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New bounds for the minimum eigenvalue of M-matrices

Abstract: Some new bounds for the minimum eigenvalue of M-matrices are obtained. These inequalities improve existing results, and the estimating formulas are easier to calculate since they only depend on the entries of matrices. Finally, some examples are also given to show that the bounds are better than some previous results.

Keywords: M-matrix, Nonnegative matrix, Hadamard product, Spectral radius, Minimum eigenvalue

MSC: 15A18, 15A42

1 Introduction

Let $C^{n\times n}$ ($R^{n\times n}$) denote the set of all $n \times n$ complex (real) matrices, $A = (a_{ij}) \in C^{n\times n}$, $N = \{1, 2, \ldots, n\}$. We write $A \geq 0$ if all $a_{ij} \geq 0 (i, j \in N)$. A is called nonnegative if $A \geq 0$. Let Z_n denote the class of all $n \times n$ real matrices all of whose off-diagonal entries are nonpositive. A matrix A is called an M-matrix [1] if $A \in Z_n$ and the inverse of A, denoted by A^{-1}, is nonnegative. M_n will be used to denote the set of all $n \times n$ M-matrices.

Let A be an M-matrix. Then there exists a positive eigenvalue of A, $\tau(A) = \rho(A^{-1})^{-1}$, where $\rho(A^{-1})$ is the spectral radius of the nonnegative matrix A^{-1}, $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, $\sigma(A)$ denotes the spectrum of A. $\tau(A)$ is called the minimum eigenvalue of A [2, 3].

The Hadamard product of two matrices $A = (a_{ij}) \in R^{n\times n}$ and $B = (b_{ij}) \in R^{n\times n}$ is the matrix $A \circ B = (a_{ij}b_{ij}) \in R^{n\times n}$.

An $n \times n$ matrix A is said to be reducible if there exists a permutation matrix P such that

$$P^T A P = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11}, A_{22} are square matrices of order at least one. We call A irreducible if it is not reducible. Note that any nonzero 1×1 matrix is irreducible.

A matrix $A = (a_{ij}) \in C^{n\times n}$ is called a weakly chained diagonally dominant M-matrix [4] if A satisfies the following conditions:

(i) For all $i, j \in N$ with $i \neq j$, $a_{ij} \leq 0$ and $a_{ii} > 0$;

(ii) For all $i \in N$, $|a_{ii}| \geq \sum\limits_{j=1,\neq i}^{n} |a_{ij}|$ and $J(A) = \left\{ i \in N : |a_{ii}| > \sum\limits_{j=1,\neq i}^{n} |a_{ij}| \right\} \neq \emptyset$;

(iii) For all $i \in N$, $i \notin J(A)$, there exist indices $i_1, i_2, \cdots, i_k$ in N with $a_{i_r,i_{r+1}} \neq 0$, $0 \leq r \leq k-1$, where $i_0 = i$ and $i_k \in J(A)$.

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Estimating the bounds for the minimum eigenvalue $\tau(A)$ of an M-matrix A is an interesting subject in matrix theory, and has important applications in many practical problems [4–6]. Hence, it is necessary to estimate the bounds for $\tau(A)$.

In [4], Shivakumar et al. obtained the following bound for $\tau(A)$: Let $A = (a_{ij}) \in R^{n \times n}$ be a weakly chained diagonally dominant M-matrix, and let $A^{-1} = (\alpha_{ij})$. Then

$$\min_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\} \le \tau(A) \le \max_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\}, \quad \tau(A) \le \min_{i \in N}\{a_{ii}\}, \quad \frac{1}{M} \le \tau(A) \le \frac{1}{m}, \tag{1}$$

where

$$M = \max_{i \in N} \left\{ \sum_{j \in N} \alpha_{ij} \right\}, \quad m = \min_{i \in N} \left\{ \sum_{j \in N} \alpha_{ij} \right\}.$$

Subsequently, Tian *et al.* [7] provided a lower bound for $\tau(A)$ by using the spectral radius of the Jacobi iterative matrix J_A of A: Let $A = (a_{ij}) \in R^{n \times n}$ be an M-matrix and $A^{-1} = (\alpha_{ij})$. Then

$$\tau(A) \ge \frac{1}{1 + (n-1)\rho(J_A)} \frac{1}{\max_{i \in N}\{\alpha_{ii}\}}. \tag{2}$$

Recently, Li *et al.* [8] improved (2) and gave the following result: Let $B = (b_{ij}) \in R^{n \times n}$ be an M-matrix and $B^{-1} = (\beta_{ij})$. Then

$$\tau(B) \ge \frac{2}{\max_{i \ne j}\{\beta_{ii} + \beta_{jj} + [(\beta_{ii} - \beta_{jj})^2 + 4(n-1)^2 \beta_{ii}\beta_{jj}\rho^2(J_B)]^{\frac{1}{2}}\}}. \tag{3}$$

In this paper, we continue to research the problems mentioned above. For M-matrix B, we establish some new inequalities on the bounds for $\tau(B)$. Finally, some examples are given to illustrate our results.

