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An Ulam stability result on quasi- b -metric-like spaces

DOI 10.1515/math-2016-0097

Received August 8, 2016; accepted October 10, 2016.

Abstract: In this paper a class of general type α -admissible contraction mappings on quasi- b -metric-like spaces are defined. Existence and uniqueness of fixed points for this class of mappings is discussed and the results are applied to Ulam stability problems. Various consequences of the main results are obtained and illustrative examples are presented.

Keywords: Fixed point, Quasi- b -metric like, α -admissible contraction mappings, Ulam-Hyers stability

MSC: 47H10, 54C60, 54H25, 55M20

1 Introduction and preliminaries

The problem of stability of functional equations was motivated by a question of S.M. Ulam asked in 1940: "When is it true that the solution of an equation differing slightly from a given one, must of necessity be close to the solution of the given equation?" The answer of D.H. Hyers, published in [1], initiates the theory of Ulam stability theory. In this paper, we investigate the Ulam stability results by using the fixed point techniques. For further information and results on the subject we refer the reader to [2–4].

First of all, we give a set of the tools in the context of fixed point theory that will be used for Ulam stability results.

The concept of α -admissible mapping introduced by Samet et al. [5] in 2012 has been one of the most important generalizations in fixed point theory in the recent years. It is a very general concept and can be used in various abstract spaces. On the other hand, b -metric spaces received considerable interest recently and combined with α -admissibility they also proved to be one of the attractive studies in fixed point theory. In their recent study, Bota et al. [6] investigated $\alpha - \psi$ -contractive mapping of type- (b) in the framework of b -metric spaces. In another work, Bilgili et al. [7] discussed $\alpha - \psi$ contractive mappings in the setting of quasi-metric spaces and proved the existence and uniqueness of a fixed point of such mappings. Also very recently, Gülyaz [8] presented some results about α -admissible mappings on quasi- b -metric like spaces.

In this work, a class of α -admissible contractive mappings of general types are considered in the context of a quasi- b -metric like space and the existence and uniqueness of their fixed points are discussed.

First, we define some basic notions and concepts to be used in the sequel.

Metric-like spaces or, also known as dislocated spaces have been introduced by Amini-Harandi in [9].

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Definition 1.1. Let X be a nonempty set and let $\gamma : X \times X \rightarrow [0, +\infty)$ be a mapping satisfying the following conditions for all $x, y, z \in X$:

$$(ML1) \gamma(x, y) = 0 \Rightarrow x = y;$$

$$(ML2) \gamma(x, y) = \gamma(y, x);$$

$$(ML3) \gamma(x, y) \leq \gamma(x, z) + \gamma(z, y).$$

Then the mapping γ is called a metric-like and the pair (X, γ) is called a metric-like space (dislocated space).

Clearly, the conditions of a metric-like on X coincide with the conditions of a metric except that $\gamma(x, x)$ does not need to be 0 in general. Therefore, a metric space is always a metric-like space but a metric-like space is not metric space in general. Alghamdi et al. [10] combined the concepts of b -metric space and metric-like space recently and introduced the so-called b -metric-like spaces.

Definition 1.2 ([10]). Let X be a nonempty set and let $\gamma_b : X \times X \rightarrow [0, +\infty)$ be a function such that for all $x, y, z \in X$ and a constant $s \geq 1$, the following conditions are satisfied:

$$(BML1) \gamma_b(x, y) = 0 \Rightarrow x = y;$$

$$(BML2) \gamma_b(x, y) = \gamma_b(y, x);$$

$$(BML3) \gamma_b(x, y) \leq s[\gamma_b(x, z) + \gamma_b(z, y)].$$

The map γ_b is called a b -metric-like and the pair (X, γ_b) is called a b -metric-like space with constant s .

Let (X, γ_b) be a b -metric-like space. For any $x \in X$ and $r > 0$, the set $B(x, r) = \{y \in X : |\gamma_b(x, y) - \gamma_b(x, x)| < r\}$ is called an open ball centered at x of radius $r > 0$.

Also very recently, Zhu et al. [11] defined the concept of quasi-metric-like spaces and proved some fixed point theorems on quasi-metric-like spaces. The definition of quasi-metric-like is given next.

Definition 1.3 ([11]). Let X be a nonempty set. A mapping $\rho : X \times X \rightarrow [0, +\infty)$ is called a quasi-metric-like if for all $x, y, z \in X$,

$$(QML1) \rho(x, y) = 0 \Rightarrow x = y;$$

$$(QML2) \rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

The pair (X, ρ) is called a quasi-metric-like space.

Remark 1.4. Since the symmetry condition is not required for quasi-metric-like spaces, whenever $\rho(x, y) = 0$ we may have $\rho(y, x) \neq 0$. Therefore, instead of (QML1) we use the condition given in [12],

$$(QML1') \quad \rho(x, y) = \rho(y, x) = 0 \Rightarrow x = y.$$

After their results on quasi-metric-like spaces, Zhu et al. [13] combined b -metric space and quasi-metric-like space and introduced the concept of quasi- b -metric-like spaces. The authors also presented some fixed point results on quasi- b -metric-like spaces. On the other hand, also very recently Klin-eam and Suanoom [12] studied cyclic Banach and Kannan type contraction mappings on quasi- b -metric-like. We give next the definition of quasi- b -metric-like spaces.

Definition 1.5 ([12]). A quasi- b -metric-like on a nonempty set X is a function $\rho_b : X \times X \rightarrow [0, +\infty)$ such that for all $x, y, z \in X$ and a constant $s \geq 1$,

$$(QBML1) \rho_b(x, y) = \rho_b(y, x) = 0 \Rightarrow x = y;$$

$$(QBML2) \rho_b(x, y) \leq s[\rho_b(x, z) + \rho_b(z, y)].$$

The pair (X, ρ_b) is called a quasi- b -metric-like space with constant s .

Inspired by the work of Klin-eam and Suanoom [12] we present the following examples of quasi- b -metric-like.

Example 1.6. The function $\rho_b : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ defined as

$$\rho_b(x, y) = |x - y|^2 + |2x + y|^2$$

for all $x, y \in \mathbb{R}$ is a quasi- b -metric-like on $\mathbb{R}$ with constant $s = 2$.

Example 1.7. The function $\rho_b : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ defined as

$$\rho_b(x, y) = |x - y|^4 + |x|^2 + 2|y|^2$$

for all $x, y \in \mathbb{R}$ is a quasi- b -metric-like on $\mathbb{R}$ with constant $s = 8$.

Some topological concepts on quasi- b -metric-like spaces are defined next.

Definition 1.8 ([13]). Let (X, ρ_b) be a quasi- b -metric-like space.

1. A sequence $\{\xi_n\} \subset X$ is said to converge to a point $\xi \in X$ if and only if

$$\lim_{n \rightarrow \infty} \rho_b(\xi, \xi_n) = \lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi) = \rho_b(\xi, \xi).$$

2. A sequence $\{\xi_n\} \subset X$ is said to be a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m)$ and $\lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n)$ exist and are finite.
3. (X, ρ_b) is said to be a complete quasi- b -metric-like space if and only if for every Cauchy sequence $\{\xi_n\}$ in X there exists some $\xi \in X$ such that

$$\lim_{n \rightarrow \infty} \rho_b(\xi, \xi_n) = \lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi) = \rho_b(\xi, \xi) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m).$$

4. A sequence $\{\xi_n\} \subset X$ is said to be a 0-Cauchy sequence if

$$\lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = 0.$$

5. (X, ρ_b) is said to be a 0-complete quasi- b -metric-like space if for every 0-Cauchy sequence $\{\xi_n\}$ in X there exists a $\xi \in X$ such that

$$\lim_{n \rightarrow \infty} \rho_b(\xi, \xi_n) = \lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi) = \rho_b(\xi, \xi) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m).$$

6. A mapping $f : X \rightarrow X$ is said to be continuous at $\xi_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(\xi_0, \delta)) \subset B(f(\xi_0), \varepsilon)$.

The above definition makes it clear that in the quasi- b -metric like space (X, ρ_b) , every 0-Cauchy sequence is a Cauchy sequence, and that very complete quasi- b -metric-like space is also 0-complete. Note, however, that the converses of these statements may not be true.

