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Convergence theorems for a family of multivalued nonexpansive mappings in hyperbolic spaces

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Abstract: In this article we modify an iteration process to prove strong convergence and $\Delta-$ convergence theorems for a finite family of nonexpansive multivalued mappings in hyperbolic spaces. The results presented here extend some existing results in the literature.

Keywords: Hyperbolic spaces, $\Delta-$ convergence, Nonexpansive multivalued mapping

MSC: 47H10, 49M05, 54H25

1 Introduction

Many important problems of mathematics, including boundary value problems for nonlinear ordinary or partial differential equation, can be translated in terms of a fixed point equation $Tx = x$ for a given mapping T on a Banach space. The class of nonexpansive mappings contains contractions as a subclass and its study has remained a popular area of research ever since its introduction. The iterative construction of fixed points of these mappings is a fascinating field of research. The fixed point problem for one or a family of nonexpansive mappings has been studied in Banach spaces, metric spaces and hyperbolic spaces [1–12].

Most of the fundamental early results discovered for nonexpansive mappings were done in the context of Banach spaces. It is then natural to try to develop a similar theory in the nonlinear spaces. The closest class of sets considered was the class of hyperbolic spaces that enjoys convexity properties very similar to the linear one. This class of metric spaces includes all normed vector spaces, Hadamard manifolds, as well as the Hilbert ball and the cartesian product of Hilbert balls.

Multivalued mappings arise in optimal control theory, especially inclusions and related subjects like game theory and economics. In physics, multivalued mappings play an increasingly important role. They form the mathematical basis for Dirac's magnetic monopoles, for the theory of defects in crystals and the resulting plasticity of materials, for vortices in superfluids and superconductors, and for phase transitions in these systems, for instance melting and quark confinement. They are the origin of gauge field structures in many branches of physics.

In 2009 Yildirim and Ozdemir [13] used the following iteration to approximate fixed point of nonself asymptotically nonexpansive mappings.

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Let $x_1 \in C$ and $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_{n+1} &= (1-\alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1-\alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ \vdots & \\ y_{2n} &= (1-\alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})y_{0n} + \alpha_{1n}T_1^n y_{0n} \end{cases} \tag{1}$$

where $y_{0n} = x_n$. For $k = 3$ the iterative process (1) is reduced to SP iteration which is defined by Phuengrattana and Suantai [14] in 2011 and iteration process of Thianwan [15, 16] for $k = 2$. Also, the iterative process (1) is the generalized form of the modified Mann (one-step) iterative process which is given by Schu [17].

In 2010 Kettapun et all [18] studied the iteration process (1) for self mapping in Banach spaces. Recently, Gunduz and Akbulut [19] studied this iteration for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces by using the following modified version of it.

$$\begin{cases} x_{n+1} &= W(T_k^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}) \\ y_{(k-1)n} &= W(T_{k-1}^n y_{(k-1)n}, y_{(k-1)n}, \alpha_{(k-1)n}) \\ \vdots & \\ y_{2n} &= W(T_2^n y_{1n}, y_{1n}, \alpha_{2n}) \\ y_{1n} &= W(T_1^n y_{0n}, y_{0n}, \alpha_{1n}) \end{cases} \tag{2}$$

where $\alpha_{in} \in [0,1]$, for all $i = 1, 2, \ldots, k$ and any $x_1 \in C$.

Now, we use the iteration (2) for a finite family of nonexpansive multivalued mappings in hyperbolic spaces and get some convergence results.

