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Research Article

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The general solution of impulsive systems with Riemann-Liouville fractional derivatives

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Abstract: In this paper, we study a kind of fractional differential system with impulsive effect and find the formula of general solution for the impulsive fractional-order system by analysis of the limit case (as impulse tends to zero). The obtained result shows that the deviation caused by impulses for fractional-order system is undetermined. An example is also provided to illustrate the result.

Keywords: Fractional differential equations, Riemann-Liouville fractional derivative, Impulse, General solution

MSC: 34A08, 34A37

1 Introduction

Fractional calculus was utilized as a powerful tool to reveal the hidden aspects of the dynamics of the complex or hypercomplex systems [1-3], and the subject of fractional differential equations is gaining much attention. For details see [4-14] and the references therein.

Impulsive effects exist widely in many processes in which their states can be described by impulsive differential equations. Moreover, in case of impulsive differential equations with Caputo fractional derivative there have been numerous works about the subject [15-23], and impulsive fractional partial differential equations are widely considered in [24-29].

Motivated by the above-mentioned works, we will study the following impulsive Cauchy problem with Riemann-Liouville fractional derivative:

$$\begin{cases} D_{a+}^{q}u(t) = f(t,u(t)), \quad t \in (a,T] \text{ and } t \neq t_i \ (i = 1,2,...,m), \\ \Delta\left(\mathcal{J}_{a+}^{1-q}u\right)\Big|_{t=t_i} = \mathcal{J}_{a+}^{1-q}u(t_i^+) - \mathcal{J}_{a+}^{1-q}u(t_i^-) = \Delta_i\left(u(t_i^-)\right), \quad i = 1,2,...,m, \\ \mathcal{J}_{a+}^{1-q}u(a) = u_a, \quad u_a \in \mathbb{C}, \end{cases} \tag{1}$$

where $q \in \mathbb{C}$ and $\Re(q) \in (0,1)$, D_{a+}^{q} denotes left-sided Riemann-Liouville fractional derivative of order q and $\mathcal{J}_{a+}^{1-q}$ denotes left-sided Riemann-Liouville fractional integral of order $1-q$. $f : J \times \mathbb{C} \to \mathbb{C}$ is an appropriate continuous function, and $a = t_0 < t_1 < ... < t_m < t_{m+1} = T$. Here $\mathcal{J}_{a+}^{1-q}u(t_i^+) = \lim_{\varepsilon\to 0^+} \mathcal{J}_{a+}^{1-q}u(t_i+\varepsilon)$ and $\mathcal{J}_{a+}^{1-q}u(t_i^-) = \lim_{\varepsilon\to 0^-} \mathcal{J}_{a+}^{1-q}u(t_i+\varepsilon)$ represent the right and left limits of $\mathcal{J}_{a+}^{1-q}u(t)$ at $t = t_i$, respectively.

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For impulsive system (1), we have

$$\lim_{\Delta_1\to 0,\Delta_2\to 0,\cdots,\Delta_m\to 0}\{\text{impulsive system (1)}\} = \begin{cases} D^q_{a+}u(t) = f(t,u(t)), & q\in(0,1),\ t\in(a,T], \\ \mathcal{J}^{1-q}_{a+}u(a) = u_a, & u_0\in\mathbb{C}, \end{cases} \tag{2}$$

Therefore, it means that there exists a hidden condition

$$\lim_{\Delta_1\to 0,\Delta_2\to 0,\cdots,\Delta_m\to 0}\{\text{the solution of impulsive system (1)}\} = \{\text{the solution of system (2)}\} \tag{3}$$

Therefore, the definition of solution for impulsive system (1) is provided as follows:

Definition 1.1. *A function $z(t) : [a,T] \to \mathbb{C}$ is said to be a solution of the fractional Cauchy problem* (1) *if $\mathcal{J}^{1-q}_{a+}z(a) = u_a$, the equation condition $D^q_{a+}z(t) = f(t,z(t))$ for each $t\in(a,T]$ is verified, the impulsive conditions $\Delta\left(\mathcal{J}^{1-q}_{a+}z\right)\Big|_{t=t_i} = \Delta_i(z(t_i^-)$ (here $i = 1,2,...,m$)are satisfied, the restriction of $z(t)$ to the interval $(t_k,t_{k+1}]$ (here $k = 0,1,2,...,m$) is continuous, and the condition* (3) *holds.*

Therefore, we will consider impulsive system (1) and seek some solutions of impulsive system (1) according to Definition 1.1.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we give the formula of general solution for impulsive differential equations with Riemann-Liouville fractional derivatives. In Section 4, an example is provided to expound the main result.

2 Preliminaries

Firstly, we recall some concepts of fractional calculus [2] and a property for nonlinear fractional differential equations.

Definition 2.1. *The left-sided Riemann-Liouville fractional integral of order $\alpha\in\mathbb{C}$ ($\Re(\alpha)>0$) of function $x(t)$ is defined by*

$$\mathcal{J}^\alpha_{a+}x(t) = \frac{1}{\Gamma(q)}\int_a^t (t-s)^{\alpha-1}x(s)ds, \quad t>a,$$

where Γ is the gamma function.

