Open Math. 2016; 14: 1014–1022 **DE GRUYTER** OPEN

Open Mathematics **Open Access**

Research Article

Moiz ud Din Khan*, Rafaqat Noreen, and Muhammad Siddique Bosan

Semi-quotient mappings and spaces

DOI 10.1515/math-2016-0093
Received April 30, 2016; accepted October 6, 2016.

Abstract: In this paper, we continue the study of s-topological and irresolute-topological groups. We define semi-quotient mappings which are stronger than semi-continuous mappings, and then consider semi-quotient spaces and groups. It is proved that for some classes of irresolute-topological groups $(G, *, \tau)$ the semi-quotient space G/H is regular. Semi-isomorphisms of s-topological groups are also discussed.

Keywords: Semi-continuity, Semi-homeomorphism, s-topological group, Irresolute topological group, Semi-quotient space, Semi-quotient group

MSC: 54H11, 22A05, 54C08, 54H99

1 Introduction

The basic aim of this article is to study properties of topological spaces and mappings between them by weakening the continuity and openness conditions. Semi-continuity [1] and irresolute mappings [2] were a consequence of the study of semi-open sets in topological spaces. In [3] Bohn and Lee defined and investigated the notion of s-topological groups and in [4] Siddique et. al. defined the notion of S-topological groups. In [5] Siab et. al. defined and studied the notion of irresolute-topological groups by using irresolute mappings. Study of s-paratopological groups and irresolute-paratopological groups is a consequence of the study of paratopological groups (see [6]). For the study of semi-topological groups with respect to semi-continuity and irresoluteness we refer the reader to Oner's papers [7–9].

In this paper we continue the study of properties of s-topological and irresolute-topological groups. Keeping in mind the existing concepts, semi-quotient topology on a set is defined as a generalization of the quotient topology for spaces and groups. Various results on semi-quotients of topologized groups are proved. A counter example is given to show that the quotient topology is properly contained in the semi-quotient structure. We define also semi-isomorphisms and S-isomorpisms between topologized groups and prove that if certain irresolute-topological groups G and H are semi-isomorphic or S-isomorphic, then their semi-quotients are semi-isomorphic. Investigation of s-openness and s-closedness of mappings on s-topological groups is also presented.

2 Definitions and preliminaries

Throughout this paper X and Y are always topological spaces on which no separation axioms are assumed. If $f : X \to Y$ is a mapping between topological spaces X and Y and B is a subset of Y, then $f^{\leftarrow}(B)$ denotes

***Corresponding Author: Moiz ud Din Khan:** COMSATS Institute of Information Technology, Chak Shahzad, Islamabad - 45550, Pakistan, E-mail: moiz@comsats.edu.pk
Rafaqat Noreen: COMSATS Institute of Information Technology, Chak Shahzad, Islamabad - 45550, Pakistan, E-mail: rafaqat_nnn@hotmail.com
Muhammad Siddique Bosan: Punjab Education Department, Pakistan, E-mail: siddiquebosan@hotmail.com

the pre-image of B. By $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ we denote the closure and interior of a set A in a space X. Our other topological notation and terminology are standard as in [10]. If $(G, *)$ is a group, then e or e_G denotes its identity element, and for a given $x \in G$, $\ell_x : G \to G$, $y \mapsto x \circ y$, and $r_x : G \to G$, $y \mapsto y \circ x$, denote the left and the right translation by x, respectively. The operation $*$ we call the multiplication mapping $m : G \times G \to G$, and the inverse operation $x \mapsto x^{-1}$ is denoted by i.

In 1963, N. Levine [1] defined semi-open sets in topological spaces. Since then, many mathematicians have explored different concepts and generalized them by using semi-open sets (see [2, 11–14]). A subset A of a topological space X is said to be *semi-open* if there exists an open set U in X such that $U \subset A \subset \mathrm{Cl}(U)$, or equivalently if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$. $\mathsf{SO}(X)$ denotes the collection of all semi-open sets in X, and $\mathsf{SO}(X, x)$ is the collection of semi-open sets in X containing the point $x \in X$. The complement of a semi-open set is said to be *semi-closed*; the *semi-closure* of $A \subset X$, denoted by $\mathrm{sCl}(A)$, is the intersection of all semi-closed subsets of X containing A [15, 16]. $x \in \mathrm{sCl}(A)$ if and only if any $U \in \mathsf{SO}(X, x)$ meets A.

Clearly, every open (resp. closed) set is semi-open (resp. semi-closed). It is known that a union of any collection of semi-open sets is again a semi-open set. The intersection of two semi-open sets need not be semi-open whereas the intersection of an open set and a semi-open set is semi-open. Basic properties of semi-open sets and semi-closed sets are given in [1], and [15, 16].

Recall that a set $U \subset X$ is a *semi-neighbourhood* of a point $x \in X$ if there exists $A \in \mathsf{SO}(X, x)$ such that $A \subset U$. If a semi-neighbourhood U of a point x is a semi-open set, we say that U is a *semi-open neighbourhood* of x. A set $A \subset X$ is semi-open in X if and only if A is a semi-open neighbourhood of each of its points. Let X be a topological space and $A \subset X$. Then $x \in X$ is called a *semi-interior point* of A if there exists a semi-open set U such that $x \in U \subset A$. The set of all semi-interior points of A is called a *semi-interior* of A and is denoted by $\mathrm{sInt}(A)$. A nonempty set A is *pre-open* (or *locally dense*) [17] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$. A space X is *s-compact* [18], if every semi-open cover of X has a finite subcover. Every s-compact space is compact but converse is not always true. For some applications of semi-open sets see [13].