Let E be a hyperbolic space and D be a nonempty convex subset of E. Let $\{T_i : \quad i = 1, 2, \ldots, k\}$ be a family of multivalued mappings such that $T_i : D \to P(D)$ and $P_{T_i}(x) = \{y \in T_i x : \quad d(x,y) = d(x, T_i x)\}$ is a nonexpansive mappping. Suppose that $\alpha_{in} \in [0,1]$, for all $n = 1, 2, \ldots$ and $i = 1, 2, \ldots, k$ for $x_0 \in D$ and let $\{x_n\}$ be the sequence generated by the following algorithm;

$$\begin{cases} x_{n+1} &= W(u_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}) \\ y_{(k-1)n} &= W(u_{(k-2)n}, y_{(k-2)n}, \alpha_{(k-1)n}) \\ \vdots & \\ y_{2n} &= W(u_{1n}, y_{1n}, \alpha_{2n}) \\ y_{1n} &= W(u_{0n}, y_{0n}, \alpha_{1n}) \end{cases} \tag{3}$$

where $u_{in} \in P_{T_{i+1}}(y_{in})$ for $i = 0, 1, 2, \ldots, k-1$ and $y_{0n} = x_n$.

2 Preliminaries

Now we need to give some notions about the concept of hyperbolic spaces and multivalued mappings.

A hyperbolic space is a triple (X, d, W) such that (X, d) is a metric space and $W : X \times X \times [0,1] \to X$ is a mapping satisfying the following conditions.

(W1) $d(z, W(x,y,\alpha) \leq (1-\alpha)d(z,x) + \alpha d(z,y)$,
(W2) $d(W(x,y,\alpha), W(x,y,\beta)) = |\alpha - \beta| d(x,y)$,
(W3) $W(x,y,\alpha) = W(y,x,(1-\alpha))$,
(W4) $d(W(x,z,\alpha), W(y,w,\alpha)) \leq (1-\alpha)d(x,y) + \alpha d(z,w)$ for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$.

Let $D \subset X$ if $W(x, y, \alpha) \in D$ for all $x, y \in K$ and $\alpha \in [0, 1]$, then D is called convex. If (X, d, W) satisfies only (W1), it is reduced to the convex metric space introduced by Takahashi [20] which incorporates all normed linear spaces, $\mathbb{R}$-trees and the Hilbert Ball with the Hyperbolic metric [21]. If (X, d, W) satisfies (W1)-(W3) then it is called the hyperbolic space in the sense of Goebel and Kirk [22]. After Itoh [23] gave the condition (III) that is equivalent (W4) condition, Reich and Shafrir [24] and Kirk [25] defined their notions of hyperbolic space by using Itoh's condition.

A hyperbolic space (X, d, W) is said to be uniformly convex [26] if there exists a $\delta \in (0, 1]$ such that

$$\left.\begin{array}{l} d(x,u) \le r \\ d(y,u) \le r \\ d(x,y) \ge \epsilon r \end{array}\right\} \Rightarrow d\Big(W(x, y, \frac{1}{2}), u\Big) \le (1-\delta) r$$

for all $u, x, y \in X$, $r > 0$ and $\epsilon \in (0, 2]$.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which satisfies such a $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$, is called modulus of uniform convexity. We call η monotone if it decreases with r for a fixed ϵ.

Let (X, d) be a metric space and K be a nonempty subset of X, K is said to be proximinal if there exists an element $y \in K$ such that

$$d(x, y) = d(x, K) := \inf_{z \in K} d(x, z)$$

for each $x \in X$. The collection of all nonempty compact subsets of K, the collection of all nonempty closed bounded subsets and nonempty proximinal bounded subsets of K are denoted by $C(K)$,$CB(K)$ and $P(K)$ respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\Big\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \Big\}$$

for all $A, B \in CB(X)$. Let $T : K \to CB(X)$ be a multivalued mapping. An element $x \in K$ is said to be a fixed point of T if $x \in Tx$. A multivalued mapping $T : K \to CB(X)$ is said to be nonexpansive if

$$H(Tx, Ty) \le d(x, y), \quad \forall x, y \in K.$$

Now,we need to give some definitions and notations to mention the concept of convergences in hyperbolic spaces.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space (X, d, W). Let r be a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ given by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center $A_K(\{x_n\})$ of a bounded sequence $\{x_n\}$ with respect to $K \subset X$ is the set

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\}), \quad \forall y \in K\}.$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$.