For convenience, we employ the following notation throughout. Let $A = (a_{ij})$ be an $n \times n$ matrix. For $i, j, k \in N, i \ne j$, denote

$$R_i = \sum_{j \ne i} |a_{ji}|, \quad d_i = \frac{R_i}{|a_{ii}|}, \quad s_{ji} = \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}|d_k}{|a_{jj}|};$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}|s_{ki}}{|a_{jj}|}, \quad v_{ji} = \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}|u_{ki}}{|a_{jj}|};$$

$$h_i = \max_{j \ne i} \left\{ \frac{|a_{ji}|}{|a_{jj}|v_{ji} - \sum_{k \ne j,i} |a_{jk}|v_{ki}} \right\}, \quad w_{ji} = \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}|v_{ki}h_i}{|a_{jj}|}, \quad w_i = \max_{j \ne i}\{w_{ij}\}.$$

2 Main results

In this section, we present our main results. Firstly, we give some notation and lemmas.

Let $A \ge 0$ and $D = diag(a_{ii})$. Denote $C = A - D$, $J_A = D_1^{-1}C$, $D_1 = diag(d_{ii})$, where

$$d_{ii} = \begin{cases} 1, & if \ a_{ii} = 0, \\ a_{ii}, & if \ a_{ii} \ne 0. \end{cases}$$

By the definition of J_A, we obtain

$$\rho(J_{A^T}) = \rho(D_1^{-1}C^T) = \rho(CD_1^{-1}) = \rho(D_1^{-1}(CD_1^{-1})D_1) = \rho(D_1^{-1}C) = \rho(J_A).$$

Lemma 2.1 ([9]). *Let $A \in C^{n \times n}$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_i \left\{ z \in C : |z - a_{ii}| \leq x_i \sum_{j \neq i} \frac{1}{x_j} |a_{ji}|, \ i \in N \right\}.$$

Lemma 2.2 ([3]). *Let $A \in C^{n \times n}$, and let $x_1, x_2, \cdots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{j \neq i} \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \left(x_i \sum_{k \neq i} \frac{1}{x_k} |a_{ki}| \right) \left(x_j \sum_{l \neq j} \frac{1}{x_l} |a_{lj}| \right) \right\}.$$

Lemma 2.3 ([3]). *Let $A, B \in R^{n \times n}$, and let $X, Y \in R^{n \times n}$ be diagonal matrices. Then*

$$X(A \circ B)Y = (XAY) \circ B = (XA) \circ (BY) = (AY) \circ (XB) = A \circ (XBY).$$

Lemma 2.4 ([3]). *Let $A = (a_{ij}) \in M_n$. Then there exists a positive diagonal matrix X such that $X^{-1}AX$ is a strictly diagonally dominant M-matrix.*

Lemma 2.5. *Let $A = (a_{ij}) \in R^{n \times n}$ be a strictly diagonally dominant matrix and let $A^{-1} = (\alpha_{ij})$. Then*

$$\alpha_{ij} \leq w_{ji}\alpha_{jj} \leq w_j\alpha_{jj}, \quad j, i \in N, \quad j \neq i.$$

Proof. This proof is similar to the one of Lemma 2.2 in [10]. $\qquad \square$

Theorem 2.6. *Let $A = (a_{ij}) \geq 0$, $B = (b_{ij}) \in M_n$ and let $B^{-1} = (\beta_{ij})$. Then*

$$\rho(A \circ B^{-1}) \leq \max_{1 \leq i \leq n} \{(a_{ii} + w_i \rho(\mathcal{J}_A)d_{ii})\beta_{ii}\}. \tag{4}$$

Proof. It is evident that the result holds with equality for $n = 1$.

We next assume that $n \geq 2$.

(1) First, we assume that A and B are irreducible matrices. Since B is an M-matrix, by Lemma 2.4, there exists a positive diagonal matrix X, such that $X^{-1}BX$ is a strictly row diagonally dominant M-matrix, and

$$\rho(A \circ B^{-1}) = \rho(X^{-1}(A \circ B^{-1})X) = \rho(A \circ (X^{-1}BX)^{-1}).$$

Hence, for convenience and without loss of generality, we assume that B is a strictly diagonally dominant matrix.