A mapping $f : X \to Y$ between topological spaces X and Y is called:

- *semi-continuous* [1] (resp. *irresolute* [2]) if for each open (resp. semi-open) set $V \subset Y$ the set $f^{\leftarrow}(V)$ is semi-open in X. Equivalently, the mapping f is semi-continuous (irresolute) if for each $x \in X$ and for each open (semi-open) neighbourhood V of $f(x)$, there exists a semi-open neighbourhood U of x such that $f(U) \subset V$;
- *pre-semi-open* [2] if for every semi-open set A of X, the set $f(A)$ is semi-open in Y;
- *s-open* (*s-closed*) if for every semi-open (semi-closed) set A of X, the set $f(A)$ is open (closed) in Y;
- *s-perfect* if it is semi-continuous, s-closed, surjective, and $f^{\leftarrow}(y)$ is s-compact relative to X, for each y in Y.
- *semi-homeomorphism* [2, 19] if f is bijective, irresolute and pre-semi-open;
- *S-homeomorphism* [4] if f is bijective, semi-continuous and pre-semi-open.

We need also some basic information on (topological) groups; for more details see the excellent monograph [20]. If G is a group and H its normal subgroup, then the canonical projection of G onto the quotient group G/H (sending each $g \in G$ to the coset in G/H containing g) will be denoted by p. A mapping $f : G \to H$ between two topological groups is called a *topological isomorphism* if f is an algebraic isomorphism and a topological homeomorphism.

Let $f : X \to Y$ be a surjection; a subset C of X is called *saturated* with respect to f (or *f-saturated*) if $f^{\leftarrow}(f(C)) = C$ [21].

3 Semi-quotient mappings

Definition 3.1. *A mapping $f : X \to Y$ from a space X onto a space Y is said to be* semi-quotient *provided a subset V of Y is open in Y if and only if $f^{\leftarrow}(V)$ is semi-open in X.*

Evidently, every semi-quotient mapping is semi-continuous and every quotient mapping is semi-quotient. The following simple examples show that semi-quotient mappings are different from semi-continuous mappings and quotient mappings.

Example 3.2. *Let $X = Y = \{1, 2, 3\}$ and let $\tau_X = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{1, 3\}\}$ and $\tau_Y = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}\}$ be topologies on X and Y. Let $f : X \to Y$ be defined by $f(x) = x$, $x \in X$. Since $\tau_Y \subset \tau_X$, the mapping f is continuous, hence semi-continuous. On the other hand, this mapping is not semi-quotient because $f^{\leftarrow}(\{1, 3\})$ is semi-open in X although $\{1, 3\}$ is not open in Y.*

Example 3.3. *Let $X = \{1, 2, 3, 4\}$, $Y = \{a, b\}$, $\tau_X = \{\emptyset, X, \{1\}, \{3\}, \{1, 3\}\}$, $\tau_Y = \{\emptyset, Y, \{a\}\}$. Define $f : X \to Y$ by; $f(1) = f(3) = f(4) = a$; $f(2) = b$. The mapping f is not a quotient mapping because it is not continuous. On the other hand, f is semi-quotient: the only proper subset of Y whose preimage is semi-open in X is the set $\{a\}$ which is open in Y.*

The following proposition is obvious.

Proposition 3.4.

(a) *Every surjective semi-continuous mapping $f : X \to Y$ which is either s-open or s-closed is a semi-quotient mapping.*

(b) *If $f : X \to Y$ is a semi-quotient mapping and $g : Y \to Z$ a quotient mapping, then $g \circ f : X \to Z$ is semi-quotient.*

Proof. We prove only (b). A subset $V \subset Z$ is open in Z if and only if $g^{\leftarrow}(V)$ is open in Y (because g is a quotient mapping), while the latter set is open in Y if and only if $f^{\leftarrow}(g^{\leftarrow}(V)$ is semi-open in X (because f is semi-quotient). So, V is open in Z if and only $(g \circ f)^{\leftarrow}(V)$ is semi-open in X, i.e. $g \circ f$ is a semi-quotient mapping. □

The restriction of a semi-quotient mapping to a subspace is not necessarily semi-quotient. Let X and Y be the spaces from Example 3.3, and $A = \{2, 4\}$. Then $\tau_A = \{\emptyset, A\}$. The restriction $f_A : A \to Y$ of f to A is not a semi-quotient mapping because $f_A^{\leftarrow}(\{a\}) = \{4\}$ is not semi-open in A.

To see when the restriction of a semi-quotient mapping is also semi-quotient we will need the following simple but useful lemmas.