Due to the fact that interchange symmetry condition is not necessary on quasi-metric type spaces, we need to recall right- and left-Cauchy sequences and right- and left-completeness definitions.

Definition 1.9. Let (X, ρ_b) be a quasi- b -metric-like space.

- (i) A sequence $\{\xi_n\}$ in X is called a left-Cauchy sequence if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $\rho_b(\xi_n, \xi_m) < \varepsilon$ for all $n \geq m > N$;
- (ii) A sequence $\{\xi_n\}$ in X is called a right-Cauchy sequence if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $\rho_b(\xi_n, \xi_m) < \varepsilon$ for all $m \geq n > N$;
- (iii) A quasi- b -metric-like space is said to be left-complete if every left-Cauchy sequence $\{\xi_n\}$ in X is convergent,
- (iv) A quasi- b -metric-like space is said to be right-complete if every right-Cauchy sequence $\{\xi_n\}$ in X is convergent,
- (v) A quasi- b -metric-like space is complete if and only if every Cauchy sequence in X is convergent.

Remark 1.10. It is obvious from the Definition 1.9 that a sequence $\{\xi_n\}$ in a quasi- b -metric-like space is Cauchy if and only if it is both left-Cauchy and right-Cauchy, and a quasi- b -metric-like space is complete if and only if it is left-complete and right-complete.

In the following discussion we will employ the so-called comparison functions and their variants. These functions have been introduced by Berinde [14] and Rus [15]. We give next the definitions of comparison, (c)-comparison and (b)-comparison functions. It should be mentioned that Berinde [14] introduced the concept of (c)-comparison functions in order to investigate convergence properties of iterative sequences generated by fixed point methods, and (b)-comparison functions to be used on b -metric space (see [16, 17] for details).

Definition 1.11. *CF (See [14, 15]). Comparison function is an increasing mapping $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ satisfying $\varphi^n(t) \rightarrow 0, n \rightarrow \infty$ for any $t \in [0, \infty)$.*

CCF (See [14]) A (c)-comparison function is a function $\varphi_c : [0, +\infty) \rightarrow [0, +\infty)$ satisfying

(c₁) φ_c is increasing,

(c₂) there exist $k_0 \in \mathbb{N}, a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $\varphi_c^{k+1}(t) \leq a\varphi_c^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

BCF (See [16, 17]) For a real number $s \geq 1$ (b)-comparison function is a function $\varphi_b : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the conditions

(b₁) φ_b is increasing,

(b₂) there exist $k_0 \in \mathbb{N}, a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \leq as^k\varphi_b^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

In the sequel, we denote the class of comparison functions by Φ , the class of (c)-comparison functions by Φ_c and the class of (b)-comparison functions by Φ_b . Obviously, a (b)-comparison function reduces to (c)-comparison function whenever $s = 1$. In addition, every (c)-comparison function is a comparison function.

We will need the following essential properties in our further discussion.

Lemma 1.12 (Berinde [14], Rus [15]). *For a comparison function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ the following hold:*

- (1) *each iterate φ^k of φ $k \geq 1$, is also a comparison function;*
- (2) *φ is continuous at 0;*
- (3) *$\varphi(t) < t$, for any $t > 0$.*

Lemma 1.13 ([17]). *For a (b)-comparison function $\varphi_b : [0, +\infty) \rightarrow [0, +\infty)$ the following hold:*

- (1) *the series $\sum_{k=0}^{\infty} s^k \varphi_b^k(t)$ converges for any $t \in [0, +\infty)$;*
- (2) *the function $b_s : [0, +\infty) \rightarrow [0, +\infty)$ defined by $b_s(t) = \sum_{k=0}^{\infty} s^k \varphi_b^k(t)$, $t \in [0, \infty)$ is increasing and continuous at 0.*

For more details on comparison functions and examples we refer the reader to [14, 15].

Finally, we recall the concept of α -admissible mappings introduced by Samet [5].

Definition 1.14. *A mapping $T : X \rightarrow X$ is called α -admissible if for all $\xi, \eta \in X$ we have*

$$\alpha(\xi, \eta) \geq 1 \Rightarrow \alpha(T\xi, T\eta) \geq 1, \quad (1)$$

where $\alpha : X \times X \rightarrow [0, \infty)$ is a given function.

For recent results related with α -admissible mappings, see [18, 19]. In their work Samet et.al [5], studied $\alpha - \psi$ -contractive mappings and their fixed points.

Recently, Gülyaz [8] defined $\alpha - \psi$ -contractive mapping in the framework of quasi- b -metric-like spaces.

Definition 1.15. *Let (X, ρ_b) be a quasi- b -metric-like space and $T : X \rightarrow X$ be a given mapping. We say that T is an $\alpha - \psi$ -contractive mapping if there exist two functions $\alpha : X \times X \rightarrow [0, \infty)$ and $\psi \in \Phi_b$ such that for all $\xi, \eta \in X$, we have*

$$\alpha(\xi, \eta)\rho_b(T\xi, T\eta) \leq \psi(\rho_b(\xi, \eta)). \quad (2)$$

Gülyaz [8] also proved the existence of fixed points for $\alpha - \psi$ -contractive mapping on quasi- b -metric-like spaces.

Theorem 1.16. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is an $\alpha - \psi$ -contractive mapping. Suppose also that*

- (i) *T is α -admissible;*
 - (ii) *there exists $\xi_0 \in X$ such that $\alpha(T\xi_0, \xi_0) \geq 1$ and $\alpha(\xi_0, T\xi_0) \geq 1$;*
 - (iii) *T is continuous,*
- Then T has a fixed point.*

Theorem 1.17. Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is an $\alpha - \psi$ -contractive mapping. Suppose also that

- (i) T is α -admissible;
- (ii) there exists $\xi_0 \in X$ such that $\alpha(T\xi_0, \xi_0) \geq 1$ and $\alpha(\xi_0, T\xi_0) \geq 1$;
- (iii) If $\{\xi_n\}$ is a sequence in X which converges to ξ and satisfies $\alpha(\xi_{n+1}, \xi_n) \geq 1$ and $\alpha(\xi_n, \xi_{n+1}) \geq 1$ for all n then, there exists a subsequence $\{\xi_{n(k)}\}$ of $\{\xi_n\}$ such that $\alpha(\xi, \xi_{n(k)}) \geq 1$ and $\alpha(\xi_{n(k)}, \xi) \geq 1$ for all k .

Then T has a fixed point.

2 Existence and uniqueness theorems on complete quasi- b -metric-like spaces

In this section we present our main results. We concentrate on existence and uniqueness of fixed points for a general class of α -admissible contractive mappings.

Definition 2.1. Let (X, ρ_b) be a quasi- b -metric-like space with a constant $s \geq 1$ and let $\alpha : X \times X \rightarrow [0, \infty)$ and $\varphi_b \in \Phi_b$ be two functions.

- (i) An $\alpha - \varphi_b$ contractive mapping $T : X \rightarrow X$ is of type (A) if

$$\alpha(\xi, \eta) \rho_b(T\xi, T\eta) \leq \varphi_b(M(\xi, \eta)), \text{ for all } \xi, \eta \in X, \quad (3)$$

where

$$M(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta), \frac{1}{4s}[\rho_b(T\xi, \eta) + \rho_b(T\eta, \xi)]\}.$$

- (ii) An $\alpha - \varphi_b$ contractive mapping $T : X \rightarrow X$ is of type (B) if

$$\alpha(\xi, \eta) \rho_b(T\xi, T\eta) \leq \varphi_b(N(\xi, \eta)), \text{ for all } \xi, \eta \in X \quad (4)$$

where

$$N(\xi, \eta) = \max\{\rho_b(\xi, \eta), \frac{1}{2s}[\rho_b(T\xi, \xi) + \rho_b(T\eta, \eta)], \frac{1}{4s}[\rho_b(T\xi, \eta) + \rho_b(T\eta, \xi)]\}.$$

- (iii) An $\alpha - \varphi_b$ contractive mapping $T : X \rightarrow X$ is of type (C) if

$$\alpha(\xi, \eta) \rho_b(T\xi, T\eta) \leq \varphi_b(K(\xi, \eta)), \text{ for all } \xi, \eta \in X, \quad (5)$$

where

$$K(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta)\}.$$

Remark 2.2. Note that $\rho_b(\xi, \eta) \leq K(\xi, \eta) \leq N(\xi, \eta) \leq M(\xi, \eta)$ for all $\xi, \eta \in X$.