Definition 2.2. *The left-sided Riemann-Liouville fractional derivative of order $q\in\mathbb{C}$ ($\Re(q)\geq 0$) of function $x(t)$ is defined by*

$$D^q_{a+}x(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n\int_a^t (t-s)^{n-q-1}x(s)ds,\ \ n=[\Re(q)]+1,\ t>a.$$

By Lemma 2.2 in [11], the initial value problem

$$\begin{cases} D^q_{a+}u(t) = f(t,u(t)), & q\in\mathbb{C}\ and\ \Re(q)\in(0,1),\ t\in(a,T], \\ \mathcal{J}^{1-q}_{a+}u(a) = u_a, & u_a\in\mathbb{C}, \end{cases} \tag{4}$$

is equivalent to the following nonlinear Volterra integral equation of the second kind,

$$u(t) = \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1}f(s,u(s))ds. \tag{5}$$

3 Main results

Define a piecewise function

$$\tilde{u}(t) = \frac{1}{\Gamma(q)} \mathcal{J}_{a+}^{1-q} u(t_k^+) (t - t_k)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s)) ds$$
$$\text{for } t \in (t_k, t_{k+1}] \text{ (where } k = 0, 1, 2, ..., m)$$

with $\mathcal{J}_{a+}^{1-q} u(t_k^+) = \mathcal{J}_{a+}^{1-q} u(t_k^-) + \Delta_k \left(u(t_k^-)\right)$. By Definition 2.2, we have

$$\begin{aligned} D_{a+}^{q} \tilde{u}(t) &= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \left\{ \int_{a}^{t} (t-\eta)^{1-q-1} \left[\mathcal{J}_{a+}^{1-q} u(t_k^+) (\eta - t_k)^{q-1} + \int_{t_k}^{\eta} (\eta - s)^{q-1} f(s, u(s)) ds \right] d\eta \right\} \\ &= \frac{1}{\Gamma(1-q)\Gamma(q)} \frac{d}{dt} \left\{ \int_{t_k}^{t} (t-\eta)^{1-q-1} \left[\mathcal{J}_{a+}^{1-q} u(t_k^+) (\eta - t_k)^{q-1} + \int_{t_k}^{\eta} (\eta - s)^{q-1} f(s, u(s)) ds \right] d\eta \right\} \\ &= f(t, u(t))|_{t \in (t_k, t_{k+1}]} \end{aligned}$$

So, $\tilde{u}(t)$ satisfies the condition of fractional derivative of (1), and it doesn't satisfy the condition (3). Thus, we assume that $\tilde{u}(t)$ is an approximate solution to seek the exact solution of impulsive system (1).

Theorem 3.1. *Let ξ be a constant. A function $u(t)$ is a general solution of system* (1) *if and only if $u(t)$ satisfies the fractional integral equation*

$$u(t) = \begin{cases} \dfrac{u_a}{\Gamma(q)} (t-a)^{q-1} + \dfrac{1}{\Gamma(q)} \displaystyle\int_{a}^{t} (t-s)^{q-1} f(s, u(s)) ds \ \ \textit{for } t \in (a, t_1], \\ \dfrac{u_a}{\Gamma(q)} (t-a)^{q-1} + \dfrac{1}{\Gamma(q)} \displaystyle\int_{a}^{t} (t-s)^{q-1} f(s, u(s)) ds + \sum_{i=1}^{k} \dfrac{\Delta_i \left(u(t_i^-)\right)}{\Gamma(q)} (t - t_i)^{q-1} \\ \quad - \displaystyle\sum_{i=1}^{k} \dfrac{\xi \Delta_i \left(u(t_i^-)\right)}{\Gamma(q)} \Bigg\{ u_a (t-a)^{q-1} + \int_{a}^{t} (t-s)^{q-1} f(s, u(s)) ds \\ \quad - \left[u_a + \displaystyle\int_{a}^{t_i} f(s, u(s)) ds \right] (t - t_i)^{q-1} - \displaystyle\int_{t_i}^{t} (t-s)^{q-1} f(s, u(s)) ds \Bigg\} \ \ \textit{for } t \in (t_k, t_{k+1}], \end{cases} \tag{6}$$

provided that the integral in (6) *exists.*

Proof. "Necessity". First we can easily verify that Eq. (6) satisfies the hidden condition (3).

Next, Taking Riemann-Liouville fractional derivative to Eq. (6) for each $t \in (t_k, t_{k+1}]$ (where $k = 0, 1, 2, ..., m$), we have

$$\begin{aligned} D_{a+}^{q} u(t) = D_{a+}^{q} \Bigg\{ & \frac{u_a}{\Gamma(q)} (t-a)^{q-1} + \frac{1}{\Gamma(q)} \int_{a}^{t} (t-s)^{q-1} f(s, u(s)) ds + \sum_{i=1}^{k} \frac{\Delta_i \left(u(t_i^-)\right)}{\Gamma(q)} (t - t_i)^{q-1} \\ & - \sum_{i=1}^{k} \frac{\xi \Delta_i \left(u(t_i^-)\right)}{\Gamma(q)} \Bigg[u_a (t-a)^{q-1} + \int_{a}^{t} (t-s)^{q-1} f(s, u(s)) ds \\ & - \left(u_a + \int_{a}^{t_i} f(s, u(s)) ds \right) (t - t_i)^{q-1} - \int_{t_i}^{t} (t-s)^{q-1} f(s, u(s)) ds \Bigg] \Bigg\} \end{aligned}$$

$$= \left\{ f(t,u(t))_{t\geq a} - \xi \sum_{i=1}^{k} \Delta_i \left(u(t_i^-)\right) \left[f(t,u(t))_{t\geq a} - f(t,u(t))_{t\geq t_i} \right] \right\}_{t\in(t_k,t_{k+1}]}$$
$$= f(t,u(t))|_{t\in(t_k,t_{k+1}]} .$$