Lemma 3.5 ([22, Theorem 1])**.** *Let X be a topological space, $X_0 \in \mathsf{SO}(X)$ and $A \subset X_0$. Then $A \in \mathsf{SO}(X_0)$ if and only if $A \in \mathsf{SO}(X)$.*

Lemma 3.6 ([23, Lemma 2.1])**.** *Let X be a topological space, X_0 a subspace of X. If $A \in \mathsf{SO}(X_0)$, then $A = B \cap X_0$, for some $B \in \mathsf{SO}(X)$.*

Lemma 3.7 ([21])**.** *Let $f : X \to Y$ be a mapping, A a subspace of X saturated with respect to f, B a subset of X. If $g : A \to f(A)$ is the restriction of f to A, then:*

(1) *$g^{\leftarrow}(C) = f^{\leftarrow}(C)$ for any $C \subset f(A)$;*

(2) *$f(A \cap B) = f(A) \cap f(B)$.*

Now we have this result.

Theorem 3.8. *Let $f : X \to Y$ be a semi-quotient mapping and let A be a subspace of X saturated with respect to f, and let $g : A \to f(A)$ be the restriction of f to A. Then:*

(a) *If A is open in X, then g is a semi-quotient mapping;*

(b) *If f is an s-open mapping, then g is semi-quotient.*

Proof. (a) Let V be an open subset of $f(A)$. Then $V = W \cap f(A)$ for some open subset W of X, so that $g^{\leftarrow}(V) = f^{\leftarrow}(W \cap f(A)) = f^{\leftarrow}(W) \cap A$ is a semi-open set in A.

Let now V be a subset of $f(A)$ such that $g^{\leftarrow}(V)$ is semi-open in A. We have to prove that V is open in $f(A)$. Since $g^{\leftarrow}(V)$ is semi-open in A and A is open in X we have that $g^{\leftarrow}(V)$ is semi-open in X. By Lemma 3.7, $g^{\leftarrow}(V) = f^{\leftarrow}(V)$; the set $f^{\leftarrow}(V)$ is semi-open in X since f is semi-quotient, hence $g^{\leftarrow}(V)$ is semi-open in $f(A)$. This means that V is open in Y and thus in $f(A)$. This completes the proof that g is a semi-quotient mapping.

(b) Let now f be s-open and V a subset of $f(A)$ such that $g^{\leftarrow}(V)$ is semi-open in A. Again we must prove that V is open in $f(A)$. Since $g^{\leftarrow}(V) = f^{\leftarrow}(V)$ and $g^{\leftarrow}(V)$ is semi-open in A, by Lemma 3.6 we have $f^{\leftarrow}(V) = U \cap A$, for some U semi-open in X. As f is surjective, it holds $f(f^{\leftarrow}(V)) = V$. By Lemma 3.7, then $V = f(f^{\leftarrow}(V)) = f(U \cap A) = f(U) \cap f(A)$. The set $f(U)$ is open in Y because f is s-open, so that V is open in $f(A)$. Other part is the same as in (a). □

As a complement to Proposition 3.4 we have the following two theorems.

Theorem 3.9. *Let X, Y and Z be topological spaces, $f : X \to Y$ a semi-quotient mapping, $g : Y \to Z$ a mapping. Then the mapping $g \circ f : X \to Z$ is semi-quotient if and only if g is a quotient mapping.*

Proof. If g is a quotient mapping, then $g \circ f$ is semi-quotient as the composition of a semi-quotient and a quotient mapping (Proposition 3.4).

Conversely, let $g \circ f$ be semi-quotient. We have to prove that a subset V of Z is open in Z if and only if $g^{\leftarrow}(V)$ is open in Y. For, the set $(g \circ f)^{\leftarrow}(V)$ is semi-open in X and since f is a semi-quotient mapping we conclude that it will be if and only if $g^{\leftarrow}(V)$ is open in Y. □

Theorem 3.10. *Let $f : X \to Y$ be a mapping and $g : X \to Z$ a mapping which is constant on each set $f^{\leftarrow}(\{y\})$, $y \in Y$. Then g induces a mapping $h : Y \to Z$ such that $g = h \circ f$. Then:*
(1) If f is pre-semi-open and irresolute, then h is a semi-continuous mapping if and only if g is semi-continuous;
(2) If f is semi-quotient, then h is continuous if and only if g is semi-continuous.

Proof. Since g is constant on the set $f^{\leftarrow}(\{y\})$, $y \in Y$, then for each $y \in Y$, the set $g(f^{\leftarrow}(\{y\}))$ is a one-point set in Z, say $h(y)$. Define now $h : Y \to Z$ by the rule

$$h(y) = g(f^{\leftarrow}(y)) \quad (y \in Y).$$

Then for each $x \in X$ we have

$$g(x) = g(f^{\leftarrow}(f(x))) = h(f(x)),$$

i.e. $g = h \circ f$.

(1) Suppose g is a semi-continuous mapping. If V is an open set in Z, then $h^{\leftarrow}(V) = f(g^{\leftarrow}(V)) \in \mathsf{SO}(Y)$ because f is pre-semi-open and $g^{\leftarrow}(V)$ is semi-open in X. Thus h is a semi-continuous mapping.

Conversely, suppose h is a semi-continuous. Let V be an open set in Z. The set $g^{\leftarrow}(V) = f^{\leftarrow}(h^{\leftarrow}(V))$ is semi-open in X because f is irresolute and $h^{\leftarrow}(V) \in \mathsf{SO}(Y)$. So, g is semi-continuous.

(2) If g is semi-continuous, then for any open set V in Z we have $g^{\leftarrow}(V)$ is a semi-open set in X. But, $g^{\leftarrow}(V) = f^{\leftarrow}(h^{\leftarrow}(V))$. Since f is semi-quotient it follows that $h^{\leftarrow}(V)$ is open in Y. So, h is continuous.

