so there exists $x_i \in A_i(i = 1,2)$ such that $x_i \in [a_i]_{(\mu,t)} \cap A_i(i = 1,2)$. Since $U(\mu,t)$ is a congruence relation, we have $x_1x_2 \in [a_1a_2]_{(\mu,t)}$, $x_1x_2 \in A_1A_2$. So we have $x_1x_2 \in [a_1a_2]_{(\mu,t)} \cap A_1A_2$, so $x = x_1x_2 \in \overline{U}(\mu,t,A_1A_2)$.
Hence, $\overline{U}(\mu,t,A_1) \cdot \overline{U}(\mu,t,A_2) \subseteq \overline{U}(\mu,t,A_1A_2)$. □

Next, we give an example to prove that the converse of containment in Proposition 3.9 does not hold.

Example 3.10. *Consider the subsets* $\{a\}$ *and* $\{c\}$ *of* S *as in Example 3.4. We have* $\overline{U}(\mu, 0.4, a)\overline{U}(\mu, 0.4, c) = ac = b$. *However,* $\overline{U}(\mu, 0.4, ac) = \{b, d\}$. *This shows that the converse of containment in Proposition 3.9 does not hold.*

Proposition 3.11. *Let* $U(\mu, t)$ *be a complete congruence relation on* S. *If* A_1, A_2 *are non-empty subsets of* S, *then we have* $\underline{U}(\mu, t, A_1) \cdot \underline{U}(\mu, t, A_2) \subseteq \underline{U}(\mu, t, A_1A_2)$.

Proof. Suppose that $x \in \underline{U}(\mu, t, A_1) \cdot \underline{U}(\mu, t, A_2)$, then there exists $a_i \in \underline{U}(\mu, t, A_i)(i = 1, 2)$ such that $x = a_1a_2$, and because $[a_i]_{(\mu,t)} \subseteq A_i(i = 1, 2)$. Since $U(\mu, t)$ is a complete congruence relation, we have $[a_1]_{(\mu,t)}[a_2]_{(\mu,t)} = [a_1a_2]_{(\mu,t)} \subseteq A_1A_2$, and so $x = a_1a_2 \in \underline{U}(\mu, t, A_1A_2)$.

Hence, $\underline{U}(\mu, t, A_1) \cdot \underline{U}(\mu, t, A_2) \subseteq \underline{U}(\mu, t, A_1A_2)$. □

Here, we prove that if $U(\mu, t)$ is not a complete congruence relation on S, the containment in Proposition 3.11 may not be true as in the following example.

Example 3.12. *Consider the subsets* $\{b, c\}$ *and* $\{b, c\}$ *of* S *in Example 3.4. We have* $\underline{U}(\mu, 0.4, \{a, c\}) \cdot \underline{U}(\mu, 0.4, \{a, c\}) = c \cdot c = b$. *However,* $\underline{U}(\mu, 0.4, \{b, c\}\{b, c\}) = \underline{U}(\mu, 0.4, b) = \emptyset$. *This shows that the containment in Proposition 3.11 does not hold.*

In the following, we use the above conclusions to study the properties of rough subsemigroups and rough (prime) ideals.

Theorem 3.13. *Let* μ *be a fuzzy ideal of* S *and* $t \in [0, 1]$, *then*

(1) *If* A *is a subsemigroup of* S, A *is an upper rough subsemigroup of* S;

(2) *If* A *is an ideal of* S, A *is an upper rough ideal of* S.

Proof. (1) Let A be a subsemigroup of S, we have $AA \subseteq A$, according to Propositions 3.8 and 3.9, we have $\overline{U}(\mu, t, A) \cdot \overline{U}(\mu, t, A) \subseteq \overline{U}(\mu, t, AA) \subseteq \overline{U}(\mu, t, A)$, so we know that $\overline{U}(\mu, t, A)$ is a subsemigroup. By Definition 3.7, we have that A is an upper rough subsemigroup of S.