Lemma 2.1 ([26, Proposition 3.3]). *Let (X, d, W) be a complete uniformly convex hyperbolic space. Every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Recall that if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ then the sequence $\{x_n\}$ in X is said to be $\Delta-$converge to $x \in X$. In this case, we write $\Delta - \lim_{n \to \infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

This concept in general metric spaces was coined by Lim [3] and Kirk and Panyanak [27].

Lemma 2.2 ([28, Lemma 2.5]). *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in E$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n \to \infty} d(x_n, x) \le r$, $\limsup_{n \to \infty} d(y_n, x) \le r$ and $\lim_{n \to \infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n \to \infty} d(x_n, y_n) = 0$.*

Lemma 2.3 ([28, Lemma 2.6]). *Let D be a nonempty closed subset of a uniformly convex hyperbolic space X and $\{x_n\}$ be a bounded sequence in D such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in D such that $\lim_{n\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 2.4 ([29, Lemma 1]). *Let $T : D \to P(D)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : d(x, y) = d(x, Tx)\}$. Then the followings are equivalent.*
(1) *$x \in F(T)$, that is, $x \in Tx$,*
(2) *$P_T(x) = \{x\}$, that is, $x = y$ for each $y \in P_T(x)$,*
(3) *$x \in F(P_T)$, that is, $x \in P_T(x)$. Moreover, $F(T) = F(P_T)$.*

Lemma 2.5 ([30, p. 480]). *Let $A, B \in CB(X)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \leq H(A, B) + \eta$.*

3 Main results

Now we give two useful lemmas to prove our main results.

Lemma 3.1. *Let D be a nonempty closed convex subset of a hyperbolic space X and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of nonexpansive multivalued mappings such that P_{T_i} is nonexpansive mapping and $p \in F(T_i)$ for $i = 1, 2, \ldots, k$. Suppose that $F = \cap_{i=1}^{k} F(T_i) \neq \emptyset$, $x_1 \in D$, and the iterative sequence $\{x_n\}$ is defined by* (3). *Then for $p \in F$, we get*
(1) *$d(x_n, P_{T_i} x_n) \leq 2d(x_n, p)$ for all $i = 1, 2, \ldots, k$,*
(2) *$d(y_{(i-1)n}, u_{(i-1)n}) \leq 2d(y_{(i-1)n}, p)$ for all $i = 1, 2, \ldots, k$,*
(3) *$d(u_{(i-1)n}, p) \leq d(y_{(i-1)n}, p)$ for all $i = 1, 2, \ldots, k$,*
(4) *$d(y_{in}, p) \leq d(x_n, p)$ for all $i = 1, 2, \ldots, k - 1$,*
(5) *$d(x_{n+1}, p) \leq d(x_n, p)$,*
(6) *$\lim_{n\to\infty} d(x_n, p)$ exists,*

Proof. Let $p \in F$.

(1). For $i = 1, 2, 3, \ldots, k$, we have

$$\begin{aligned} d(x_n, P_{T_i} x_n) &\leq d(x_n, p) + d(p, P_{T_i}(x_n)) \\ &\leq d(x_n, p) + H(P_{T_i}(p), P_{T_i}(x_n)) \\ &\leq d(x_n, p) + d(p, x_n) \\ &= 2d(x_n, p). \end{aligned} \tag{4}$$

(2). In a similar way with part (1), we get

$$\begin{aligned} d(y_{(i-1)n}, u_{(i-1)n}) &\leq d(y_{(i-1)n}, p) + d(p, u_{(i-1)n}) \\ &\leq d(y_{(i-1)n}, p) + d(p, P_{T_i}(y_{(i-1)n})) \\ &\leq d(y_{(i-1)n}, p) + H(P_{T_i}(p), P_{T_i}(y_{(i-1)n})) \\ &\leq d(y_{(i-1)n}, p) + d(p, y_{(i-1)n}) \\ &= 2d(y_{(i-1)n}, p). \end{aligned} \tag{5}$$