Conversely, suppose h is continuous. For a given open set V in Z, $h^{\leftarrow}(V)$ is an open set in Y. We have then $g^{\leftarrow}(V) = f^{\leftarrow}(h^{\leftarrow}(V))$ is semi-open in X because f is semi-quotient. Hence g is semi-continuous. □

At the end of this section we describe now a typical construction which shows how the notion of semi-quotient mappings may be used to get a topology or a topology-like structure on a set.

Construction: Let X be a topological space and Y a set. Let $f : X \to Y$ be a mapping. Define

$$s\tau_Q := \{V \subset Y : f^{\leftarrow}(V) \in \mathsf{SO}(X)\}.$$

It is easy to see that the family $s\tau_Q$ ia a generalized topology on Y (i.e. $\emptyset \in s\tau_Q$ and union of any collection of sets in $s\tau_Q$ is again in $s\tau_Q$) generated by f; we call it the *semi-quotient generalized topology*. But $s\tau_Q$ need not be a topology on Y. It happens if X is an extremally disconnected space, because in this case the intersection of two semi-open sets in X is semi-open [24]. It is trivial fact that in the latter case $s\tau_Q$ is the finest topology σ on Y such that $f : X \to (Y, \sigma)$ is semi-continuous. In fact, $f : X \to (Y, s\tau_Q)$ is a quotient mapping in this case.

In particular, let ρ be an equivalence relation on X. Let $p : X \to X/\rho$ be the natural (or canonical) projection from X onto the quotient set X/ρ: for each x in X, p sends x to the equivalence class $\rho(x)$. Then the family $s\tau_Q$

generated by p is a generalized topology on the quotient set Y/ρ, and a topology when X is extremally disconnected. This topology will be called the *semi-quotient topology* on X/ρ. Observe, that we forced the mapping p to be semi-continuous, that is semi-quotient.

This kind of construction will be applied here to topologized groups: to s-topological groups and irresolute-topological groups.

The following example shows that a quotient topology on a set generated by a mapping and the semi-quotient (generalized) topology generated by the same mapping are different.

Example 3.11. *Let the set* $X = \{1, 2, 3, 4\}$ *be endowed with the topology*

$$\tau = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Then the set $\mathsf{SO}(X)$ *is*

$$\{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

Define the relation R *on* X *by* xRy *if and only if* $x + y$ *is even. Therefore,*

$$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$$

is an equivalence relation, and $X/R = \{R(1), R(2)\} = \{\{1, 3\}, \{2, 4\}\}$. *Let* $p : X \to X/R$ *be the canonical projection. Then,* $p^{\leftarrow}(R(1)) = \{1, 3\} \in \mathsf{SO}(X)$, *and* $p^{\leftarrow}(R(2)) = \{2, 4\} \in \mathsf{SO}(X)$, *so that*

$$s\tau_Q = \{\emptyset, X/R, \{R(1)\}, \{R(2)\}\}$$

is the semi-quotient topology on X/R. *On the other hand,* $p^{\leftarrow}(\{R(1)\}) = \{1, 3\} \in \tau$, *but* $p^{\leftarrow}(\{R(2)\}) = \{2, 4\} \notin \tau$. *Therefore, the quotient topology on* X/R *is*

$$\tau_Q = \{\emptyset, X/R, \{R(1)\}\}.$$

4 Topologized groups

In this section we give some information on s-topological groups and irresolute-topological groups introduced and studied first in [4] and [5], respectively.

Definition 4.1 ([3]). *An* s-topological group *is a group* $(G, *)$ *with a topology* τ *such that for each* $x, y \in G$ *and each neighbourhood* W *of* $x * y^{-1}$ *there are semi-open neighbourhoods* U *of* x *and* V *of* y *such that* $U * V^{-1} \subset W$.

Definition 4.2 ([5]). *A triple* $(G, *, \tau)$ *is an* irresolute-topological group *if* $(G, *)$ *is a group, and* τ *a topology on* G *such that for each* $x, y \in G$ *and each semi-open neighbourhood* W *of* $x * y^{-1}$ *there are semi-open neighbourhoods* U *of* x *and* V *of* y *such that* $U * V^{-1} \subset W$.

Lemma 4.3 ([25]). *If* $(G, *, \tau)$ *is an* s*-topological group,* $y \in G$, *and* K *an* s*-compact subset of* G, *then* $y * K^{-1}$ *is* s*-compact in* G. *In particular,* K^{-1} *is* s*-compact.*

Theorem 4.4. *If a mapping* $f : X \to Y$ *between topological spaces* X *and* Y *is* s*-perfect, then for any compact subset* K *of* Y, *the pre-image* $f^{\leftarrow}(K)$ *is an* s*-compact subset of* X.