(2) Let A be an ideal of S, then $SAS \subseteq A$. By Proposition 3.9, we have $\overline{U}(\mu, t, S) \cdot \overline{U}(\mu, t, A) \cdot \overline{U}(\mu, t, S) \subseteq \overline{U}(\mu, t, SAS) \subseteq \overline{U}(\mu, t, A)$, so $\overline{U}(\mu, t, A)$ is an ideal of S. Therefore, according to Definition 3.7, we have that A is an upper rough ideal of S. □

Remark 3.14. *The above theorem shows that the notion of an upper rough subsemigroup (ideal) is an extended notion of a usual subsemigroup (ideal) of a semigroup. The following example shows that the converse of Theorem 3.13 does not hold in general.*

Example 3.15. *Consider the subset* $\{c, d\}$ *as in Example 3.4. We have* $\overline{U}(\mu, 0.4, \{c, d\}) = \{b, c, d\}$, *here* $\{b, c, d\}$ *is a subsemigroup, so* $\overline{U}(\mu, 0.4, \{c, d\})$ *is an upper rough subsemigroup, but* $\{c, d\}$ *is not a subsemigroup.*

Proposition 3.16. *Let* $U(\mu, t)$ *be a complete congruence relation on* S, *then*

(1) *If* A *is a subsemigroup of* S, A *is a lower rough subsemigroup of* S;

(2) *If* A *is an ideal of* S, A *is a lower rough ideal of* S.

Proof. It is similar to the proof of Theorem 3.13. □

The following example shows that even if A is not a subsemigroup of S, then $\underline{U}(\mu, t, A)$ may be a subsemigroup of S when $U(\mu, t)$ is complete congruence relation on S.

Example 3.17. *Consider the semigroup* S *and the congruence class as in Example 3.6. Then* $A = \{0, a, c\}$ *is not a subsemigroup of* S *but* $\underline{U}(\mu, t, A) = \{0, a\}$ *is a subsemigroup of* S.

Lemma 3.18. *Let* μ *be a fuzzy ideal of* S *and* $t \in [0, 1]$. *Then* $[0]_{(\mu,t)}$ *is an ideal of* S.

Proof. For all $x_1, x_2 \in S$ and $a \in [0]_{(\mu,t)}$, then we have $x_1 a x_2 \in [x_1]_{(\mu,t)}[0]_{(\mu,t)}[x_2]_{(\mu,t)} \subseteq [x_1 0 x_2]_{(\mu,t)} = [0]_{(\mu,t)}$. This means $x_1 a x_2 \in [0]_{(\mu,t)}$. Hence $[0]_{(\mu,t)}$ is an ideal of S. □

Proposition 3.19. *Let μ be a fuzzy ideal of S and $t \in [0, \mu(0)]$. Then $\underline{U}(\mu, t, [0]_{(\mu,t)}) = [0]_{(\mu,t)}$.*

Proof. By Proposition 3.8, we have $\underline{U}(\mu, t, [0]_{(\mu,t)}) \subseteq [0]_{(\mu,t)}$. Now, we show that $[0]_{(\mu,t)} \subseteq \underline{U}(\mu, t, [0]_{(\mu,t)})$. For every $x \in [0]_{(\mu,t)}$, then $(0, x) \in U(\mu, t)$. Let $y \in [x]_{(\mu,t)}$, then $(x, y) \in U(\mu, t)$. Since $U(\mu, t)$ is a congruence relation, we have $(0, y) \in U(\mu, t)$, this implies $y \in [0]_{(\mu,t)}$. Hence $[x]_{(\mu,t)} \subseteq [0]_{(\mu,t)}$. This means $x \in \underline{U}(\mu, t, [0]_{(\mu,t)})$. Therefore $\underline{U}(\mu, t, [0]_{(\mu,t)}) = [0]_{(\mu,t)}$. □

Corollary 3.20. *Let μ be a fuzzy ideal of S and $t \in [0, \mu(0)]$. Then $[0]_{(\mu,t)}$ is a lower rough ideal of S.*

Proposition 3.21. *Let μ be a fuzzy ideal of S and $t \in [0, \mu(0)]$. Then $\mu_t = [0]_{(\mu,t)}$.*

Proof. For all $x \in \mu_t$, then $\mu(x) \geq t$ and $\mu(0) \geq \mu(x) \geq t$. So $(\mu(x) \wedge \mu(0)) \vee Id_S(x, 0) \geq \mu(x) \wedge \mu(0) \geq t$, by Definition 3.1, we have $(x, 0) \in U(\mu, t)$. It implies $x \in [0]_{(\mu,t)}$. This means $\mu_t \subseteq [0]_{(\mu,t)}$.