Our first theorem gives conditions for the existence of a fixed point for maps in class (A).

Theorem 2.3. Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is an $\alpha - \varphi_b$ contractive mapping of type (A) satisfying the following:

- (i) T is α -admissible;
- (ii) there exists $\xi_0 \in X$ such that $\alpha(T\xi_0, \xi_0) \geq 1$ and $\alpha(\xi_0, T\xi_0) \geq 1$;
- (iii) T is continuous;

Then T has a fixed point.

Proof. As usual, we take $\xi_0 \in X$ such that $\alpha(T\xi_0, \xi_0) \geq 1$ and $\alpha(\xi_0, T\xi_0) \geq 1$ and construct the sequence $\{\xi_n\}$ as

$$\xi_{n+1} = T\xi_n \text{ for } n \in \mathbb{N}.$$

Notice that if for some $n_0 \geq 0$ we have $\xi_{n_0} = \xi_{n_0+1}$ then the proof is done, i.e., ξ_{n_0} is a fixed point of T . Assume that $\xi_n \neq \xi_{n+1}$ for all $n \geq 0$. Because of the lack of symmetry condition in quasi-b-metric-like spaces, we will show that the sequence $\{\xi_n\}$ is both left- and right-Cauchy.

From the conditions (i) and (ii), we have

$$\alpha(\xi_1, \xi_0) = \alpha(T\xi_0, \xi_0) \geq 1 \Rightarrow \alpha(T\xi_1, T\xi_0) = \alpha(\xi_2, \xi_1) \geq 1, \quad (6)$$

and

$$\alpha(\xi_0, \xi_1) = \alpha(\xi_0, T\xi_0) \geq 1 \Rightarrow \alpha(T\xi_0, T\xi_1) = \alpha(\xi_1, \xi_2) \geq 1, \quad (7)$$

or, in general

$$\alpha(\xi_{n+1}, \xi_n) \geq 1 \text{ and } \alpha(\xi_n, \xi_{n+1}) \geq 1 \quad \forall n \in \mathbb{N}. \quad (8)$$

Regarding (8), the contractive condition (3) with $\xi = \xi_{n+1}$ and $\eta = \xi_n$ becomes

$$\rho_b(\xi_{n+1}, \xi_n) = \rho_b(T\xi_n, T\xi_{n-1}) \leq \alpha(\xi_n, \xi_{n-1})\rho_b(T\xi_n, T\xi_{n-1}) \leq \varphi_b(M(\xi_n, \xi_{n-1})), \quad (9)$$

where

$$\begin{aligned} M(\xi_n, \xi_{n-1}) &= \max\{\rho_b(\xi_n, \xi_{n-1}), \rho_b(T\xi_n, \xi_n), \rho_b(T\xi_{n-1}, \xi_{n-1}), \\ &\quad \frac{1}{4s}[\rho_b(T\xi_n, \xi_{n-1}) + \rho_b(T\xi_{n-1}, \xi_n)]\} \\ &= \max\{\rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \\ &\quad \frac{1}{4s}[\rho_b(\xi_{n+1}, \xi_{n-1}) + \rho_b(\xi_n, \xi_n)]\} \end{aligned}$$

Observe that for the last term in $M(\xi_n, \xi_{n-1})$, by the triangle inequality we have

$$\begin{aligned} &\frac{1}{4s}[\rho_b(\xi_{n+1}, \xi_{n-1}) + \rho_b(\xi_n, \xi_n)] \\ &\leq \frac{1}{4}[\rho_b(\xi_{n+1}, \xi_n) + \rho_b(\xi_n, \xi_{n-1}) + \rho_b(\xi_n, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_n)] \\ &= \frac{1}{4}[2\rho_b(\xi_{n+1}, \xi_n) + \rho_b(\xi_n, \xi_{n-1}) + \rho_b(\xi_n, \xi_{n+1})] \\ &\leq \max\{\rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_n, \xi_{n+1})\}, \end{aligned} \quad (10)$$

and hence, either $M(\xi_n, \xi_{n-1}) = \max\{\rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n-1})\}$ or $M(\xi_n, \xi_{n-1}) \leq \rho_b(\xi_n, \xi_{n+1})$. Now, we will examine all three cases.

Case 1. Suppose that $M(\xi_n, \xi_{n-1}) = \rho_b(\xi_{n+1}, \xi_n)$ for some $n \geq 1$. Since $\rho_b(\xi_{n+1}, \xi_n) > 0$, from (9), we have

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b(M(\xi_n, \xi_{n-1})) = \varphi_b(\rho_b(\xi_{n+1}, \xi_n)) < \rho_b(\xi_{n+1}, \xi_n)$$

which is a contradiction. Hence, for all $n \geq 1$ either $M(\xi_n, \xi_{n-1}) = \rho_b(\xi_n, \xi_{n-1})$ or $M(\xi_n, \xi_{n-1}) \leq \rho_b(\xi_n, \xi_{n+1})$.

Case 2. Suppose that $M(\xi_n, \xi_{n-1}) = \rho_b(\xi_n, \xi_{n-1})$ for some $n \geq 1$. Regarding the properties of $\varphi_b \in \Phi_b$ and (9), we get

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b(M(\xi_n, \xi_{n-1})) \leq \varphi_b(\rho_b(\xi_n, \xi_{n-1})) < \rho_b(\xi_n, \xi_{n-1})$$

for all $n \geq 1$. Inductively, we obtain

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b^n(\rho_b(\xi_1, \xi_0)), \text{ for all } n \geq 1. \quad (11)$$

Applying repeatedly triangle inequality (QBL_2) and regarding (11), for all $k \geq 1$, we get

$$\begin{aligned} \rho_b(\xi_{n+k}, \xi_n) &\leq s[\rho_b(\xi_{n+k}, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_n)] \\ &\leq s^2[\rho_b(\xi_{n+k}, \xi_{n+2}) + \rho_b(\xi_{n+2}, \xi_{n+1})] + s\rho_b(\xi_{n+1}, \xi_n) \\ &\vdots \\ &\leq s^k \rho_b(\xi_{n+k}, \xi_{n+k-1}) + s^{k-1} \rho_b(\xi_{n+k-1}, \xi_{n+k-2}) \\ &\quad + \dots + s^2 \rho_b(\xi_{n+2}, \xi_{n+1}) + s\rho_b(\xi_{n+1}, \xi_n) \\ &\leq s^k \varphi_b^{n+k-1}(\rho_b(\xi_1, \xi_0)) + s^{k-1} \varphi_b^{n+k-2}(\rho_b(\xi_1, \xi_0)) \\ &\quad + \dots + s\varphi_b^n(\rho_b(\xi_1, \xi_0)) \\ &= \frac{1}{s^{n+1}} \left[s^{n+k-1} \varphi_b^{n+k-1}(\rho_b(\xi_1, \xi_0)) \right. \\ &\quad \left. + s^{n+k-2} \varphi_b^{n+k-2}(\rho_b(\xi_1, \xi_0)) + \dots + s^n \varphi_b^n(\rho_b(\xi_1, \xi_0)) \right] \end{aligned} \quad (12)$$

Define

$$\mathcal{S}_n = \sum_{p=0}^n s^p \varphi_b^p(\rho_b(\xi_1, \xi_0)) \text{ for } n \geq 1. \quad (13)$$

We obtain

$$\rho_b(\xi_{n+k}, \xi_n) \leq \frac{1}{s^{n-1}} [S_{n+k-1} - S_{n-1}], \quad n \geq 1, k \geq 1. \quad (14)$$

Due to the assumption $\xi_n \neq \xi_{n+1}$ for all $n \in \mathbb{N}$ and Lemma 1.13, we conclude that the series $\sum_{p=0}^{\infty} s^p \varphi_b^p(\rho_b(\xi_1, \xi_0))$ is convergent to some $\mathcal{S} \geq 0$. Thus, $\lim_{n \rightarrow \infty} \rho_b(\xi_{n+k}, \xi_n) = 0$ or, in other words, for $m > n$,

$$\lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = 0. \quad (15)$$

Case 3. Suppose that $M(\xi_n, \xi_{n-1}) \leq \rho_b(\xi_n, \xi_{n+1})$ for some $n \geq 1$. Using the fact that $\varphi_b \in \Phi_b$ and the inequality (9), we get