Proof. Let $\{U_i : i \in \Lambda\}$ be a semi-open cover of $f^{\leftarrow}(K)$. Then for each $x \in K$ the set $f^{\leftarrow}(x)$ can be covered by finitely many U_i; let $U(x)$ denote their union. Then $O(x) = Y \setminus f[X \setminus U(x)]$ is an open neighbourhood of x in Y because f is an s-closed map. So, $K \subset \bigcup_{x \in K} O(x)$, and because K is assumed to be compact, there are finitely many points $x_1, x_2, \cdots, x_n$ in K such that $K \subset \bigcup_{i=1}^{n} O(x_i)$. It follows that $f^{\leftarrow}(K) \subset \bigcup_{i=1}^{n} f^{\leftarrow}(O(x_i)) \subset \bigcup_{i=1}^{n} U(x_i)$, hence $f^{\leftarrow}(K)$ is s-compact in X. □

The following results are related to s-topological groups, and they are generalizations of some results for topological groups.

Theorem 4.5. *Let G, H and K be s-topological groups, $\varphi : G \to H$ a semi-continuous homomorphism, $\psi : G \to K$ an irresolute endomorphisms, such that* $\ker\psi \subset \ker\varphi$*. Assume also that for each open neighbourhood U of e_H there is a semi-open neighbourhood V of e_K with $\psi^{\leftarrow}(V) \subset \varphi^{\leftarrow}(U)$. Then there is a semi-continuous homomorphism $f : K \to H$ such that $\varphi = f \circ \psi$.*

Proof. The existence of a homomorphism f such that $\varphi = f \circ \psi$ is well-known fact in group theory. We verify the semi-continuity of f. Suppose U is an open neighbourhood of e_H in H. By our assumption, there is a semi-open neighbourhood V of e_K in K such that $\psi^{\leftarrow}(V) \subset \varphi^{\leftarrow}(U)$. Then $\varphi = f \circ \psi$ implies $f(V) = \varphi(\psi^{\leftarrow}(V)) \subset \varphi(\varphi^{\leftarrow}U) \subset U$, which means that f is semi-continuous at the identity element e_K of K. By [4, Theorem 3.5], f is semi-continuous on K. □

Theorem 4.6. *Suppose that G, H and K are s-topological groups. Let $\varphi : G \to H$ be a semi-continuous homomorphism, $\psi : G \to K$ an irresolute endomorphism such that* $\ker\psi \subset \ker\varphi$*. If ψ is pre-semi-open, then there is a semi-continuous homomorphism $f : K \to H$ such that $\varphi = f \circ \psi$.*

Proof. By Theorem 4.5, there exists a homomorphism $f : K \to H$ satisfying $\varphi = f \circ \psi$. We prove that f is semi-continuous. Let V be an open set in H. From $\varphi = f \circ \psi$ it follows $f^{\leftarrow}(V) = \psi(\varphi^{\leftarrow}(V))$. Since, φ is semi-continuous, the set $\varphi^{\leftarrow}(V)$ is semi-open in G, and pre-semi-openness of ψ implies that $\psi(\varphi^{\leftarrow}(V))$ is semi-open, i.e. $f^{\leftarrow}(V)$ is semi-open in K. This means that f is semi-continuous. □

Theorem 4.7. *Let $(G, *, \tau_G)$ and $(H, \cdot, \tau_H)$ be s-topological groups, and $f : G \to H$ a homomorphism of G onto H such that for some non-empty open set $U \subset G$, the set $f(U)$ is semi-open in H and the restriction $f|_U : U \to f(U)$ is a pre-semi-open mapping. Then f is pre-semi-open.*

Proof. We have to prove that if $x \in G$ and $W \in \mathsf{SO}(G, x)$, then $f(W) \in \mathsf{SO}(H, f(x))$. Pick a fixed point $y \in U$ and consider the mapping $\ell_{y*x^{-1}} : G \to G$. Evidently, $\ell_{y*x^{-1}}(x) = y$, and by [4, Theorem 3.1], $\ell_{y*x^{-1}}$ is an S-homeomorphism of G onto itself. Thus the set $V = U \cap \ell_{y*x^{-1}}(W)$ is semi-open as the intersection of an open set U and a semi-open set $\ell_{y*x^{-1}}(W)$, i.e. V is a semi-open neighbourhood of y in U. By assumption on f, the set $f(V)$ is semi-open in $f(U)$ and also in H by Lemma 3.5. Set $z = f(x * y^{-1})$ and consider the mapping $\ell_z : H \to H$. We have $\ell_z(f(y)) = z \cdot f(y) = f(x)$. Clearly, $\ell_z \circ f \circ \ell_{y*x^{-1}} = f$, hence $(\ell_z \circ f \circ \ell_{y*x^{-1}})(W) = f(W)$. However, ℓ_z is an S-homeomorphism of H onto itself. As $f(V)$ is semi-open in H, the set $\ell_z(f(V))$ is also semi-open in H. Therefore, the set $f(W)$ contains a semi-open neighbourhood $\ell_z(f(V))$ of $f(x)$ in H, so that $f(W) \in \mathsf{SO}(H, f(x))$ as required. □

5 Semi-quotients of topologized groups

In this section we apply the construction of $s\tau_Q$ described in Section 3 to topologized groups and establish some properties of their semi-quotients.

If G is a topological group and H a subgroup of G, we can look at the collection G/H of left cosets of H in G (or the collection $H\backslash G$ of right cosets of H in G), and endow G/H (or $G\backslash H$) with the semi-quotient structure induced by the natural projection $p : G \to G/H$. Recall that G/H is not a group under coset multiplication unless H is a normal subgroup of G.

The following simple lemmas may be quite useful in what follows.