On the other hand, for all $x \in [0]_{(\mu,t)}$, then $(x, 0) \in U(\mu, t)$ and $(\mu(x) \wedge \mu(0)) \vee Id_S(x, 0) \geq t$. If $x = 0$, $\mu(x) = \mu(0) \geq t$ is obvious, so $x \in \mu_t$, it implies $[0]_{(\mu,t)} \subseteq \mu_t$. If $x \neq 0$, then $\mu(x) \wedge \mu(0) \geq t$, $\mu(x) \geq t$, this means $x \in \mu_t$. Hence $[0]_{(\mu,t)} \subseteq \mu_t$. Therefore $\mu_t = [0]_{(\mu,t)}$. □

Theorem 3.22. *Let $U(\mu, t)$ be a complete congruence relation on S. If A is a prime ideal of S, then $\underline{U}(\mu, t, A)$ is a prime ideal of S if $\underline{U}(\mu, t, A) \neq \emptyset$.*

Proof. Since A is an ideal of S, by Proposition 3.16, we know that $\underline{U}(\mu, t, A)$ is an ideal of S. Let $x_1, x_2 \in \underline{U}(\mu, t, A)$ for some $x_1, x_2 \in S$, then we have $[x_1]_{(\mu,t)}[x_2]_{(\mu,t)} \subseteq [x_1 x_2]_{(\mu,t)} \subseteq A$. We suppose that $\underline{U}(\mu, t, A)$ is not a prime ideal, then there exist $x_1, x_2 \in S$ such that $x_1 x_2 \in \underline{U}(\mu, t, A)$ but $x_1 \notin \underline{U}(\mu, t, A)$, $x_2 \notin \underline{U}(\mu, t, A)$. Thus $[x_1]_{(\mu,t)} \nsubseteq A$, $[x_2]_{(\mu,t)} \nsubseteq A$, then exist $x_1' \in [x_1]_{(\mu,t)}$, $x_1' \notin A$, $x_2' \in [x_2]_{(\mu,t)}$, $x_2' \notin A$. Thus $x_1' x_2' \in [x_1]_{(\mu,t)}[x_2]_{(\mu,t)} \subseteq A$. Since A is a prime ideal of S, we have $x_i' \in A$ for some $1 \leq i \leq 2$. It contradicts with the supposition. Hence $\underline{U}(\mu, t, A) \neq \emptyset$ is a prime ideal of S. □

Theorem 3.23. *Let $U(\mu, t)$ be a complete congruence relation on S. If A is a prime ideal of S, then A is an upper rough prime ideal of S.*

Proof. Since A is a prime ideal of S, by Theorem 3.13, we know that $\overline{U}(\mu, t, A)$ is an ideal of S. Let $x_1 x_2 \in \overline{U}(\mu, t, A)$ for some $x_1, x_2 \in S$, then we have $[x_1 x_2]_{(\mu,t)} \cap A = [x_1]_{(\mu,t)}[x_2]_{(\mu,t)} \cap A \neq \emptyset$. So there exist $x_1' \in [x_1]_{(\mu,t)}$, $x_2' \in [x_2]_{(\mu,t)}$ such that $x_1' x_2' \in A$. Since A is a prime ideal, we have $x_i' \in A$ for some $1 \leq i \leq 2$. Thus $[x_i]_{(\mu,t)} \cap A \neq \emptyset$, and so $x_i \in \overline{U}(\mu, t, A)$, and so $\overline{U}(\mu, t, A)$ is a prime ideal of S. This means that A is an upper rough prime ideal of S. □

4 Rough fuzzy semigroups based on fuzzy ideals

In Section 3, we discuss rough semigroups and their properties of semigroups. Now, in this section, we introduce roughness of fuzzy semigroups based on a fuzzy ideal of semigroups. First of all, we give the definition of rough fuzzy semigroups of semigroups. Here, we also use $U(\mu, t)$ as a congruence relation.

Definition 4.1. *Let $U(\mu, t)$ be a complete congruence relation on S and β be a fuzzy subset of S, we can define a new approximation space (μ, t, β) and the lower and upper approximations of β, respectively*

$$\overline{U}(\mu, t, \beta) = \vee\{\beta(y) \mid y \in [x]_{(\mu,t)}\} \text{ and } \underline{U}(\mu, t, \beta) = \wedge\{\beta(y) \mid y \in [x]_{(\mu,t)}\}.$$

(1) *If $\overline{U}(\mu, t, \beta)(\underline{U}(\mu, t, \beta))$ is a fuzzy subsemigroup of S, then we call β an upper (lower) rough fuzzy subsemigroup of S;*

(2) *If* $\overline{U}(\mu,t,\beta)(\underline{U}(\mu,t,\beta))$ *is a fuzzy left (right, two-sided) ideal of* S*, then we call* β *an upper (lower) rough fuzzy left (right, two-sided) ideal of* S.