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b(M(\xi_n, \xi_{n-1})) \leq \varphi_b(\rho_b(\xi_n, \xi_{n+1})) < \rho_b(\xi_n, \xi_{n+1}) \quad (16)$$

for all $n \geq 1$. On the other hand, if we put $\xi = \xi_n$ and $\eta = \xi_{n+1}$ in (3), taking into account (8), we find

$$\begin{aligned} \rho_b(\xi_n, \xi_{n+1}) &= \rho_b(T\xi_{n-1}, T\xi_n) \\ &\leq \alpha(\xi_{n-1}, \xi_n) \rho_b(T\xi_{n-1}, T\xi_n) \leq \varphi_b(M(\xi_{n-1}, \xi_n)), \end{aligned} \quad (17)$$

where

$$\begin{aligned} M(\xi_{n-1}, \xi_n) &= \max \left\{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(T\xi_{n-1}, \xi_{n-1}), \rho_b(T\xi_n, \xi_n), \right. \\ &\quad \left. \frac{1}{4s} [\rho_b(T\xi_{n-1}, \xi_n) + \rho_b(T\xi_n, \xi_{n-1})] \right\} \\ &= \max \left\{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n), \right. \\ &\quad \left. \frac{1}{4s} [\rho_b(\xi_n, \xi_n) + \rho_b(\xi_{n+1}, \xi_{n-1})] \right\} \end{aligned}$$

for all $n \geq 1$. Observe that, applying triangle inequality (QBL_2) to the last term in $M(\xi_{n-1}, \xi_n)$ we have

$$\begin{aligned} &\frac{1}{4s} [\rho_b(\xi_n, \xi_n) + \rho_b(\xi_{n+1}, \xi_{n-1})] \\ &\leq \frac{1}{4} [\rho_b(\xi_n, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_n) + \rho_b(\xi_{n+1}, \xi_n) + \rho_b(\xi_n, \xi_{n-1})] \\ &= \frac{1}{4} [\rho_b(\xi_n, \xi_{n+1}) + 2\rho_b(\xi_{n+1}, \xi_n) + \rho_b(\xi_n, \xi_{n-1})] \\ &\leq \max\{\rho_b(\xi_n, \xi_{n+1}), \rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n-1})\} = \rho_b(\xi_n, \xi_{n+1}), \end{aligned} \quad (18)$$

since we are in Case 3. On the other hand, from (16) we see that

$$\rho_b(\xi_n, \xi_{n-1}) \leq \rho_b(\xi_{n-1}, \xi_n), \text{ and } \rho_b(\xi_{n+1}, \xi_n) \leq \rho_b(\xi_n, \xi_{n+1}).$$

Therefore, either $M(\xi_{n-1}, \xi_n) \leq \rho_b(\xi_n, \xi_{n+1})$ or $M(\xi_{n-1}, \xi_n) = \rho_b(\xi_{n-1}, \xi_n)$.

If $M(\xi_{n-1}, \xi_n) \leq \rho_b(\xi_n, \xi_{n+1})$ for some $n \in \mathbb{N}$, since $\rho_b(\xi_{n+1}, \xi_n) > 0$, the inequality (17) implies

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_n, \xi_{n+1})) < \rho_b(\xi_n, \xi_{n+1}),$$

which is a contradiction.

Thus, we should have $M(\xi_{n-1}, \xi_n) = \rho_b(\xi_{n-1}, \xi_n)$ for all $n \geq 1$. The inequality (17) becomes

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_{n-1}, \xi_n)) < \rho_b(\xi_{n-1}, \xi_n)$$

for all $n \geq 1$, using the fact that $\varphi_b \in \Phi_b$.

By induction, we get

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b^n(\rho_b(\xi_0, \xi_1)), \text{ for all } n \geq 1. \quad (19)$$

Hence, combining (16) and (19) we deduce

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b^n(\rho_b(\xi_0, \xi_1)), \text{ for all } n \geq 1. \quad (20)$$

Due to $(QBML_2)$, together with (20), for all $k \geq 1$, we get

$$\begin{aligned} \rho_b(\xi_{n+k}, \xi_n) &\leq s[\rho_b(\xi_{n+k}, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_n)] \\ &\leq s^2[\rho_b(\xi_{n+k}, \xi_{n+2}) + \rho_b(\xi_{n+2}, \xi_{n+1})] + s\rho_b(\xi_{n+1}, \xi_n) \\ &\vdots \\ &\leq s^k \rho_b(\xi_{n+k}, \xi_{n+k-1}) + s^{k-1} \rho_b(\xi_{n+k-1}, \xi_{n+k-2}) \\ &\quad + \dots + s^2 \rho_b(\xi_{n+2}, \xi_{n+1}) + s\rho_b(\xi_{n+1}, \xi_n) \\ &\leq s^k \varphi_b^{n+k-1}(\rho_b(\xi_0, \xi_1)) + s^{k-1} \varphi_b^{n+k-2}(\rho_b(\xi_0, \xi_1)) \\ &\quad + \dots + s\varphi_b^n(\rho_b(\xi_0, \xi_1)) \\ &= \frac{1}{s^{n-1}} \left[s^{n+k-1} \varphi_b^{n+k-1}(\rho_b(\xi_0, \xi_1)) \right. \\ &\quad \left. + s^{n+k-2} \varphi_b^{n+k-2}(\rho_b(\xi_0, \xi_1)) + \dots + s^n \varphi_b^n(\rho_b(\xi_0, \xi_1)) \right] \end{aligned}$$

Define

$$\mathcal{Q}_n = \sum_{p=0}^n s^p \varphi_b^p(\rho_b(\xi_0, \xi_1)) \text{ for } n \geq 1. \quad (21)$$

Then, employing (21), we get

$$\rho_b(\xi_{n+k}, \xi_n) \leq \frac{1}{s^{n-1}} [\mathcal{Q}_{n+k-1} - \mathcal{Q}_{n-1}], \quad n \geq 1, k \geq 1.$$

Using the fact that the series $\sum_{p=0}^{\infty} s^p \varphi_b^p(\rho_b(\xi_0, \xi_1))$ converges to some $\mathcal{Q} \geq 0$, we deduce, $\lim_{n \rightarrow \infty} \rho_b(\xi_{n+k}, \xi_n) = 0$, which means that for $m > n$,

$$\lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = 0. \quad (22)$$

Therefore, in all possible cases, we conclude that the sequence $\{\xi_n\}$ is a left Cauchy sequence.

To show that it is also right Cauchy, we proceed as follows.

Let $\xi = \xi_n$ and $\eta = \xi_{n+1}$ in (3). Using (8), we get

$$\begin{aligned} \rho_b(\xi_n, \xi_{n+1}) &= \rho_b(T\xi_{n-1}, T\xi_n) \\ &\leq \alpha(\xi_{n-1}, \xi_n) \rho_b(T\xi_{n-1}, T\xi_n) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \end{aligned} \quad (23)$$

where

$$\begin{aligned} M(\xi_{n-1}, \xi_n) &= \max \{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(T\xi_{n-1}, \xi_{n-1}), \rho_b(T\xi_n, \xi_n), \\ &\quad \frac{1}{4s} [\rho_b(T\xi_{n-1}, \xi_n) + \rho_b(T\xi_n, \xi_{n-1})] \} \\ &\leq \max \{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n), \\ &\quad \frac{1}{4s} [\rho_b(\xi_n, \xi_n) + \rho_b(\xi_{n+1}, \xi_{n-1})] \}. \end{aligned}$$

Regarding the inequality (18), we have

$$\frac{1}{4s} [\rho_b(\xi_n, \xi_n) + \rho_b(\xi_{n+1}, \xi_{n-1})] \leq \max \{ \rho_b(\xi_n, \xi_{n+1}), \rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}) \},$$

and hence, either

$$M(\xi_{n-1}, \xi_n) = \max \{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n) \}, \quad (24)$$

or $M(\xi_{n-1}, \xi_n) \leq \rho_b(\xi_n, \xi_{n+1})$ for all $n \geq 1$. We will discuss separately these four cases.