Lemma 5.1 ([20]). *Let $p : G \to G/H$ be a canonical projection map. Then for any subset U of G, $p^{\leftarrow}(p(U)) = U * H$.*

Lemma 5.2 ([25]). *Let $(G,*,\tau)$ be an s-topological group, K an s-compact subset of G, and F a semi-closed subset of G. Then $F*K$ and $K*F$ are semi-closed subsets of G.*

Lemma 5.3 ([4]). *Let $(G,*,\tau)$ be an s-topological group. Then each left (right) translation in G is an S-homeomorphism. Moreover they and symmetry mappings are actually semi-homeomorphisim (see [1, Remark 1]).*

Lemma 5.4 ([26]). *If $f: X \to Y$ is a semi-continuous mapping and X_0 is an open set in X, then the restriction $f|_{X_0}: X_0 \to Y$ is semi-continuous.*

Theorem 5.5. *Let $(G,*,\tau)$ be an extremally disconnected irresolute-topological group and H its invariant subgroup. Then $p:(G,*,\tau)\to(G/H,\bar{*},s\tau_Q)$ is pre-semi-open.*

Proof. Let $V\subset G$ be semi-open. By the definition of semi-quotient topology, $p(V)\subset G/H$ is open if and only if $p^{\leftarrow}(p(V))\subset G$ is open. By Lemma 5.1 $p^{\leftarrow}(p(V))=V*H$. Since V is semi-open, $V*H$ is semi-open and so $p(V)$ is semi-open. Hence p is pre-semi-open. □

Theorem 5.6. *Let $(G,*,\tau)$ be an extremally disconnected irresolute-topological group, H its invariant subgroup. Then $(G/H,\bar{*},s\tau_Q)$ is an irresolute-topological group.*

Proof. First, we observe that $s\tau_Q$ is a topology on G/H. Let $x*H, y*H\in G/H$ and let $W\subset G/H$ be a semi-open neighbourhood of $(x*H)\bar{*}(y*H)^{-1}$. By the definition of $s\tau_Q$ (induced by p), the set $p^{\leftarrow}(W)$ is a semi-open neighbourhooud of $x*y^{-1}$ in G, and since G is an irresolute-topological group, there are semi-open sets $U\subset \mathrm{SO}(G,x)$ and $V\subset\mathrm{SO}(G,y)$ such that $U*V^{-1}\subset p^{\leftarrow}(W)$. By Theorem 5.5, the sets $p(U)=U*H$ and $p(V)=V*H$ are semi-open in G/H, contain $x*H$ and $y*H$, respectively, and satisfy

$$(U*H)\bar{*}(V*H)^{-1}=(U*V^{-1})*H=p(U*V^{-1})\subset p(p^{\leftarrow}(W))\subset W.$$

This just means that $(G/H,\bar{*},s\tau_Q)$ is an irresolute-topological group. □

Theorem 5.7. *Let $(G,*,\tau)$ be an s-topological group and H a subgroup of G. Then for every semi-open set $U\subset G$, the set $p(U)$ belongs to $s\tau_Q$. In particular, if G is extremally disconnected, then p is an s-open mapping from G to $(G/H,s\tau_Q)$.*

Proof. Let $V\subset G$ be semi-open. By definition of $s\tau_Q$, $p(V)\in s\tau_Q$ if and only if $p^{\leftarrow}(p(V))\subset G$ is semi-open, i.e. $V*H$ is semi-open in G. But $V*H$ is semi-open in G because $V\in\mathrm{SO}(G)$ and $(G,*,\tau)$ is an s-topological group. Clearly, if $s\tau_Q$ is a topology, the last condition actually says that p is an s-open mapping. □

The following theorem is similar to Theorem 5.7.

Theorem 5.8. *If H is an s-compact subgroup of an s-topological group $(G,*,\tau)$, then for every semi-closed set $F\subset G$, the set $p(G\setminus F)$ belongs to $s\tau_Q$. If $s\tau_Q$ is a topology, then p is an s-perfect mapping.*

Proof. Let $F\subset G$ be semi-closed. By Lemma 5.2 the set $p^{\leftarrow}(p(F))=F*H\subset G$ is semi-closed. By definition of $s\tau_Q$, $G/H\setminus(F*H)\in s\tau_Q$.

Let now $s\tau_Q$ be a topology on G/H. Take any semi-closed subset F of G. The set $F*H$ is semi-closed in G and $F*H=p^{\leftarrow}(p(F))$. This implies, $p(F)$ is closed in the semi-quotient space G/H. Thus p is an s-closed mapping. On the other hand, if $z*H\in G/H$ and $p(x)=z*H$ for some $x\in G$, then $p^{\leftarrow}(z*H)=p^{\leftarrow}(p(x))=x*H$, and by Lemmas 4.3 and 5.3 this set is s-compact in G. Therefore, p is s-perfect. □

Corollary 5.9. *Let $(G,*,\tau)$ be an extremally disconnected s-topological group and H its s-compact subgroup. If the semi-quotient space $(G/H,s\tau_Q)$ is compact, then G is s-compact.*

Proof. By Theorem 5.8, the projection $p: G\to G/H$ is s-perfect. Then by Theorem 4.4 we obtain that $p^{\leftarrow}(p(G))=G*H=G$ is s-compact. □

Theorem 5.10. *Suppose that $(G, *, \tau)$ is an extremally disconnected s-topological group, H an invariant subgroup of G, $p : G \to (G/H, s\tau_Q)$ the canonical projection. Let U and V be semi-open neighbourhoods of e in G such that $V^{-1} * V \subset U$. Then* $\mathrm{Cl}(p(V)) \subset p(U)$.