Proposition 4.2. *Let* μ *be any fuzzy ideal of* S *and* β *and* γ *be any two fuzzy subsets, then*
(1) $\underline{U}(\mu,t,\beta)\subseteq\beta\subseteq\overline{U}(\mu,t,\beta)$,
(2) $\overline{U}(\mu,t,\beta\cap\gamma)\subseteq\overline{U}(\mu,t,\beta)\cap\overline{U}(\mu,t,\gamma)$,
(3) $\underline{U}(\mu,t,\beta\cap\gamma)=\underline{U}(\mu,t,\beta)\cap\underline{U}(\mu,t,\gamma)$,
(4) $\overline{U}(\mu,t,\beta\cup\gamma)=\overline{U}(\mu,t,\beta)\cup\overline{U}(\mu,t,\gamma)$,
(5) $\underline{U}(\mu,t,\beta\cup\gamma)\supseteq\underline{U}(\mu,t,\beta)\cup\underline{U}(\mu,t,\gamma)$,
(6) $\beta\subseteq\gamma$, *then* $\overline{U}(\mu,t,\beta)\subseteq\overline{U}(\mu,t,\gamma)$ *and* $\underline{U}(\mu,t,\beta)\subseteq\underline{U}(\mu,t,\gamma)$,
(7) $\overline{U}(\mu,t,\overline{U}(\mu,t,\beta))=\overline{U}(\mu,t,\beta)$,
(8) $\underline{U}(\mu,t,\underline{U}(\mu,t,\beta))=\underline{U}(\mu,t,\beta)$,
(9) $\overline{U}(\mu,t,\underline{U}(\mu,t,\beta))=\overline{U}(\mu,t,\beta)$,
(10) $\underline{U}(\mu,t,\overline{U}(\mu,t,\beta))=\underline{U}(\mu,t,\beta)$,
(11) $\overline{U}(\mu,t,\beta_\alpha)=\overline{U}(\mu,t,\beta)_\alpha$,
(12) $\underline{U}(\mu,t,\beta_\alpha)=\underline{U}(\mu,t,\beta)_\alpha$,
(13) $\overline{U}(\mu,t,\beta_{\bar\alpha})=\overline{U}(\mu,t,\beta)_{\bar\alpha}$,
(14) $\underline{U}(\mu,t,\beta_{\bar\alpha})=\underline{U}(\mu,t,\beta)_{\bar\alpha}$.

Proof. Here we only prove that (12) and (13) hold and others are trivial.

(12) $x\in\underline{U}(\mu,t,\beta)_\alpha\Leftrightarrow\underline{U}(\mu,t,\beta)(x)\ge\alpha\Leftrightarrow\wedge\{\beta(y)\mid y\in[x]_{(\mu,t)}\}\ge\alpha\Leftrightarrow y\in[x]_{(\mu,t)}$, then $\beta(y)\ge\alpha\Leftrightarrow y\in[x]_{(\mu,t)}, y\in\beta_\alpha\Leftrightarrow[x]_{(\mu,t)}\subseteq\beta_\alpha\Leftrightarrow x\in\underline{U}(\mu,t,\beta_\alpha)$.

(13) $x\in\overline{U}(\mu,t,\beta)_\alpha\Leftrightarrow\overline{U}(\mu,t,\beta)(x)\ge\alpha\Leftrightarrow\vee\{\beta(y)\mid y\in[x]_{(\mu,t)}\}\ge\alpha\Leftrightarrow$ there exists $y\in[x]_{(\mu,t)}$, and $\beta(y)\ge\alpha$ and there exists $y\in[x]_{(\mu,t)}$, and $y\in\beta_\alpha\Leftrightarrow x\in\overline{U}(\mu,t,\beta_{\bar\alpha})$. □

Theorem 4.3. *Let* μ *be a fuzzy ideal of* S *and* $t\in[0,1]$*, then*
(1) *If* β *is a fuzzy subsemigroup of* S*,* β *is an upper rough fuzzy subsemigroup of* S*;*
(2) *If* β *is a fuzzy left (right, two-sided) ideal of* S*,* β *is an upper rough fuzzy left (right, two-sided) ideal of* S.

Proof. (1) Let β be a fuzzy subsemigroup of S, for every $\alpha\in[0,1]$, β_α is a subsemigroup of S. According to Propositions 3.9 and 4.2, we have $\overline{U}(\mu,t,\beta)_\alpha\cdot\overline{U}(\mu,t,\beta)_\alpha=\overline{U}(\mu,t,\beta_\alpha)\overline{U}(\mu,t,\beta_\alpha)\subseteq\overline{U}(\mu,t,\beta_\alpha\beta_\alpha)\subseteq\overline{U}(\mu,t,\beta_\alpha)=\overline{U}(\mu,t,\beta)_\alpha$, obviously, for any $\alpha\in[0,1]$, $\overline{U}(\mu,t,\beta)_\alpha$ is a subsemigroup of S. So $\overline{U}(\mu,t,\beta)$ is a fuzzy subsemigroup of S. Thus, β is an upper rough fuzzy subsemigroup of S.