So, Eq. (6) satisfies Riemann-Liouville fractional derivative of system (1). Using (6) for each t_k (here $k = 1, 2, ..., m$), we get

$$\begin{aligned}
&\mathcal{J}_{a+}^{1-q} u(t_k^+) - \mathcal{J}_{a+}^{1-q} u(t_k^-) \\
&= \left\{ \frac{1}{\Gamma(1-q)} \int_a^t (t-\eta)^{1-q-1} u(\eta) d\eta \right\}_{t\to t_k^+} - \left\{ \frac{1}{\Gamma(1-q)} \int_a^t (t-\eta)^{1-q-1} u(\eta) d\eta \right\}_{t=t_k} \\
&= \Delta_k \left(u(t_k^-)\right) - \xi \Delta_k \left(u(t_k^-)\right) \left[u_a + \int_a^t f(s,u(s))ds - \left(u_a + \int_a^{t_k} f(s,u(s))ds \right) - \int_{t_k}^t f(s,u(s))ds \right]_{t\to t_k} \\
&= \Delta_k \left(u(t_k^-)\right).
\end{aligned}$$

Therefore, Eq. (6) satisfies impulsive conditions of (1). Then, Eq. (6) satisfies the conditions of system (1).

"Sufficiency". We prove that the solutions of system (1) satisfy Eq. (6) by mathematical induction. By Definition 2.1, the solution of (1) satisfies

$$u(t) = \frac{u_a}{\Gamma(q)} (t-a)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds \quad \text{for } t \in (a, t_1]. \tag{7}$$

By (7), we have $\mathcal{J}_{a+}^{1-q} u(t_1^+) = \mathcal{J}_{a+}^{1-q} u(t_1^-) + \Delta_1 \left(u(t_1^-)\right) = u_a + \Delta_1 \left(u(t_1^-)\right) + \int_a^{t_1} f(s,u(s))ds$, and the approximate solution $\tilde{u}(t)$ (for $t \in (t_1, t_2]$) is given by

$$\begin{aligned}
\tilde{u}(t) &= \frac{1}{\Gamma(q)} \mathcal{J}_{a+}^{1-q} u(t_1^+)(t-t_1)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds \\
&= \frac{u_a + \Delta_1 \left(u(t_1^-)\right) + \int_a^{t_1} f(s,u(s))ds}{\Gamma(q)} (t-t_1)^{q-1} + \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds \\
&\text{for } t \in (t_1, t_2],
\end{aligned} \tag{8}$$

with $e_1(t) = u(t) - \tilde{u}(t)$ for $t \in (t_1, t_2]$. By

$$\lim_{\Delta_1(u(t_1^-))\to 0} u(t) = \tfrac{u_a}{\Gamma(q)} (t-a)^{q-1} + \tfrac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds \text{ (for } t \in (t_1, t_2]),$$

we get

$$\begin{aligned}
\lim_{\Delta_1(u(t_1^-))\to 0} e_1(t) &= \lim_{\Delta_1(u(t_1^-))\to 0} \{u(t) - \tilde{u}(t)\} \\
&= \frac{u_a}{\Gamma(q)} (t-a)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds - \frac{u_a + \int_a^{t_1} f(s,u(s))ds}{\Gamma(q)} (t-t_1)^{q-1} \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds.
\end{aligned}$$

Then, we assume

$$\begin{aligned} e_1(t) &= \sigma\left(\Delta_1(u(t_1^-))\right) \lim_{\Delta_1(u(t_1^-))\to 0} e_1(t) \\ &= \frac{\sigma\left(\Delta_1(u(t_1^-))\right)}{\Gamma(q)}\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds - \left(u_a + \int_a^{t_1} f(s,u(s))ds\right)(t-t_1)^{q-1}\right. \\ &\quad \left. - \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds\right]. \end{aligned}$$

where function $\sigma(.)$ is an undetermined function with $\sigma(0) = 1$. Thus,

$$\begin{aligned} u(t) &= \tilde{u}(t) + e_1(t) \\ &= \frac{1}{\Gamma(q)}\left\{\sigma\left(\Delta_1(u(t_1^-))\right)\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right] + \Delta_1(u(t_1^-))(t-t_1)^{q-1}\right. \\ &\quad \left. + \left[1-\sigma\left(\Delta_1(u(t_1^-))\right)\right]\left[\left(u_a + \int_a^{t_1} f(s,u(s))ds\right)(t-t_1)^{q-1} + \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds\right]\right\} \\ &\quad \text{for } t \in (t_1, t_2]. \end{aligned} \tag{9}$$