Proof. Let $p(x) \in \mathrm{Cl}(p(V))$. Since $V * x$ is a semi-open neighbourhood of $x \in G$ and, by Theorem 5.7, p is s-open, we have that $p(V * x)$ is an open neighbourhood of $p(x)$. Therefore, $p(V * x) \cap p(V) \neq \emptyset$. It follows that for some $a, b \in V$ we have $p(a * x) = p(b)$, that is $a * x * h_1 = b * h_2$ for some $h_1, h_2 \in H$. Hence,

$$x = a^{-1} * b * h_2 * h_1^{-1} = (a^{-1} * b) * (h_2 * h_1^{-1}) \in U * H$$

since $a^{-1} * b \in V^{-1} * V \subset U$ and H is a subgroup of G. Therefore, $p(x) \in p(U * H) = U * H * H = U * H = p(U)$. □

Theorem 5.11. *Let $(G, *, \tau)$ be an extremally disconnected irresolute-topological group and H an invariant subgroup of G. Then the semi-quotient space $(G/H, s\tau_Q)$ is regular.*

Proof. Let W be an open neighbourhood of $p(e_G) = H$ in G/H. By semi-continuity of p, we can find a semi-open neighbourhood U of e_G such that $p(U) \subset W$. As G is extremally disconnected and irresolute-topological group, it follows from $e_G * e_G^{-1} = e_G$ that there is a semi-open neighbourhood V of e_G such that $V * V^{-1} \subset U$. By Theorem 5.10 we have $\mathrm{Cl}(p(V)) \subset p(U) \subset W$. By Theorem 5.7, $p(V)$ is an open neighbourhood of $p(e_G)$. This proves that $(G/H, s\tau_Q)$ is a regular space. □

If $(G, *)$ is a group, H its subgroup, and $a \in G$, then we define the mapping $\lambda_a : G/H \to G/H$ by $\lambda_a(x * H) = a * (x * H)$. This mapping is called a *left translation of G/H by a* [20].

Theorem 5.12. *If $(G, *, \tau)$ is an extremally disconnected irresolute-topological group, H a subgroup of G, and $a \in G$, then the mapping λ_a is a semi-homeomorphism and $p \circ \ell_a = \lambda_a \circ p$ holds.*

Proof. Since G is a group, it is easy to see that λ_a is a (well defined) bijection on G/H. We prove that $\lambda_a \circ p = p \circ \ell_a$. Indeed, for each $x \in G$ we have $(p \circ \ell_a)(x) = p(a * x) = (a * x) * H = a * (x * H) = \lambda_a(p(x)) = (\lambda_a \circ p)(x)$. This is required. It remains to prove that λ_a is irresolute and pre-semi-open.

This follows from the following facts. Let $x * H \in G/H$. For any semi-open neighbourhood U of e_G, $p(x * U * H)$ is a semi-open neighbourhood of $x * H$ in G/H. Similarly, the set $p(a * x * U * H)$ is a semi-open neighbourhood of $a * x * H$ in G/H. Since

$$\lambda_a(p(x * U * H)) = p(l_a(x * U * H)) = p(a * x * U * H),$$

it follows that λ_a is a semi-homeomorphism. □

Definition 5.13. *A mapping $f : X \to Y$ is:*
- *an* S-isomorphism *if it is an algebraic isomorphism and (topologically) an S-homeomorphism;*
- *a* semi-isomorphism *if it is an algebraic isomorphism and a semi-homeomorphism.*

Theorem 5.14. *Let $(G, *, \tau_G)$ and $(H, \cdot, \tau_H)$ be extremally disconnected irresolute-topological groups and $f : G \to H$ a semi-isomorphism. If G_0 is an invariant subgroup of G and $H_0 = f(G_0)$, then the semi-quotient irresolute-topological groups $(G/G_0, s\tau_Q)$ and $(H/H_0, s\tau_Q)$ are semi-isomorphic.*

Proof. Let $p : G \to G/G_0$, $x \mapsto x * G_0$, and $\pi : H \to H/H_0$, $f(x_0) \mapsto f(x_0) \cdot H_0$ $(x_0 \in G_0)$ be the canonical projections. Consider the mapping $\varphi : G/G_0 \to H/H_0$ defined by

$$\varphi(x * G_0) = f(x) \cdot f(G_0), \ \ x \in G, \ \ y = f(x).$$

Then for $x_1 * G_0, x_2 * G_0 \in G/G_0$ we have

$$\varphi(x_1 * G_0 * x_2 * G_0) = \varphi(x_1 * x_2 * G_0) = f(x_1 * x_2) \cdot f(G_0) = y_1 \cdot y_2 \cdot H_0 = \varphi(x_1 * G_0) \cdot \varphi(x_2 * G_0),$$

i.e. φ is a homomorphism. Let us prove that φ is one-to-one. Let $x * G_0$ be an arbitrary element of G/G_0. Set $y = f(x)$. If $\varphi(x * G_0) = H_0$, then $\pi(y) = H_0$, which implies $x \in G_0$, $y \in H_0$, and $\ker\varphi = G_0$. So, φ is one-to-one.