(3). For $i = 1, 2, 3, \ldots, k$, we have

$$\begin{aligned} d(u_{(i-1)n}, p) &\leq d(u_{(i-1)n}, P_{T_i}(p)) \\ &\leq H(P_{T_i}(y_{(i-1)n}), P_{T_i}(p)) \\ &\leq d(y_{(i-1)n}, p). \end{aligned} \tag{6}$$

(4). We prove this item in three parts. Firstly

$$\begin{aligned} d(y_{1n}, p) =& d(W(u_{0n}, y_{0n}, \alpha_{1n}), p) \\ \leq& (1-\alpha_{1n})d(u_{0n}, p) + \alpha_{1n} d(y_{0n}, p) \\ \leq& (1-\alpha_{1n})d(u_{0n}, P_{T_1}(p)) + \alpha_{1n} d(y_{0n}, p) \\ \leq& (1-\alpha_{1n})H(P_{T_1}(y_{0n}), P_{T_1}(p)) + \alpha_{1n} d(y_{0n}, p) \\ \leq& (1-\alpha_{1n})d(y_{0n}, p) + \alpha_{1n} d(y_{0n}, p) \\ =& d(x_n, p). \end{aligned} \tag{7}$$

Secondly, we assume that $d(y_{jn}, p) \leq d(x_n, p)$ holds for some $1 \leq j \leq k-2$. Then

$$\begin{aligned} d(y_{(j+1)n}, p) =& d(W(u_{jn}, y_{jn}, \alpha_{(j+1)n}), p) \\ \leq& (1-\alpha_{(j+1)n})d(u_{jn}, p) + \alpha_{(j+1)n} d(y_{jn}, p) \\ \leq& (1-\alpha_{(j+1)n})H(P_{T_{(j+1)n}}(y_{jn}), P_{T_{(j+1)n}}(p)) + \alpha_{(j+1)n} d(y_{jn}, p) \\ \leq& (1-\alpha_{(j+1)n} d(y_{jn}, p) + \alpha_{(j+1)n} d(y_{jn}, p) \\ \leq& d(y_{jn}, p) \\ \leq& d(x_n, p). \end{aligned} \tag{8}$$

Lastly,

$$\begin{aligned} d(y_{(k-1)n}, p) =& d(W(u_{(k-2)n}, y_{(k-2)n}, \alpha_{(k-1)n}), p) \\ \leq& (1-\alpha_{(k-1)n})d(u_{(k-2)n}, p) + \alpha_{(k-1)n} d(y_{(k-2)n}, p) \\ \leq& (1-\alpha_{(k-1)n})d(u_{(k-2)n}, P_{T_{k-1}}(p)) + \alpha_{(k-1)n} d(y_{(k-2)n}, p) \\ \leq& (1-\alpha_{(k-1)n})H(P_{T_{k-1}}(y_{(k-2)n}), P_{T_{k-1}}(p)) + \alpha_{(k-1)n} d(y_{(k-2)n}, p) \\ \leq& (1-\alpha_{(k-1)n})d(y_{(k-2)n}, p) + \alpha_{(k-1)n} d(y_{(k-2)n}, p) \\ =& d(y_{(k-2)n}, p). \end{aligned} \tag{9}$$

So, by induction, we get

$$d(y_{in}, p) \leq d(x_n, p) \tag{10}$$

for all $i = 1, 2, \ldots, k-1$.

(5). By part (4), we have

$$\begin{aligned} d(x_{n+1}, p) =& d(W(u_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}), p) \\ \leq& (1-\alpha_{kn})d(u_{(k-1)n}, p) + \alpha_{kn} d(y_{(k-1)n}, p) \\ \leq& (1-\alpha_{kn})d(u_{(k-1)n}, P_{T_k}(p)) + \alpha_{kn} d(y_{(k-1)n}, p) \\ \leq& (1-\alpha_{kn})H(P_{T_k}(y_{(k-1)n}), P_{T_k}(p)) + \alpha_{kn} d(y_{(k-1)n}, p) \\ \leq& (1-\alpha_{kn})d(y_{(k-1)n}, p) + \alpha_{kn} d(y_{(k-1)n}, p) \\ =& d(y_{(k-1)n}, p) \\ \leq& d(x_n, p). \end{aligned} \tag{11}$$