Using (9), we get $\mathcal{J}_{a+}^{1-q}u(t_2^+) = \mathcal{J}_a^{1-q}u(t_2^-) + \Delta_2\left(u(t_2^-)\right) = u_a + \Delta_1\left(u(t_1^-)\right) + \Delta_2\left(u(t_2^-)\right) + \int_a^{t_2} f(s,u(s))ds$. Therefore, the approximate solution $\tilde{u}(t)$ (for $t \in (t_2, t_3]$)is given by

$$\begin{aligned} \tilde{u}(t) &= \frac{1}{\Gamma(q)}\left(\mathcal{J}_{a+}^{1-q}u(t_2^+)\right)(t-t_2)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds \\ &= \frac{u_a + \Delta_1\left(u(t_1^-)\right) + \Delta_2\left(u(t_2^-)\right) + \int_a^{t_2} f(s,u(s))ds}{\Gamma(q)}(t-t_2)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds \\ &\quad \text{for } t \in (t_2, t_3] \end{aligned} \tag{10}$$

with $e_2(t) = u(t) - \tilde{u}(t)$ for $t \in (t_2, t_3]$. Moreover, by (9), the exact solution $u(t)$ of (1) satisfies

$$\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \Delta_2(u(t_2^-))\to 0}} u(t) = \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s))ds \quad \text{for } t \in (t_2, t_3],$$

$$\begin{aligned} &\lim_{\Delta_1(u(t_1^-))\to 0} u(t) \\ &= \frac{1}{\Gamma(q)}\left\{\sigma(\Delta_2(u(t_2^-)))\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right] + \Delta_2(u(t_2^-))(t-t_2)^{q-1}\right. \\ &\quad \left. + \left[1-\sigma(\Delta_2(u(t_2^-)))\right]\left[\left(u_a + \int_a^{t_2} f(s,u(s))ds\right)(t-t_2)^{q-1} + \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds\right]\right\} \\ &\quad \text{for } t \in (t_2, t_3], \end{aligned}$$

$$\lim_{\Delta_2(u(t_2^-))\to 0} u(t)$$
$$= \frac{1}{\Gamma(q)}\Bigg\{\sigma\left(\Delta_1(u(t_1^-))\right)\left[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right] + \Delta_1(u(t_1^-))\,(t-t_1)^{q-1}$$
$$+\left[1-\sigma\left(\Delta_1(u(t_1^-))\right)\right]\left[\left(u_a + \int_a^{t_1} f(s,u(s))ds\right)(t-t_1)^{q-1} + \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds\right]\Bigg\}$$

for $t \in (t_2, t_3]$.

Therefore,

$$\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \Delta_2(u(t_2^-))\to 0}} e_2(t) = \lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \Delta_2(u(t_2^-))\to 0}} \{u(t)-\tilde{u}(t)\}$$
$$= \frac{1}{\Gamma(q)}\Bigg[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds - \left(u_a + \int_a^{t_2} f(s,u(s))ds\right)(t-t_2)^{q-1} \tag{11}$$
$$-\int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds\Bigg],$$

$$\lim_{\Delta_1(u(t_1^-))\to 0} e_2(t) = \lim_{\Delta_1(u(t_1^-))\to 0} \{u(t)-\tilde{u}(t)\}$$
$$= \frac{\sigma\left(\Delta_2(u(t_2^-))\right)}{\Gamma(q)}\Bigg[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \tag{12}$$
$$-\left(u_a + \int_a^{t_2} f(s,u(s))ds\right)(t-t_2)^{q-1} - \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds\Bigg],$$

$$\lim_{\Delta_2(u(t_2^-))\to 0} e_2(t) = \lim_{\Delta_2(u(t_2^-))\to 0} \{u(t)-\tilde{u}(t)\}$$
$$= \frac{1}{\Gamma(q)}\Bigg\{\sigma\left(\Delta_1(u(t_1^-))\right)\left[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right]$$
$$+ \Delta_1(u(t_1^-))\,(t-t_1)^{q-1} - \Delta_1(u(t_1^-))\,(t-t_2)^{q-1} \tag{13}$$
$$+\left[1-\sigma(\Delta_1(u(t_1^-)))\right]\left[\left(u_a + \int_a^{t_1} f(s,u(s))ds\right)(t-t_1)^{q-1} + \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds\right]$$
$$-\left(u_a + \Delta_1(u(t_1^-)) + \int_a^{t_2} f(s,u(s))ds\right)(t-t_2)^{q-1} - \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds\Bigg\}.$$