Case I. Suppose that $M(\xi_{n-1}, \xi_n) \leq \rho_b(\xi_n, \xi_{n+1})$ for some $n \geq 1$. Then the inequality (23) implies

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_{n+1}, \xi_n)) < \rho_b(\xi_n, \xi_{n+1}) \quad (25)$$

which is a contradiction because $\rho_b(\xi_n, \xi_{n+1}) > 0$.

Case II. If for some $n \geq 1$ we have $M(\xi_{n-1}, \xi_n) = \rho_b(\xi_{n+1}, \xi_n)$, then the inequality (23) becomes

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_{n+1}, \xi_n)) < \rho_b(\xi_{n+1}, \xi_n). \quad (26)$$

Recalling (9) and (10), that is,

$$\rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b(M(\xi_n, \xi_{n-1})) \quad (27)$$

where

$$M(\xi_n, \xi_{n-1}) \leq \max \{ \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n), \rho_b(\xi_n, \xi_{n+1}) \},$$

and since by the assumption

$$\max \{ \rho_b(\xi_{n-1}, \xi_n), \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n+1}, \xi_n) \} = \rho_b(\xi_{n+1}, \xi_n) \geq \rho_b(\xi_n, \xi_{n+1}),$$

then, inequalities (26) and (27) yield

$$\rho_b(\xi_n, \xi_{n+1}) \leq \rho_b(\xi_{n+1}, \xi_n) \leq \varphi_b(M(\xi_n, \xi_{n-1})) \leq \varphi_b(\rho_b(\xi_{n+1}, \xi_n)) < \rho_b(\xi_{n+1}, \xi_n)$$

due to the properties of φ_b . This is clearly impossible since $\rho_b(\xi_{n+1}, \xi_n) > 0$ and we end up with a contradiction.

Case III. Assume that $M(\xi_{n-1}, \xi_n) = \rho_b(\xi_{n-1}, \xi_n)$ for some $n \geq 1$. Since $\rho_b(\xi_{n-1}, \xi_n) > 0$, from (9), we get

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_{n-1}, \xi_n)) < \rho_b(\xi_{n-1}, \xi_n)$$

for all $n \geq 1$. Recursively, we derive

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b^n(\rho_b(\xi_0, \xi_1)), \quad \forall n \geq 1. \quad (28)$$

Applying repeatedly triangle inequality ($QBM L_2$) and regarding (28), we get for all $k > 0$,

$$\begin{aligned} \rho_b(\xi_n, \xi_{n+k}) &\leq s[\rho_b(\xi_n, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_{n+k})] \\ &\vdots \\ &\leq s\varphi_b^n(\rho_b(\xi_0, \xi_1)) + s^2\varphi_b^{n+1}(\rho_b(\xi_0, \xi_1)) + \\ &+ \dots + s^{k-1}\varphi_b^{n+k-2}(\rho_b(\xi_0, \xi_1)) + s^k\varphi_b^{n+k-1}(\rho_b(\xi_0, \xi_1)) \\ &= \frac{1}{s^{n-1}} [s^n\varphi_b^n(\rho_b(\xi_0, \xi_1)) + \dots + \\ &+ s^{n+k-2}\varphi_b^{n+k-2}(\rho_b(\xi_0, \xi_1)) + s^{n+k-1}\varphi_b^{n+k-1}(\rho_b(\xi_0, \xi_1))] \end{aligned} \quad (29)$$

Recalling (21), we see that

$$\rho_b(\xi_n, \xi_{n+k}) \leq \frac{1}{s^{n-1}} [\mathcal{Q}_{n+k-1} - \mathcal{Q}_{n-1}], \quad n \geq 1, k \geq 1.$$

As a result, we have, $\lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi_{n+k}) = 0$ or, in other words for $m > n$,

$$\lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m) = 0.$$

Case IV. Suppose that $M(\xi_{n-1}, \xi_n) = \rho_b(\xi_n, \xi_{n-1})$ for some $n \geq 1$. Then, (23) gives

$$\rho_b(\xi_n, \xi_{n+1}) \leq \varphi_b(M(\xi_{n-1}, \xi_n)) \leq \varphi_b(\rho_b(\xi_n, \xi_{n-1})) < \rho_b(\xi_n, \xi_{n-1}) \quad (30)$$

for all $n \geq 1$. Now, rewriting (9) and (10) for $n - 1$ we have

$$\begin{aligned} \rho_b(\xi_n, \xi_{n-1}) &= \rho_b(T\xi_{n-1}, T\xi_{n-2}) \\ &\leq \alpha(\xi_{n-1}, \xi_{n-2})\rho_b(T\xi_{n-1}, T\xi_{n-2}) \leq \varphi_b(M(\xi_{n-1}, \xi_{n-2})) \end{aligned} \quad (31)$$

where

$$M(\xi_{n-1}, \xi_{n-2}) \leq \max \{ \rho_b(\xi_n, \xi_{n-1}), \rho_b(\xi_{n-1}, \xi_{n-2}), \rho_b(\xi_{n-1}, \xi_n) \},$$

where obviously, the maximum can be either $\rho_b(\xi_n, \xi_{n-1})$ or $\rho_b(\xi_{n-1}, \xi_{n-2})$.

The first possibility, that is, if $M(\xi_{n-1}, \xi_{n-2}) \leq \rho_b(\xi_n, \xi_{n-1})$ for some $n \geq 1$, results in

$$\rho_b(\xi_n, \xi_{n-1}) \leq \varphi_b(M(\xi_{n-1}, \xi_{n-2}) \leq \varphi_b(\rho_b(\xi_n, \xi_{n-1})) < \rho_b(\xi_n, \xi_{n-1}),$$

which is a contradiction. Therefore, we should have $M(\xi_{n-1}, \xi_{n-2}) \leq \rho_b(\xi_{n-1}, \xi_{n-2})$ for all $n \geq 1$. Then the inequality (31) yields

$$\rho_b(\xi_n, \xi_{n-1}) \leq \varphi_b(M(\xi_{n-1}, \xi_{n-2}) \leq \varphi_b(\rho_b(\xi_{n-1}, \xi_{n-2})) < \rho_b(\xi_{n-1}, \xi_{n-2}) \quad (32)$$

for all $n \geq 1$. Thus, we deduce

$$\rho_b(\xi_n, \xi_{n-1}) \leq \varphi_b^{n-1}(\rho_b(\xi_1, \xi_0)), \text{ for all } n \geq 1. \quad (33)$$

If we combine the inequalities (30) with (33), we derive

$$\rho_b(\xi_n, \xi_{n+1}) < \rho_b(\xi_n, \xi_{n-1}) \leq \varphi_b^{n-1}(\rho_b(\xi_1, \xi_0)), \quad (34)$$

for all $n \geq 1$. As done in Case 2, when applying triangle inequality repeatedly for every $k \geq 1$, we get

$$\begin{aligned} \rho_b(\xi_n, \xi_{n+k}) &\leq s[\rho_b(\xi_n, \xi_{n+1}) + \rho_b(\xi_{n+1}, \xi_{n+k})] \\ &\leq s\rho_b(\xi_n, \xi_{n+1}) + s^2[\rho_b(\xi_{n+1}, \xi_{n+2}) + \rho_b(\xi_{n+2}, \xi_{n+k})] \\ &\vdots \\ &\leq s\rho_b(\xi_n, \xi_{n+1}) + s^2\rho_b(\xi_{n+1}, \xi_{n+2}) + \\ &\quad + \dots + s^{k-1}\rho_b(\xi_{n+k-2}, \xi_{n+k-1}) + s^k\rho_b(\xi_{n+k-1}, \xi_{n+k}) \\ &\leq s\varphi_b^{n-1}(\rho_b(\xi_1, \xi_0)) + s^2\varphi_b^n(\rho_b(\xi_1, \xi_0)) + \\ &\quad + \dots + s^{k-1}\varphi_b^{n+k-3}(\rho_b(\xi_1, \xi_0)) + s^k\varphi_b^{n+k-2}(\rho_b(\xi_1, \xi_0)) \\ &\leq \frac{1}{s^{n-2}} \left[s^{n-1}\varphi_b^{n-1}(\rho_b(\xi_1, \xi_0)) + \dots \right. \\ &\quad \left. + s^{n+k-3}\varphi_b^{n+k-3}(\rho_b(\xi_1, \xi_0)) + s^{n+k-2}\varphi_b^{n+k-2}(\rho_b(\xi_1, \xi_0)) \right] \\ &\leq \frac{1}{s^{n-2}} [S_{n+k-2} - S_{n-2}], \quad n \geq 1, k \geq 1, \end{aligned}$$

where S_n is defined in (13). Therefore, we end up with $\lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi_{n+k}) = 0$, or, equivalently, for $m > n$,

$$\lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m) = 0.$$

We conclude that $\{\xi_n\}$ is a right-Cauchy sequence, and hence, a Cauchy sequence. Moreover, it is a 0-Cauchy sequence in 0-complete quasi b -metric like space (X, ρ_b) . Thus, there exists $\xi \in X$ such that for $m > n$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \rho_b(\xi_n, \xi) &= \lim_{n \rightarrow \infty} \rho_b(\xi, \xi_n) = \rho_b(\xi, \xi) = 0 \\ &= \lim_{m, n \rightarrow \infty} \rho_b(\xi_m, \xi_n) = \lim_{m, n \rightarrow \infty} \rho_b(\xi_n, \xi_m). \end{aligned}$$