(6). By part (5), we get

$$d(x_{n+1}, p) \leq d(x_n, p)$$

for $i = 1, 2, \ldots, k$. Thus, we obtain $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. □

Lemma 3.2. *Let D be a nonempty closed subset of a uniformly convex hyperbolic space X and $T_i : D \to P(D)$ be a family of multivalued mappings such that P_{T_i} is nonexpansive mapping for $i = 1, 2, , \ldots, k$ with a set of $F \neq \emptyset$. Then for the iterative process $\{x_n\}$ defined in* (3)*, we have*

$$\lim_{n\to\infty} d(x_n, P_{T_i} x_n) = 0$$

for $i = 1, 2, \ldots, k$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. Therefore, for a $c \geq 0$, we have

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{12}$$

By taking lim sup on both sides of (10) , we have

$$\limsup_{n\to\infty} d(y_{in}, p) \leq c \tag{13}$$

for all $i = 1, 2, \ldots, k-1$.

So, by (6) and (13), we get

$$\limsup_{n\to\infty} d(u_{(i-1)n}, p) \leq c \tag{14}$$

for all $i = 1, 2, \ldots, k$.

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, we have $\lim_{n\to\infty} d(W(u_{(k-1)n}, y_{(k-1)n}, \alpha_{kn}), p) = c$. From Lemma 2.2, (13) and (14), we have

$$\lim_{n\to\infty} d(y_{(k-1)n}, u_{(k-1)n}) = 0.$$

We claim that,

$$\lim_{n\to\infty} d(y_{(j-1)n}, u_{(j-1)n}) = 0 \tag{15}$$

for all $j = 2, 3, \ldots, k$. By Lemma 3.1 we get

$$d(x_{n+1}, p) \leq d(y_{in}, p) \tag{16}$$

for all $i = 1, 2, \ldots, k-1$. Therefore, from (16), we obtain

$$c \leq \liminf_{n\to\infty} d(y_{(i-1)n}, p) \tag{17}$$

for $i = 2, 3, \ldots, k$. By (3), (13) and (17), we obtain

$$\lim_{n\to\infty} d(W(u_{(j-2)n}, y_{(j-2)n}, \alpha_{(j-1)n}), p) = \lim_{n\to\infty} d(y_{(j-1)n}, p) = c.$$

Using (13), (14) and Lemma 2.2, we get $\lim_{n\to\infty} d(u_{(j-2)n}, y_{(j-2)n}) = 0$. So, by induction

$$\lim_{n\to\infty} d(y_{(i-1)n}, u_{(i-1)n}) = 0 \tag{18}$$

for all $i = 1, 2, \ldots, k$. From (3), we get

$$\begin{aligned} d(y_{in}, y_{(i-1)n}) =& d(W(u_{(i-1)n}, y_{(i-1)n}, \alpha_{in}), y_{(i-1)n}) \\ \leq& (1-\alpha_{in}) d(u_{(i-1)n}, y_{(i-1)n}). \end{aligned}$$

By (18), we have

$$\lim_{n\to\infty} d(y_{in}, y_{(i-1)n}) = 0 \tag{19}$$

for $i = 1, 2, \ldots, k$. Since

$$\begin{aligned} d(x_n, y_{1n}) =& d(x_n, W(u_{0n}, y_{0n}, \alpha_{1n}) \\ \leq& (1-\alpha_{1n}) d(x_n, u_{0n}) + \alpha_{1n} d(x_n, y_{0n}) \\ =& (1-\alpha_{1n}) d(x_n, u_{0n}). \end{aligned}$$

by taking $i = 1$ in (18), we get

$$\lim_{n\to\infty} d(x_n, y_{1n}) = 0. \tag{20}$$

By known triangle inequality,

$$d(x_n, y_{in}) \leq d(x_n, y_{1n}) + d(y_{1n}, y_{2n}) + \cdots + d(y_{(i-1)n}, y_{in}).$$