Then, by $(11)-(13)$, we obtain

$$
\begin{aligned}
e_2(t) = \frac{1}{\Gamma(q)} \Bigg\{ & \left[\sigma\left(\Delta_1(u(t_1^-))\right) + \sigma\left(\Delta_2(u(t_2^-))\right) - 1\right] \left[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \right] \\
& + \Delta_1(u(t_1^-))\,(t-t_1)^{q-1} - \Delta_1(u(t_1^-))\,(t-t_2)^{q-1} \\
& + \left[1-\sigma(\Delta_1(u(t_1^-))\right] \left[\left(u_a + \int_a^{t_1} f(s,u(s))ds \right)(t-t_1)^{q-1} + \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds \right] \\
& -\sigma\left(\Delta_2(u(t_2^-))\right) \left[\left(u_a + \int_a^{t_2} f(s,u(s))ds \right)(t-t_2)^{q-1} + \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds \right] \Bigg\}.
\end{aligned} \tag{14}
$$

Thus,

$$
\begin{aligned}
u(t) &= \tilde{u}(t) + e_2(t) \\
&= \frac{1}{\Gamma(q)} \Bigg\{ \left[\sigma\left(\Delta_1(u(t_1^-))\right) + \sigma\left(\Delta_2(u(t_2^-))\right) - 1\right] \left[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \right] \\
&\quad + \Delta_1(u(t_1^-))\,(t-t_1)^{q-1} + \Delta_2(u(t_2^-))\,(t-t_2)^{q-1} \\
&\quad + \left[1-\sigma(\Delta_1(u(t_1^-))\right] \left[\left(u_a + \int_a^{t_1} f(s,u(s))ds \right)(t-t_1)^{q-1} + \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds \right] \\
&\quad + \left[1-\sigma(\Delta_2(u(t_2^-)))\right] \left[\left(u_a + \int_a^{t_2} f(s,u(s))ds \right)(t-t_2)^{q-1} + \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds \right] \Bigg\} \\
&\quad \text{for } t \in (t_2, t_3].
\end{aligned} \tag{15}
$$

Moreover, letting $t_2 \to t_1$, we have

$$
\lim_{t_2 \to t_1} \begin{cases} D_{a+}^q u(t) = f(t,u(t)), \;\; q \in \mathbb{C} \text{ and } \Re(q) \in (0,1), \; t \in (a,t_3] \text{ and } t \neq t_1 \text{ and } t \neq t_2, \\ \Delta\left(\mathcal{J}_{a+}^{1-q} u\right)\Big|_{t=t_k} = \mathcal{J}_{a+}^{1-q} u(t_k^+) - \mathcal{J}_{a+}^{1-q} u(t_k^-) = \Delta_k(u(t_k^-)), \;\; k = 1,2, \\ \mathcal{J}_{a+}^{1-q} u(a) = u_a, \;\; u_a \in \mathbb{C}, \end{cases} \tag{16}
$$

$$
= \begin{cases} D_{a+}^q u(t) = f(t,u(t)), \;\; q \in \mathbb{C} \text{ and } \Re(q) \in (0,1), \; t \in (a,t_3] \text{ and } t \neq t_1, \\ \Delta\left(\mathcal{J}_{a+}^{1-q} u\right)\Big|_{t=t_1} = \mathcal{J}_{a+}^{1-q} u(t_1^+) - \mathcal{J}_{a+}^{1-q} u(t_1^-) + \mathcal{J}_{a+}^{1-q} u(t_2^+) - \mathcal{J}_{a+}^{1-q} u(t_2^-) \\ \qquad\qquad = \Delta_1(u(t_1^-)) + \Delta_2(u(t_2^-)), \\ \mathcal{J}_{a+}^{1-q} u(a) = u_a, \;\; u_a \in \mathbb{C}, \end{cases} \tag{17}
$$

Using (9) and (15), we have $1-\sigma(\Delta_1+\Delta_2) = 1-\sigma(\Delta_1) + 1-\sigma(\Delta_2)$. Letting $\rho(z) = 1-\sigma(z)$, we get $\rho(z+w) = \rho(z)+\rho(w)$ for $\forall z, w \in \mathbb{C}$. So, $\rho(z) = \xi z$, here ξ is a constant. Thus,

$$
\begin{aligned}
u(t) = & \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds + \frac{\Delta_1(u(t_1^-))}{\Gamma(q)}(t-t_1)^{q-1} \\
& - \frac{\xi \Delta_1(u(t_1^-))}{\Gamma(q)} \Bigg[u_a\,(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \\
& - \left(u_a + \int_a^{t_1} f(s,u(s))ds \right)(t-t_1)^{q-1} - \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds \Bigg] \quad \text{for } t \in (t_1, t_2].
\end{aligned} \tag{18}
$$

and

$$u(t) = \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s))ds + \frac{\Delta_1(u(t_1^-))}{\Gamma(q)}(t-t_1)^{q-1}$$
$$+ \frac{\Delta_2(u(t_2^-))}{\Gamma(q)}(t-t_2)^{q-1} - \frac{\xi\Delta_1(u(t_1^-))}{\Gamma(q)}\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right.$$
$$\left. - \left(u_a + \int_a^{t_1} f(s,u(s))ds\right)(t-t_1)^{q-1} - \int_{t_1}^t (t-s)^{q-1} f(s,u(s))ds\right] \tag{19}$$
$$- \frac{\xi\Delta_2(u(t_2^-))}{\Gamma(q)}\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right.$$
$$\left. - \left(u_a + \int_a^{t_2} f(s,u(s))ds\right)(t-t_2)^{q-1} - \int_{t_2}^t (t-s)^{q-1} f(s,u(s))ds\right] \text{ for } t \in (t_2,t_3].$$