By the continuity of T , we obtain

$$\xi = \lim_{n \rightarrow \infty} \xi_{n+1} = \lim_{n \rightarrow \infty} T\xi_n = T \lim_{n \rightarrow \infty} \xi_n = T\xi,$$

that is, ξ is a fixed point of T . □

Theorem 2.4. Adding the condition

(U) For every pair ξ and η of fixed points of T , $\alpha(\xi, \eta) \geq 1$ and $\alpha(\eta, \xi) \geq 1$.
to the statement of Theorem 2.3 we obtain the uniqueness of the fixed point.

Proof. By Theorem 2.3 we know that the mapping T has at least one fixed point. To show the uniqueness, we assume that η is another fixed point of T , such that $\xi \neq \eta$. From the condition (U) we have, $\alpha(\xi, \eta) \geq 1$, hence, the contractive condition for the fixed points ξ and η , that is,

$$\rho_b(\xi, \eta) = \rho(T\xi, T\eta) \leq \alpha(\xi, \eta)\rho_b(T\xi, T\eta) \leq \varphi_b(M(\xi, \eta)),$$

where,

$$M(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta), \frac{1}{4s}[\rho_b(T\xi, \eta) + \rho_b(T\eta, \xi)]\} = \rho_b(\xi, \eta).$$

This yields

$$\rho_b(\xi, \eta) \leq \varphi_b(\rho_b(\xi, \eta)) < \rho_b(\xi, \eta),$$

which implies $\rho_b(\xi, \eta) = 0$. In a similar way, by changing the roles of ξ and η in the contractive condition we get

$$\rho_b(\eta, \xi) = \rho_b(T\eta, T\xi) \leq \alpha(\eta, \xi)\rho_b(T\eta, T\xi) \leq \varphi_b(M(\eta, \xi)),$$

where,

$$M(\eta, \xi) = \max\{\rho_b(\eta, \xi), \rho_b(T\eta, \eta), \rho_b(T\xi, \xi), \frac{1}{4s}[\rho_b(T\eta, \xi) + \rho_b(T\xi, \eta)]\} = \rho_b(\eta, \xi),$$

and thus,

$$\rho_b(\eta, \xi) \leq \varphi_b(\rho_b(\eta, \xi)) < \rho_b(\eta, \xi).$$

Therefore, we deduce $\rho_b(\eta, \xi) = 0$ and hence, $\xi = \eta$. This completes the uniqueness proof of the Theorem. $\square$

Regarding Remark 2.2, we can state the following result as an immediate consequence.

Corollary 2.5. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is an $\alpha - \varphi_b$ contractive mapping of type (B) or type (C) satisfying the following:*

- (i) T is α -admissible;
- (ii) there exists $\xi_0 \in X$ such that $\alpha(T\xi_0, \xi_0) \geq 1$ and $\alpha(\xi_0, T\xi_0) \geq 1$;
- (iii) T is continuous;
- (iv) For every pair ξ and η of fixed points of T , $\alpha(\xi, \eta) \geq 1$ and $\alpha(\eta, \xi) \geq 1$.

Then T has a unique fixed point.

We also state another consequence of the main result obtained by taking the function $\alpha(x, y) = 1$.

Corollary 2.6. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is a continuous mapping satisfying*

$$\rho_b(T\xi, T\eta) \leq \varphi_b(M(\xi, \eta)), \text{ for all } \xi, \eta \in X, \quad (35)$$

where

$$M(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta), \frac{1}{4s}[\rho_b(T\xi, \eta) + \rho_b(T\eta, \xi)]\}.$$

Then T has a unique fixed point.

Corollary 2.7. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is a continuous mapping satisfying*

$$\rho_b(T\xi, T\eta) \leq \varphi_b(N(\xi, \eta)), \text{ for all } \xi, \eta \in X \quad (36)$$

where

$$N(\xi, \eta) = \max\{\rho_b(\xi, \eta), \frac{1}{2s}[\rho_b(T\xi, \xi) + \rho_b(T\eta, \eta)], \frac{1}{4s}[\rho_b(T\xi, \eta) + \rho_b(T\eta, \xi)]\}.$$

or

$$\rho_b(T\xi, T\eta) \leq \varphi_b(K(\xi, \eta)), \text{ for all } \xi, \eta \in X, \quad (37)$$

where

$$K(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta)\}.$$

Then T has a unique fixed point.

We also point out that the main result of Gülyaz [8], that is the Theorem 1.16, follows from our main Theorem 2.3. In addition, by taking $\alpha(x, y) = 1$, we deduce another corollary.

Corollary 2.8. Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : X \rightarrow X$ is a continuous mapping satisfying

$$\rho_b(T\xi, T\eta) \leq \varphi_b(\rho_b(\xi, \eta)), \text{ for all } \xi, \eta \in X, \quad (38)$$

where $\varphi_b \in \Phi_b$. Then T has a unique fixed point.

It has been shown in some recent studies that by a suitable choice of the function α , the fixed point results on partially ordered spaces and fixed point results of cyclic contraction mappings can be concluded from the fixed point results of α -admissible mappings (see [19] for details). Using the idea of these studies we present some theorems and their consequences below.

We recall the definition of cyclic contraction mappings on quasi b -metric-like spaces introduced originally by Kirk et.al. [20].

Definition 2.9. Let X be nonempty set and let A and B be nonempty subsets of X . A mapping $T : A \cup B \rightarrow A \cup B$ is cyclic if $T(A) \subset B$ and $T(B) \subset A$.

We define generalized cyclic contraction of type as follows.

Definition 2.10. Let (X, ρ_b) be a quasi- b -metric-like space with a constant $s \geq 1$ and let $\varphi_b \in \Phi_b$ be a given function. A generalized cyclic contractive mapping $T : A \cup B \rightarrow A \cup B$ is a cyclic mapping satisfying

$$\rho_b(T\xi, T\eta) \leq \varphi_b(K(\xi, \eta)), \text{ for all } \xi \in A, \eta \in B, \quad (39)$$

where

$$K(\xi, \eta) = \max\{\rho_b(\xi, \eta), \rho_b(T\xi, \xi), \rho_b(T\eta, \eta)\}.$$

A fixed point theorem for the generalized cyclic contractions defined above is given next.

Theorem 2.11. Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$. Suppose that $T : A \cup B \rightarrow A \cup B$ is a generalized cyclic contractive mapping. Then T has a unique fixed point in $A \cap B$.

Proof. Define $\alpha : X \rightarrow X$ as follows.

$$\alpha(\xi, \eta) = \begin{cases} 1 & \text{if } \xi \in A, \eta \in B \\ 0 & \text{otherwise} \end{cases}.$$

Then all conditions of Corollary 2.5 are satisfied and T has a unique fixed point. $\square$

Finally, we observe that the main result (Theorem 2.9) in [12] is a consequence of Theorem 2.11. Indeed, by choosing $\varphi_b(t) = \frac{k}{s}t$ and noticing that $\rho_b(\xi, \eta) \leq M(\xi, \eta)$, the main result of Klin-eam and Suanoom [12] follows immediately from Theorem 2.11.

We end this section with illustrative examples.