for all $i = 1, 2, \ldots, k-1$. It follows by (19) and (20) that

$$\lim_{n\to\infty} d(x_n, y_{in}) = 0. \tag{21}$$

For $i = 1, 2, 3, \ldots, k$

$$\begin{aligned} d(x_n, P_{T_i}x_n) \le & d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + d(u_{(i-1)n}, P_{T_i}x_n) \\ \le & d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}) + H(P_{T_i}y_{(i-1)n}, P_{T_i}x_n) \\ \le & 2d(x_n, y_{(i-1)n}) + d(y_{(i-1)n}, u_{(i-1)n}). \end{aligned}$$

From (18) and (21), we conclude

$$\lim_{n\to\infty} d(x_n, P_{T_i}x_n) = 0.$$ □

Theorem 3.3. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η. Let T_i, P_{T_i} and F be as in Lemma 3.2, Then the iterative process $\{x_n\}$, $\Delta-$ converges to p in F.*

Proof. Let $p \in F$. Then $p \in F(T_i) = F(P_{T_i})$, for $i = 1, 2, \ldots, k$. By the Lemma 3.1, $\{x_n\}$ is bounded and so $\lim_{n\to\infty} d(x_n, p)$ exists. Thus $\{x_n\}$ has a unique asymptotic center. In other words, we have $A(\{x_n\}) = \{x_n\}$. Let $\{v_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. From Lemma 3.2, we get $\lim_{n\to\infty}(v_n, P_{T_1}(v_n)) = 0$. We claim that v is a fixed point of P_{T_1}.

To prove this, we take another sequence $\{z_m\}$ in $P_{T_1}(v)$. Then,

$$\begin{aligned} r(z_m, \{v_n\}) =& \limsup_{n\to\infty} d(z_m, v_n) \\ \le& \lim_{n\to\infty}\{d(z_m, P_{T_1}(v_n) + d(P_{T_1}(v_n), v_n)\} \\ \le& \lim_{n\to\infty}\{H(P_{T_1}(v), P_{T_1}(v_n)) + d(P_{T_1}(v_n), v_n))\} \\ \le& \limsup_{n\to\infty} d(v, v_n) \\ =& r(v, \{v_n\}) \end{aligned}$$

this gives $|r(z_m, \{v_n\}) - r(v, \{v_n\})| \to 0$ for $m \to \infty$. By Lemma 2.3, we get $\lim_{m\to\infty} z_m = v$. Note that $T_1(v) \in P(D)$ being proximinal is closed, hence $P_{T_1}(v)$ is either closed or bounded. Consequently $\lim_{m\to\infty} z_m = v \in P_{T_1}(v)$. Similarly $v \in P_{T_2}(v)$,$v \in P_{T_3}(v) \ldots v \in P_{T_k}(v)$. So $v \in F$. □

Theorem 3.4. *Let D be a nonempty closed convex subset of a hyperbolic space E and Let T_i, P_{T_i} and F be as in Lemma 3.2 and let $\{x_n\}$ be the iterative process defined in 3, then $\{x_n\}$ converges to p in F if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. since $0 \le d(x_n, F) \le d(x_n, p)$, we have $\lim_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\lim_{n\to\infty} \inf d(x_n, F) = 0$. By Lemma 3.1, we have

$$d(x_{n+1}, p) \le d(x_n, p)$$

which implies

$$d(x_{n+1}, F) \le d(x_n, F).$$

This gives that $\lim_{n\to\infty} d(x_n, F)$ exists. Therefore, by the hypothesis of our theorem, $\liminf_{n\to\infty} d(x_n, F) = 0$. Thus we have $\lim_{n\to\infty} d(x_n, F) = 0$. Let us show that $\{x_n\}$ is a Cauchy sequence in D. Let $m, n \in \mathbb{N}$ and assume $m > n$. Then it follows that $d(x_m, p) \le d(x_n, p)$ for all $p \in F$. Thus we get,

$$d(x_m, x_n) \le d(x_m, p) + d(p, x_n) \le 2d(x_n, p).$$

Taking inf on the set F, we have $d(x_m, x_n) \le d(x_n, F)$. We show that $\{x_n\}$ is a Cauchy sequence in D. By taking as $m, n \to \infty$ in the inequality $d(x_m, x_n) \le d(x_n, F)$. So, it converges to a $q \in D$. Now it is left to show that