Next, we have $\varphi(x * G_0) = y \cdot H_0$, i.e. $\varphi(p(x)) = \pi(y) = \pi(f(x))$. This implies $\varphi \circ p = \pi \circ f$. Since f is a semi-homeomorphism, and p and π are s-open, semi-continuous homomorphisms, we conclude that φ is open and continuous. Hence φ is semi-homeomorphism and a semi-isomorphism. □

Theorem 5.15. *Let $(G, *, \tau)$ be an extremally disconnected irresolute-topological group, H an invariant subgroup of G, M an open subgroup of G, and $p : G \to G/H$ the canonical projection. Then the semi-quotient group $M * H/H$ is semi-isomorphic to the subgroup $p(M)$ of G/H.*

Proof. It is clear that $M * H = p^{\leftarrow}(p(M))$. As p is s-open and semi-continuous and M is open in τ, the restriction π of p to $M * H$ is an s-open and semi-continuous mapping of $M * H$ onto $p(M)$ by Lemma 5.4. Since M is a subgroup of G and p is a homomorphism it follows that $p(M)$ is a subgroup of G/H, $M * H$ is a subgroup of G, and $\pi : M * H \to p(M)$ is a homomorphism. We have $\pi^{\leftarrow}(\pi(e_G)) = p^{\leftarrow}(p(e_G)) = H$, i.e. $\ker\pi = H$. It is easy now to conclude that $M * H/H$ and $p(M)$ are semi-isomorphic. □

References

[1] Levine N., Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 1963, 70, 36-41

[2] Crossley S.G., Hildebrand S.K., Semi-topological properties, Fund. Math. 1972, 74, 233–254

[3] Bohn E., Lee J., Semi-topological groups, Amer. Math. Monthly, 1965, 72, 996-998

[4] Siddique Bosan M., Moiz ud Din Khan, Kočinac Lj. D.R., On s-topological groups, Math. Moravica, 2014, 18:2, 35-44

[5] Siab A., Moiz ud Din Khan, Kočinac Lj. D.R., Irresolute-topological groups, math. moravica, 2015, 19:1, 73-80

[6] Moiz ud Din Khan and Rafaqat Noreen, On Paratopologized Groups, Analele Universitatii din Oradea - Fasc. Matematica, 2016, Tom XXIII:2, 147-157

[7] Oner T., Kandemir M.B., Tanay B., Semi-topological groups with respect to semi-continuity and irresoluteness, J. Adv. stud. topology, 2013, 4:3, 23-28

[8] Oner T., Ozek A., A note on quasi irresolute topological groups, JLTA, 2016, 5:1, 41-46

[9] Oner T., Ozek A., On semi-topological groups with respect to irresoluteness, Int. J. Recent Sci. Res., 2015, 6:12, 7914-7916

[10] R. Engelking, General Topology, 2nd edition, Heldermann-Verlag, Berlin, 1989

[11] D.R. Anderson, J.A. Jensen, Semi-continuity on topological spaces, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat., 1967, 42, 782–783

[12] T.R. Hamlett, A correction to the paper "Semi-open sets and semi-continuity in topological spaces" by Norman Levine, Proc. Amer. Math. Soc., 1975, 49, 458–460

[13] Nour T.M., A note on some applications of semi-open sets, Internat. J. Math. Math. Sci., 1998, 21, 205-207

[14] Piotrowski Z., On semi-homeomorphisms, Boll. U.M.I., 1979, 5:16-A, 501–509

[15] Crossley S.G., Hildebrand S.K., Semi-closure, Texas J. Sci., 1971, 22, 99-112

[16] Crossley S.G., Hildebrand S.K., Semi-closed sets and semi-continuity in topological spaces, Texas J. Sci., 1971, 22, 123-126

[17] Corson H.H., Michael E., Metrizability of certain countable unions, Illinois J. Math., 1964, 8, 351-360

[18] Carnahan D.A., Some properties related to compactness in topological spaces, Ph.D. Thesis, University of Arkansas, 1973

[19] Lee J.P, On semihomeomorphisms, Internat. J. Math. Math. Sci., 1990, 13, 129-134

[20] Arhangel'skii A.V., Tkachenko M., Topological Groups and Related Structures, Atlantis Studies in Mathematics, Vol. 1, Atlantis Press/World Scientific, Amsterdam-Paris, 2008

[21] Munkres J.R., Topology (Second edition), Prentice Hall, Inc., 2000

[22] Noiri T., On semi-continuous mappings, Atti Accad. Naz. Lincei, Ser., 1973, VIII 54:2, 210-214

[23] Pipitone V., Russo G., Spazi semiconnessi e spazi semiaperti, Rend. Circ. Mat. Palermo, 1975, 24:3, 273-285

[24] Njåstad O., On some classes of nearly open sets, Pacific J. Math., 1965, 15, 961-970

[25] Siddique Bosan M., Moiz ud Din Khan, A note on s-topological groups, Life Science Journal, 2014, 11:7s, 370-374

[26] Noiri T., On semi-continuous mappings, Atti.Lin.Rend.Sc.Fis.Mat.Natur., 1973, 8:54, 210-215.