Example 2.12. Let $X = [-1, 1]$ and $\rho_b(\xi, \eta) = |\xi - \eta|^2 + |2\xi + \eta|^2$. Then ρ_b is a quasi- b -metric like on X with a constant $s = 2$. Define $T\xi = \frac{\xi}{4}$. Then we see that

$$\begin{aligned} \rho_b(T\xi, T\eta) &= \left| \frac{\xi}{4} - \frac{\eta}{4} \right|^2 + \left| 2\frac{\xi}{4} + \frac{\eta}{4} \right|^2 \\ &= \frac{1}{16}|\xi - \eta|^2 + \frac{1}{16}|2\xi + \eta|^2 \\ &= \frac{1}{16}\rho_b(\xi, \eta) \leq \varphi_b(\rho_b(\xi, \eta)), \end{aligned}$$

where $\varphi_b(t) = kt$ with $\frac{1}{16} \leq k < \frac{1}{2}$ is clearly a (b) -comparison function. By the Corollary 2.8 T has a unique fixed point which is $\xi = 0$.

Example 2.13. Let $X = [0, 1]$ and $\rho_b(\xi, \eta) = |\xi - \eta|^2 + 2|\xi|^2 + |\eta|^2$. Then ρ_b is a quasi- b -metric like on X with a constant $s = 2$. Define $T\xi = \frac{1}{8}\xi^2 e^{-\xi^2}$. Without loss of generality assume that $\xi \geq \eta$. Then we have $e^{-\xi^2} \leq e^{-\eta^2} \leq 1$ in $[0, 1]$.

$$\begin{aligned} \rho_b(T\xi, T\eta) &= \left| \frac{1}{8}\xi^2 e^{-\xi^2} - \frac{1}{8}\eta^2 e^{-\eta^2} \right|^2 + 2 \left| \frac{1}{8}\xi^2 e^{-\xi^2} \right|^2 + \left| \frac{1}{8}\eta^2 e^{-\eta^2} \right|^2 \\ &\leq \frac{1}{64} |\xi^2 e^{-\eta^2} - \eta^2 e^{-\eta^2}|^2 + \frac{1}{32} |\xi^2 e^{-\xi^2}|^2 + \frac{1}{64} |\eta^2 e^{-\eta^2}|^2 \\ &= \frac{1}{64} |\xi^2 - \eta^2|^2 e^{-\eta^2} + \frac{1}{32} |\xi^2|^2 e^{-\xi^2} + \frac{1}{64} |\eta^2|^2 e^{-\eta^2} \\ &\leq \frac{1}{64} |\xi - \eta|^2 |\xi + \eta|^2 + \frac{1}{32} |\xi|^2 |\xi|^2 + \frac{1}{64} |\eta|^2 |\eta|^2 \\ &\leq \frac{1}{64} |\xi - \eta|^2 + \frac{1}{32} |\xi|^2 + \frac{1}{64} |\eta|^2 \\ &\leq \frac{1}{16} (|\xi - \eta|^2 + 2|\xi|^2 + |\eta|^2) = \frac{1}{16} \rho_b(\xi, \eta). \end{aligned}$$

Choosing a (b) -comparison function $\varphi_b(t) = kt$ where $\frac{1}{16} \leq k < \frac{1}{2}$ we see that the mapping T satisfies the conditions of the Corollary 2.8. Hence, T has a unique fixed point which is $\xi = 0$.

3 Ulam-Hyers stability

In this section we apply our main results to Ulam-Hyers stability problems. We first recall the classical definition of Ulam-Hyers stability on metric space.

Definition 3.1. Let (X, d) be a metric space and $T : X \rightarrow X$ be an operator. The fixed point equation

$$x = Tx, \quad x \in X \quad (40)$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing function $\varphi_b : [0, \infty) \rightarrow [0, \infty)$, continuous at 0 and $\varphi_b(0) = 0$ such that for each $\varepsilon > 0$ and for each ε -solution of the fixed point equation (40), that is, for each $\xi^* \in X$, satisfying the inequality

$$d(T\xi^*, \xi^*) \leq \varepsilon, \quad (41)$$

there exists a solution $x^* \in X$ of (40) such that

$$d(x^*, \xi^*) \leq \varphi_b(\varepsilon), \quad (42)$$

If $\varphi_b(t) := ct$ for all $t \in [0, \infty)$, where $c > 0$, the fixed point equation (40) is said to be Ulam-Hyers stable.

We refer the reader to Bota-Boriceanu, Petrusel [21], Lazăr [22], and Rus [23],[24] for some Ulam-Hyers stability results in the case of fixed point problems.

In the framework of quasi- b -metric like spaces the above definition can be stated as follows.

Definition 3.2. Let (X, ρ_b) be a quasi- b -metric-like space and $T : X \rightarrow X$ be an operator. The fixed point equation

$$x = Tx, \quad x \in X \quad (43)$$

is called generalized Ulam-Hyers stable if and only if there exists an increasing function $\varphi_b : [0, \infty) \rightarrow [0, \infty)$, continuous at 0 and $\varphi_b(0) = 0$ such that for each $\varepsilon > 0$ and for each ε -solution of the fixed point equation (43), that is, $\xi^* \in X$, satisfying the inequalities

$$\rho_b(T\xi^*, \xi^*) \leq \varepsilon \text{ and } \rho_b(\xi^*, T\xi^*) \leq \varepsilon \quad (44)$$

there exists a solution $x^* \in X$ of (43) such that

$$\rho_b(x^*, \xi^*) \leq \varphi_b(\varepsilon) \text{ and } \rho_b(\xi^*, x^*) \leq \varphi_b(\varepsilon) \quad (45)$$

If $\varphi_b(t) := ct$ for all $t \in [0, \infty)$, where $c > 0$, the fixed point equation (43) is said to be Ulam-Hyers stable.

In fact, the nature of the quasi- b -metric like space makes it possible to introduce two types of stability, namely, right and left stability. We define these new concepts in the following.

Definition 3.3. Let (X, ρ_b) be a quasi- b -metric-like space and $T : X \rightarrow X$ be an operator. The fixed point equation (43) is called generalized right Ulam-Hyers stable if and only if there exists an increasing function $\varphi_b : [0, \infty) \rightarrow [0, \infty)$, continuous at 0 and $\varphi_b(0) = 0$ such that for each $\varepsilon > 0$ and for each ξ^* , satisfying the inequality

$$\rho_b(\xi^*, T\xi^*) \leq \varepsilon \quad (46)$$

there exists a solution $x^* \in X$ of (43) such that

$$\rho_b(\xi^*, x^*) \leq \varphi_b(\varepsilon). \quad (47)$$

If $\varphi_b(t) := ct$ for all $t \in [0, \infty)$, where $c > 0$, the fixed point equation (43) is said to be right Ulam-Hyers stable.

Similarly, the fixed point equation (43) is called generalized left Ulam-Hyers stable if and only if there exists an increasing function $\varphi_b : [0, \infty) \rightarrow [0, \infty)$, continuous at 0 and $\varphi_b(0) = 0$ such that for each $\varepsilon > 0$ and for each ξ^* , satisfying the inequality

$$\rho_b(T\xi^*, \xi^*) \leq \varepsilon \quad (48)$$

there exists a solution $x^* \in X$ of (43) such that

$$\rho_b(x^*, \xi^*) \leq \varphi_b(\varepsilon). \quad (49)$$

If $\varphi_b(t) := ct$ for all $t \in [0, \infty)$, where $c > 0$, the fixed point equation (43) is said to be left Ulam-Hyers stable.

Clearly, if the fixed point equation (43) is both generalized right and left Ulam-Hyers stable, then it is generalized Ulam-Hyers stable.

We also recall the definition of well posed fixed point problems and apply it to quasi- b -metric-like spaces.

Definition 3.4. Let (X, ρ_b) be a quasi- b -metric-like space and $T : X \rightarrow X$ be a mapping. The fixed point problem (43) for T is said to be well-posed if it satisfies the following conditions:

- (i) T has a unique fixed point x^* in X ;
- (ii) for any sequence $\{\xi_n\}$ in X such that

$$\lim_{n \rightarrow \infty} \rho_b(\xi_n, T\xi_n) = \lim_{n \rightarrow \infty} \rho_b(T\xi_n, \xi_n) = 0,$$

one has

$$\lim_{n \rightarrow \infty} \rho_b(\xi_n, x^*) = \lim_{n \rightarrow \infty} \rho_b(x^*, \xi_n) = 0.$$

Inspired by the Ulam-Hyers stability problem and the ideas given in Petru et al. [25] we state and prove the following results.