$q \in F(T_1)$. Indeed by $d(x_n, F(P_{T_1})) = \inf_{y \in F(P_{T_1})} d(x_n, y)$. So for each $\epsilon > 0$, there exists $p_n^{(\epsilon)} \in F(P_{T_1})$ such that,

$$d(x_n, p_n^{(\epsilon)}) < d(x_n, F(P_{T_1})) + \frac{\epsilon}{2}.$$

This implies $\lim_{n\to\infty} d(x_n, p_n^{(\epsilon)}) \leq \frac{\epsilon}{2}$. Since $d(p_n^{(\epsilon)}, q) \leq d(x_n, p_0^{(\epsilon)}) + d(x_n, q)$ it follows that $\lim_{n\to\infty} d(p_n^{(\epsilon)}, q) \leq \frac{\epsilon}{2}$. Finally

$$\begin{aligned} d(P_{T_1}(q), q) &\leq d(q, p_n^{(\epsilon)}) + d(p_n^{(\epsilon)}, P_{T_1}(q)) \\ &\leq d(q, p_n^{(\epsilon)}) + H(P_{T_1}(p_n^{(\epsilon)}), P_{T_1}(q)) \\ &\leq 2d(p_n^{(\epsilon)}, q) \end{aligned}$$

which shows $d(P_{T_1}(q), q) < \epsilon$. So, $d(P_{T_1}(q), q) = 0$. In a similar way, we get for any $i = 1, 2 \ldots, k$ we obtain $d(P_{T_i}(q), q) = 0$. Since F is closed, $q \in F$. □

Now we give the definition of condition (B) of Senter and Dotson for a finite family of multivalued mappings to complete the proof of the following theorem.

Definition 3.5 ([31])**.** *The multivalued nonexpansive mappings $T_1, T_2, \ldots, T_k : D \to CB(D)$ are said to satisfy condition (B). If there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$d(x_n, T_i x_n) \geq f(d(x_n, F)), \quad F \neq \emptyset.$$

Definition 3.6. *A map $T : D \to P(D)$ is called semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n, T_{x_n}) \to 0$ as $n \to \infty$ has a convergent subsequence.*

Theorem 3.7. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η. Let T_i, P_{T_i} and F be as in Lemma 3.2. Suppose that each P_{T_i} satisfies condition (B). Then the iterative process $\{x_n\}$ defined in* (3) *converges strongly to $p \in F$.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$. We call it c for some $c \geq 0$. Then if $c = 0$, proof is completed. Assume $c > 0$. Now $d(x_{n+1}, p) \leq d(x_n, p)$ gives that

$$\inf_{p \in F(T_i)} d(x_{n+1}, p) \leq \inf_{p \in F(T_i)} d(x_n, p)$$

which means that $d(x_{n+1}, F) \leq d(x_n, F)$. So, $\lim_{n\to\infty} d(x_n, F)$ exists. By using the condition (B) and Lemma 3.2 we obtain,

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, P_{T_i}(x_n) \to 0$$

and so $\lim_{n\to\infty} f(d(x_n, F)) = 0$. By the properties of f, we get $\lim_{n\to\infty} d(x_n, F) = 0$. Finally by applying Theorem 3.4, we obtain the result. □

Theorem 3.8. *Let D be a nonempty closed convex subset of a complete uniformly convex hyperbolic space E with monotone modulus of uniform convexity η and let T_i, P_{T_i}, F be as in Lemma 3.2. Suppose that P_{T_i} is semi-compact then the iterative process $\{x_n\}$ defined in* (3) *converges strongly to $p \in F$.*

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