Next, for $t \in (t_n, t_{n+1}]$, suppose

$$u(t) = \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s))ds + \sum_{i=1}^{n} \frac{\Delta_i\left(u(t_i^-)\right)}{\Gamma(q)}(t-t_i)^{q-1}$$
$$- \sum_{i=1}^{n} \frac{\xi\Delta_i\left(u(t_i^-)\right)}{\Gamma(q)}\left\{u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right. \tag{20}$$
$$\left. - \left[u_a + \int_a^{t_i} f(s,u(s))ds\right](t-t_i)^{q-1} - \int_{t_i}^t (t-s)^{q-1} f(s,u(s))ds\right\} \quad \text{for } t \in (t_n,t_{n+1}].$$

Using (20),we have

$$\mathcal{J}_{a+}^{1-q}u(t_{n+1}^+) = \mathcal{J}_{a+}^{1-q}u(t_{n+1}^-) + \Delta_{n+1}(u(t_{n+1}^-)) = u_a + \sum_{i=1}^{n+1}\Delta_i(u(t_i^-)) + \int_a^{t_{n+1}} f(s,u(s))ds$$

Thus, the approximate solution $\tilde{u}(t)$ for $t \in (t_{n+1}, t_{n+2}]$ is given by

$$\tilde{u}(t) = \frac{1}{\Gamma(q)}\left(\mathcal{J}_{a+}^{1-q}u(t_{n+1}^+)\right)(t-t_{n+1})^{q-1} + \frac{1}{\Gamma(q)}\int_{t_{n+1}}^t (t-s)^{q-1} f(s,u(s))ds$$
$$= \frac{u_a + \sum_{i=1}^{n+1}\Delta_i(u(t_i^-)) + \int_a^{t_{n+1}} f(s,u(s))ds}{\Gamma(q)}(t-t_{n+1})^{q-1} + \frac{1}{\Gamma(q)}\int_{t_{n+1}}^t (t-s)^{q-1} f(s,u(s))ds \tag{21}$$
$$\text{for } t \in (t_{n+1},t_{n+2}]$$

with $e_{n+1}(t) = u(t) - \tilde{u}(t)$ for $t \in (t_{n+1}, t_{n+2}]$. By (20), the exact solution $u(t)$ of (1) satisfies

$$\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \vdots \\ \Delta_{n+1}\left(u(t_{n+1}^-)\right)\to 0}} u(t) = \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s))ds \quad \text{for } t \in (t_{n+1},t_{n+2}],$$

$$\lim_{\substack{\Delta_j(u(t_j^-))\to 0,\\ 1\le j\le n+1}} u(t)$$
$$= \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f(s,u(s))ds + \sum_{\substack{1\le i\le n+1,\\ \text{and } i\ne j}} \frac{\Delta_i\left(u(t_i^-)\right)}{\Gamma(q)}(t-t_i)^{q-1}$$
$$- \sum_{\substack{1\le i\le n+1,\\ \text{and } i\ne j}} \frac{\xi\Delta_i\left(u(t_i^-)\right)}{\Gamma(q)}\left\{u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right.$$
$$\left. - \left[u_a + \int_a^{t_i} f(s,u(s))ds\right](t-t_i)^{q-1} - \int_{t_i}^t (t-s)^{q-1} f(s,u(s))ds\right\} \text{ for } \in (t_{n+1}, t_{n+2}].$$

Therefore,

$$\lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \vdots\\ \Delta_{n+1}(u(t_{n+1}^-))\to 0}} e_{n+1}(t) = \lim_{\substack{\Delta_1(u(t_1^-))\to 0,\\ \vdots\\ \Delta_{n+1}(u(t_{n+1}^-))\to 0}} \{u(t) - \tilde{u}(t)\}$$
$$= \frac{1}{\Gamma(q)}\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds - \left(u_a + \int_a^{t_{n+1}} f(s,u(s))ds\right)(t-t_{n+1})^{q-1}\right. \tag{22}$$
$$\left. - \int_{t_{n+1}}^t (t-s)^{q-1} f(s,u(s))ds\right],$$

$$\lim_{\substack{\Delta_j(u(t_j^-))\to 0,\\ 1\le j\le n+1}} e_{n+1}(t) = \lim_{\substack{\Delta_j(u(t_j^-))\to 0,\\ 1\le j\le n+1}} \{u(t) - \tilde{u}(t)\}$$
$$= \frac{1}{\Gamma(q)}\left\{\left[1 - \sum_{\substack{1\le i\le n+1\\ \text{and } i\ne j}} \xi\Delta_i\left(u(t_i^-)\right)\right]\left[u_a(t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds\right]\right.$$
$$+ \sum_{\substack{1\le i\le n+1\\ \text{and } i\ne j}} \Delta_i\left(u(t_i^-)\right)(t-t_i)^{q-1}$$
$$+ \sum_{\substack{1\le i\le n+1\\ \text{and } i\ne j}} \xi\Delta_i\left(u(t_i^-)\right)\left[\left(u_a + \int_a^{t_i} f(s,u(s))ds\right)(t-t_i)^{q-1} + \int_{t_i}^t (t-s)^{q-1} f(s,u(s))ds\right] \tag{23}$$
$$- \left[u_a + \sum_{\substack{1\le i\le n+1\\ \text{and } i\ne j}} \Delta_i\left(u(t_i^-)\right) + \int_a^{t_{n+1}} f(s,u(s))ds\right](t-t_{n+1})^{q-1}$$
$$\left. - \int_{t_{n+1}}^t (t-s)^{q-1} f(s,u(s))ds\right\}.$$