(2) If β is a fuzzy left ideal of S, for every $\alpha\in[0,1]$, β_α is a left ideal of S, that is, $S\cdot\beta_\alpha\subseteq\beta_\alpha$, since $\overline{U}(\mu,t,S)=S$, $S\cdot\overline{U}(\mu,t,\beta)_\alpha=\overline{U}(\mu,t,S)\cdot\overline{U}(\mu,t,\beta_\alpha)\subseteq\overline{U}(\mu,t,S\cdot\beta_\alpha)\subseteq\overline{U}(\mu,t,\beta_\alpha)=\overline{U}(\mu,t,A)_\alpha$, so $\overline{U}(\mu,t,\beta)_\alpha$ is a left ideal of S, then $\overline{U}(\mu,t,\beta)$ is a fuzzy left ideal of S. Hence β is an upper rough fuzzy left ideal of S. □

From this theorem, we see that upper rough fuzzy subsemigroups (left ideals, right ideals, two-sided ideals) are the generalizations of the fuzzy subsemigroups (left ideals, right ideals, two-sided ideals).

Proposition 4.4. *Let* $U(\mu,t)$ *be a complete congruence relation on* S*, then*
(1) *If* β *is a fuzzy subsemigroup of* S*,* β *is a lower rough fuzzy subsemigroup of* S*;*
(2) *If* β *is a fuzzy left (right, two-sided) ideal of* S*,* β *is a lower rough fuzzy left (right, two-sided) ideal of* S.

Proof. (1) Let β be a fuzzy subsemigroup of S, for every $\alpha\in[0,1]$, β_α is a subsemigroup of S. According to Propositions 3.9 and 4.2, we have $\underline{U}(\mu,t,\beta)_\alpha\underline{U}(\mu,t,\beta)_\alpha=\underline{U}(\mu,t,\beta_\alpha)\underline{U}(\mu,t,\beta_\alpha)\subseteq\underline{U}(\mu,t,\beta_\alpha\beta_\alpha)\subseteq\underline{U}(\mu,t,\beta_\alpha)=\underline{U}(\mu,t,\beta)_\alpha$, obviously, for any $\alpha\in[0,1]$, $\overline{U}(\mu,t,\beta)_\alpha$ is a subsemigroup of S. So $\overline{U}(\mu,t,\beta)$ is a fuzzy subsemigroup of S. So β is a lower rough fuzzy subsemigroup of S.

(2) If β is a fuzzy left ideal of S, for every $\alpha\in[0,1]$, β_α is a left ideal of S, that is $S\beta_\alpha\subseteq\beta_\alpha$, since $\underline{U}(\mu,t,S)=S$, $S\cdot\underline{U}(\mu,t,A)_\alpha=\underline{U}(\mu,t,S)\cdot\underline{U}(\mu,t,\beta_\alpha)\subseteq\underline{U}(\mu,t,S\beta_\alpha)\subseteq\underline{U}(\mu,t,\beta_\alpha)=\underline{U}(\mu,t,\beta)_\alpha$, so

$\underline{U}(\mu, t, \beta)_\alpha$ is a left ideal of S, then $\underline{U}(\mu, t, \beta)$ is a fuzzy left ideal of S. Hence β is an upper rough fuzzy left ideal. □

Theorem 4.5. *Let $U(\mu, t)$ be a complete congruence relation on S. If β is a fuzzy prime ideal of S, then we have*
(1) *If $\underline{U}(\mu, t, \beta) \neq \emptyset$, β is an lower rough fuzzy prime ideal of S.*
(2) *If $\overline{U}(\mu, t, \beta) \neq \emptyset$, β is an upper rough fuzzy prime ideal of S.*

Proof. (1) Since β is a fuzzy prime ideal of S, we know $\beta_\alpha(\alpha \in [0, 1])$ is a prime ideal of S. By Theorem 3.22 we have $\underline{U}(\mu, t, \beta_\alpha)$, if it is nonempty, is a prime ideal of S. And by Proposition 4.2, we know $\underline{U}(\mu, t, \beta)_\alpha = \underline{U}(\mu, t, \beta_\alpha)$ is also a prime ideal of S, if it is nonempty. So $\underline{U}(\mu, t, \beta)$ is a fuzzy prime ideal of S, if it is nonempty, then β is a lower rough fuzzy prime ideal of S.

(2) It can be proven as (1). □

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