Lemma 3.5. Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$ and $T : X \rightarrow X$ be a mapping. Let the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) := r - s\varphi_b(r)$ be strictly increasing and onto. Suppose that all the hypotheses of Corollary 2.8 are satisfied. Then the fixed point equation (43) is generalized right Ulam-Hyers stable.

Proof. According to the Corollary 2.8, there is a unique $x^* \in X$ such that $x^* = Tx^*$, that is, $x^* \in X$ is a solution of the fixed point equation (43). Let $\varepsilon > 0$ and let $\xi^* \in X$ satisfy

$$\rho_b(\xi^*, T\xi^*) \leq \varepsilon.$$

Using the contractive condition (38) with $x = \xi^*$ and $y = x^*$, and triangle inequality we obtain

$$\begin{aligned}\rho_b(\xi^*, x^*) &\leq s[\rho_b(\xi^*, T\xi^*) + \rho_b(T\xi^*, x^*)] \\ &= s[\rho_b(\xi^*, T\xi^*) + \rho_b(T\xi^*, Tx^*)] \\ &\leq s[\varepsilon + \varphi_b(\rho_b(\xi^*, x^*))].\end{aligned}$$

Then recalling the definition of β we have

$$\rho_b(\xi^*, x^*) - s\varphi_b(\rho_b(\xi^*, x^*)) = \beta(\rho_b(\xi^*, x^*)) \leq s\varepsilon,$$

or,

$$\rho_b(\xi^*, x^*) \leq \beta^{-1}(s\varepsilon),$$

and since β is continuous, strictly increasing and onto, then, β^{-1} is also increasing and continuous with $\beta^{-1}(0) = 0$. Hence, the fixed equation (43) is generalized left Ulam-Hyers stable. $\square$

The following Lemma regarding right Ulam-Hyers stability can be proved in a similar way, so we give the statement only without proof.

Lemma 3.6. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$ and $T : X \rightarrow X$ be a mapping. Let the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) := r - s\varphi_b(r)$ be strictly increasing and onto. Suppose that all the hypotheses of Corollary 2.8 are satisfied. Then the fixed point equation (43) is generalized left Ulam-Hyers stable.*

Using the above lemmas, we can prove the following theorem.

Theorem 3.7. *Let (X, ρ_b) be a 0-complete quasi- b -metric-like space with a constant $s \geq 1$ and $T : X \rightarrow X$ be a mapping. Let the function $\beta : [0, \infty) \rightarrow [0, \infty)$, $\beta(r) := r - s\varphi_b(r)$ be strictly increasing and onto. Suppose that all the hypotheses of Corollary 2.8 are satisfied. Then the following hold.*

- (a) *The fixed point equation (43) is generalized Ulam-Hyers stable.*
- (b) *If $\{\xi_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} \rho_b(\xi_n, T\xi_n) = \lim_{n \rightarrow \infty} \rho_b(T\xi_n, \xi_n) = 0$ and x^* is a fixed point of T , then the fixed point problem (43) is well-posed.*
- (c) *If $G : X \rightarrow X$ is a mapping such that there exists $\eta > 0$ with*

$$\rho_b(T\xi, G\xi) \leq \eta, \text{ and } \rho_b(G\xi, T\xi) \leq \eta \quad \forall \xi \in X, \quad (50)$$

then for any fixed point y^ of G we have*

$$\rho_b(x^*, y^*) \leq \beta^{-1}(s\eta). \quad (51)$$

Proof. (a) By Lemma 3.5 and Lemma 3.6 the fixed point equation (43) is generalized right and left Ulam-Hyers stable. Therefore, it is generalized Ulam-Hyers stable.

(b) Let $\{\xi_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} \rho_b(\xi_n, T\xi_n) = \lim_{n \rightarrow \infty} \rho_b(T\xi_n, \xi_n) = 0$ and let x^* be the unique fixed point of T . Employing the contractive condition and triangle inequality we get

$$\begin{aligned}\rho_b(\xi_n, x^*) &\leq s[\rho_b(\xi_n, T\xi_n) + \rho_b(T\xi_n, x^*)] \\ &= s[\rho_b(\xi_n, T\xi_n) + \rho_b(T\xi_n, Tx^*)] \\ &\leq s[\rho_b(\xi_n, T\xi_n) + \varphi_b(\rho_b(\xi_n, x^*))],\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, we have,

$$\begin{aligned}\beta(\rho_b(\xi_n, x^*)) &= \rho_b(\xi_n, x^*) - s\varphi_b(\rho_b(\xi_n, x^*)) \\ &\leq s\rho_b(\xi_n, T\xi_n)\end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \rho_b(\xi_n, T\xi_n) = 0$, we obtain

$$\lim_{n \rightarrow \infty} \beta(\rho_b(\xi_n, x^*)) = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} \rho_b(\xi_n, x^*) = 0. \quad (52)$$

On the other hand, the contractive condition with $\xi = x^*$ and $\eta = \xi_n$ and triangle inequality yields

$$\begin{aligned} \rho_b(x^*, \xi_n) &\leq s[\rho_b(x^*, T\xi_n) + \rho_b(T\xi_n, \xi_n)] \\ &= s[\rho_b(Tx^*, T\xi_n) + \rho_b(T\xi_n, \xi_n)] \\ &\leq s[\varphi_b(\rho_b(x^*, \xi_n)) + \rho_b(T\xi_n, \xi_n)], \end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} \beta(\rho_b(x^*, \xi_n)) &= \rho_b(x^*, \xi_n) - s\varphi_b(\rho_b(x^*, \xi_n)) \\ &\leq s\rho_b(T\xi_n, \xi_n), \end{aligned}$$

for all $n \in \mathbb{N}$ and regarding $\lim_{n \rightarrow \infty} \rho_b(T\xi_n, \xi_n) = 0$, we get

$$\lim_{n \rightarrow \infty} \beta(\rho_b(x^*, \xi_n)) = 0.$$

Consequently,

$$\lim_{n \rightarrow \infty} \rho_b(x^*, \xi_n) = 0. \quad (53)$$

As a result, the limits in (52) and (53) show that the fixed point problem (43) is well-posed.

(c) To show the last part of the Theorem we assume that $G : X \rightarrow X$ is a mapping such that there exists $\eta > 0$ with

$$\rho_b(T\xi, G\xi) \leq \eta, \text{ and } \rho_b(G\xi, T\xi) \leq \eta \quad \forall \xi \in X, \quad (54)$$

and let y^* be a fixed point of G . Then, by the triangle inequality, we have

$$\begin{aligned} \rho_b(x^*, y^*) &= \rho_b(Tx^*, Gy^*) \leq s[\rho_b(Tx^*, Ty^*) + \rho_b(Ty^*, Gy^*)] \\ &\leq s[\varphi_b(\rho_b(x^*, y^*)) + \rho_b(Ty^*, Gy^*)]. \end{aligned}$$

Then we have

$$\beta(\rho_b(x^*, y^*)) = \rho_b(x^*, y^*) - s\varphi_b(\rho_b(x^*, y^*)) \leq s\rho_b(Ty^*, Gy^*) \leq s\eta,$$

and hence

$$\rho_b(x^*, y^*) \leq \beta^{-1}(s\eta). \quad (55)$$

In a similar way, by changing the roles of x^* and y^* in the discussion above we obtain

$$\rho_b(y^*, x^*) \leq \beta^{-1}(s\eta), \quad (56)$$

which completes the proof. $\square$

4 Conclusion

The main contribution of this study to fixed point theory is the existence-uniqueness result given in Theorem 2.3. This theorem provides the existence and uniqueness conditions for a very large class of contractive mappings on various abstract spaces. In fact, the quasi- b -metric-like spaces cover the b -metric spaces, quasi- b -metric spaces, b -metric-like spaces as well as b -metric, usual quasi-metric and metric spaces.

However, the most significant part of the study is the definition of right and left Ulam-Hyers stability to get some new Ulam-Hyers stability results. These results have applications in many areas of applied mathematics, especially in differential and integral equations. A study related particularly to the applications of the present results in the theory of ordinary differential equations is currently in progress.

Competing interests

The authors declare that they have no competing interests.

Authors' contribution

All authors contributed equally to this work. All authors read and approved the final manuscript.

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