By (22) and (23), we obtain

$$\begin{aligned} e_{n+1}(t) &= \frac{1}{\Gamma(q)} \left\{ \left[1 - \sum_{1 \le i \le n+1} \xi \Delta_i \left(u(t_i^-) \right) \right] \left[u_a (t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \right] \right. \\ &+ \sum_{1 \le i \le n+1} \Delta_i \left(u(t_i^-) \right) (t-t_i)^{q-1} - \sum_{1 \le i \le n+1} \Delta_i \left(u(t_i^-) \right) (t-t_{n+1})^{q-1} \\ &+ \sum_{1 \le i \le n+1} \xi \Delta_i \left(u(t_i^-) \right) \left[\left(u_a + \int_a^{t_i} f(s,u(s))ds \right) (t-t_i)^{q-1} + \int_{t_i}^t (t-s)^{q-1} f(s,u(s))ds \right. \\ &\left. \left. - \left(u_a + \int_a^{t_{n+1}} f(s,u(s))ds \right) (t-t_{n+1})^{q-1} - \int_{t_{n+1}}^t (t-s)^{q-1} f(s,u(s))ds \right] \right\}. \end{aligned} \tag{24}$$

Thus,

$$\begin{aligned} u(t) &= \tilde{u}(t) + e_{n+1}(t) \\ &= \frac{u_a}{\Gamma(q)}(t-a)^{q-1} + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s,u(s))ds + \sum_{i=1}^{n+1} \frac{\Delta_i \left(u(t_i^-) \right)}{\Gamma(q)} (t-t_i)^{q-1} \\ &- \sum_{i=1}^{n+1} \frac{\xi \Delta_i \left(u(t_i^-) \right)}{\Gamma(q)} \left\{ u_a (t-a)^{q-1} + \int_a^t (t-s)^{q-1} f(s,u(s))ds \right. \\ &\left. - \left[u_a + \int_a^{t_i} f(s,u(s))ds \right] (t-t_i)^{q-1} - \int_{t_i}^t (t-s)^{q-1} f(s,u(s))ds \right\} \quad \text{for } t \in (t_{n+1}, t_{n+2}]. \end{aligned}$$

So, the solution of system (1) satisfies Eq. (6). So, impulsive system (1) is equivalent to the integral equation (6). The proof is now completed. □

4 Example

For system (1) it is difficult to get the analytical solution when f is a nonlinear function in (1). So, a linear example is given to illustrate the obtained result.

Example 4.1. *Let us consider the general solution of the impulsive fractional system*

$$\begin{cases} D_{1^+}^{\frac{1}{2}} u(t) = t, \quad t \in (1,3] \text{ and } t \neq 2, \\ \Delta \left(\mathcal{J}_{1^+}^{1-\frac{1}{2}} u \right) \Big|_{t=2} = \mathcal{J}_{1^+}^{1-\frac{1}{2}} u(2^+) - \mathcal{J}_{1^+}^{1-\frac{1}{2}} u(2^-) = \delta \in \mathbb{R}, \\ \mathcal{J}_{1^+}^{1-\frac{1}{2}} u(1) = u_0 \in \mathbb{R}, \end{cases} \tag{25}$$

By Theorem 3.1, the general solution of impulsive system (25) *is obtained as follows:*

$$u(t)=\begin{cases}\frac{u_0}{\Gamma(\frac{1}{2})}(t-1)^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^t(t-s)^{-\frac{1}{2}}sds, & \text{for } t\in(1,2],\\ \frac{u_0}{\Gamma(\frac{1}{2})}(t-1)^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^t(t-s)^{-\frac{1}{2}}sds+\frac{\delta}{\Gamma(\frac{1}{2})}(t-2)^{-\frac{1}{2}} & \\ -\frac{\xi\delta}{\Gamma(\frac{1}{2})}\left[u_0(t-1)^{-\frac{1}{2}}+\int\limits_1^t(t-s)^{-\frac{1}{2}}sds-\left(u_0+\int\limits_1^2 sds\right)(t-2)^{-\frac{1}{2}}-\int\limits_2^t(t-s)^{-\frac{1}{2}}sds\right] & \\ \text{for } t\in(2,3]. & \end{cases}\tag{26}$$

Next, it is verified that Eq. (26) *satisfies the condition of system* (25)*. Taking Riemann-Liouville fractional derivative to the both sides of Eq.* (26)*, we have*

(i) for $t\in(1,2]$

$$D_{1+}^{\frac{1}{2}}u(t)=\left\{\frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dt}\int\limits_1^t(t-\eta)^{1-\frac{1}{2}-1}\left[\frac{u_0}{\Gamma(\frac{1}{2})}(\eta-1)^{\frac{1}{2}-1}+\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^\eta(\eta-s)^{\frac{1}{2}-1}sds\right]d\eta\right\}_{t\in(1,2]}=t|_{t\in(1,2]},$$

(ii) for $t\in(2,3]$

$$\begin{aligned}D_{1+}^{\frac{1}{2}}u(t)&=\left\{\frac{1}{\Gamma(\frac{1}{2})}\frac{d}{dt}\int\limits_1^t(t-\eta)^{\frac{1}{2}-1}\left\{\frac{u_0}{\Gamma(\frac{1}{2})}(\eta-1)^{-\frac{1}{2}}+\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^\eta(\eta-s)^{-\frac{1}{2}}sds+\frac{\delta}{\Gamma(\frac{1}{2})}(\eta-2)^{-\frac{1}{2}}-\frac{\xi\delta}{\Gamma(\frac{1}{2})}\right.\right.\\&\quad\left.\left.\times\left[u_0(\eta-1)^{-\frac{1}{2}}+\int\limits_1^\eta(\eta-s)^{-\frac{1}{2}}sds-\left(u_0+\int\limits_1^2 sds\right)(\eta-2)^{-\frac{1}{2}}-\int\limits_2^\eta(\eta-s)^{-\frac{1}{2}}sds\right]\right\}d\eta\right\}_{t\in(2,3]}\\&=\left\{t|_{t\geq1}-\xi\delta\right.\\&\quad\left.\times\left\{t|_{t\geq1}-\frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\frac{d}{dt}\int\limits_2^t(t-\eta)^{\frac{1}{2}-1}\left[\left(u_0+\int\limits_1^2 sds\right)(\eta-2)^{-\frac{1}{2}}-\int\limits_2^\eta(\eta-s)^{-\frac{1}{2}}sds\right]d\eta\right\}\right\}_{t\in(2,3]}\\&=\left\{t|_{t\geq1}-\xi\delta\left[t|_{t\geq1}-t|_{t\geq2}\right]\right\}_{t\in(2,3]}\\&=t|_{t\in(2,3]}\end{aligned}$$

So, Eq. (26) *satisfies Riemann-Liouville fractional derivative condition of system* (25)*. By Definition 2.1, we obtain*

$$\begin{aligned}&\mathcal{J}_{1+}^{1-\frac{1}{2}}u(2^+)-\mathcal{J}_{1+}^{1-\frac{1}{2}}u(2^-)\\&=\left\{\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^t(t-\eta)^{\frac{1}{2}-1}u(\eta)d\eta\right\}_{t\to2^+}-\left\{\frac{1}{\Gamma(\frac{1}{2})}\int\limits_1^t(t-\eta)^{\frac{1}{2}-1}u(\eta)d\eta\right\}_{t=2^-}\\&=\left\{\frac{\delta}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\int\limits_2^t(t-\eta)^{\frac{1}{2}-1}(\eta-2)^{\frac{1}{2}-1}d\eta\right\}_{t\to2^+}-\xi\delta\left\{\frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}\int\limits_1^t(t-\eta)^{\frac{1}{2}-1}\left[u_0(\eta-1)^{-\frac{1}{2}}\right.\right.\\&\quad\left.\left.+\int\limits_1^\eta(\eta-s)^{-\frac{1}{2}}sds-\left(u_0+\int\limits_1^2 sds\right)(\eta-2)^{-\frac{1}{2}}-\int\limits_2^\eta(\eta-s)^{-\frac{1}{2}}sds\right]d\eta\right\}_{t\to2^+}\\&=\delta-\xi\delta\left\{u_0+\int\limits_1^t sds-\left(u_0+\int\limits_1^2 sds\right)-\int\limits_2^t sds\right\}_{t\to2^+}\\&=\delta.\end{aligned}$$

That is, Eq. (26) *satisfies impulsive condition in system* (25).

Finally, it is obvious that the Eq. (26) satisfies the following limit case

$$\lim_{\delta\to 0}\begin{cases} D_{1+}^{\frac{1}{2}}u(t)=t, \;\; t\in(1,3] \text{ and } t\neq 2,\\ \Delta\left(\mathcal{J}_{1+}^{1-\frac{1}{2}}u\right)\Big|_{t=2}=\mathcal{J}_{1+}^{1-\frac{1}{2}}u(2^{+})-\mathcal{J}_{1+}^{1-\frac{1}{2}}u(2^{-})=\delta\in\mathbb{R},\\ \mathcal{J}_{1+}^{1-\frac{1}{2}}u(1)=u_0\in\mathbb{R},\end{cases}$$

$$=\begin{cases} D_{1+}^{\frac{1}{2}}u(t)=t, \;\; t\in(1,3],\\ \mathcal{J}_{1+}^{1-\frac{1}{2}}u(1)=u_0\in\mathbb{R},\end{cases} \tag{27}$$

So, Eq. (26) *is the general solution of system* (25).

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