On the other hand, since A is irreducible and so is $\mathcal{J}_{A^T}$. Then there exists a positive vector $x = (x_i)$ such that $\mathcal{J}_{A^T}x = \rho(\mathcal{J}_{A^T})x = \rho(\mathcal{J}_A)x$, thus, we obtain $\sum_{j \neq i} a_{ji}x_j = \rho(\mathcal{J}_A)d_{ii}x_i$.

Let $\widetilde{A} = (\tilde{a}_{ij}) = XAX^{-1}$ in which X is the positive matrix $X = diag(x_1, x_2, \ldots, x_n)$. Then, we have

$$\widetilde{A} = (\tilde{a}_{ij}) = XAX^{-1} = \begin{pmatrix} a_{11} & \frac{a_{12}x_1}{x_2} & \cdots & \frac{a_{1n}x_1}{x_n} \\ \frac{a_{21}x_2}{x_1} & a_{22} & \cdots & \frac{a_{2n}x_2}{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}x_n}{x_1} & \frac{a_{n2}x_n}{x_2} & \cdots & a_{nn} \end{pmatrix}.$$

From Lemma 2.3, we have

$$\widetilde{A} \circ B^{-1} = (XAX^{-1}) \circ B^{-1} = X(A \circ B^{-1})X^{-1}.$$

Thus, we obtain $\rho(\widetilde{A} \circ B^{-1}) = \rho(A \circ B^{-1})$. Let $\lambda = \rho(\widetilde{A} \circ B^{-1})$, so that $\lambda \geq a_{ii}\beta_{ii}, \forall i \in N$. By Lemma 2.1, there exists $i_0 \in N$, such that

$$|\lambda - a_{i_0 i_0}\beta_{i_0 i_0}| \leq w_{i_0} \sum_{t \neq i_0} \frac{1}{w_t} \tilde{a}_{t i_0} \beta_{t i_0} \leq w_{i_0} \sum_{t \neq i_0} \frac{1}{w_t} \tilde{a}_{t i_0} w_{t i_0} \beta_{i_0 i_0}$$

$$\leq w_{i_0} \sum_{t \neq i_0} \tilde{a}_{t i_0} \beta_{i_0 i_0} = w_{i_0} \beta_{i_0 i_0} \sum_{t \neq i_0} \frac{a_{t i_0} x_t}{x_{i_0}}$$

$$= w_{i_0} \rho(\mathcal{J}_A) d_{i_0 i_0} \beta_{i_0 i_0}.$$

Therefore,

$$\lambda \leq a_{i_0 i_0} \beta_{i_0 i_0} + w_{i_0} \rho(\mathcal{J}_A) d_{i_0 i_0} \beta_{i_0 i_0} = (a_{i_0 i_0} + w_{i_0} \rho(\mathcal{J}_A) d_{i_0 i_0}) \beta_{i_0 i_0},$$

i.e.,

$$\rho(A \circ B^{-1}) \leq (a_{i_0 i_0} + w_{i_0} \rho(\mathcal{J}_A) d_{i_0 i_0}) \beta_{i_0 i_0} \leq \max_{1 \leq i \leq n} \{(a_{ii} + w_i \rho(\mathcal{J}_A) d_{ii}) \beta_{ii}\}.$$

(2) Now, assume that one of A and B is reducible. It is well known that a matrix in Z_n is a nonsingular M-matrix if and only if all its leading principal minors are positive (*see* [1]). If we denote by $T = (t_{ij})$ the $n \times n$ monomial matrix with $t_{12} = t_{23} = \ldots = t_{n-1,n} = t_{n1} = -1$, the remaining t_{ij} zero, then both $A - \varepsilon T$ and $B + \varepsilon T$ are irreducible matrices for any chosen positive real number ε, sufficiently small such that all the leading principal minors of $B + \varepsilon T$ are positive. Now, we substitute $A - \varepsilon T$ and $B + \varepsilon T$ for A and B, respectively, in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. □

Theorem 2.7. *Let $B = (b_{ij}) \in M_n$ and $B^{-1} = (\beta_{ij})$. Then*

$$\tau(B) \geq \frac{1}{\max\limits_{1 \leq i \leq n} \{(1 + w_i (n-1)) \beta_{ii}\}}. \tag{5}$$

Proof. Let all entries of A in (4) be 1. Then $a_{ii} = 1 (\forall i \in N)$, $\rho(\mathcal{J}_A) = n - 1$. Therefore, by (4), we have

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{1}{\max\limits_{1 \leq i \leq n} \{(1 + w_i (n-1)) \beta_{ii}\}}.$$

The proof is completed. □

Theorem 2.8. *Let $A = (a_{ij}) \geq 0$, $B = (b_{ij}) \in M_n$ and let $B^{-1} = (\beta_{ij})$. Then*

$$\rho(A \circ B^{-1}) \leq \frac{1}{2} \max_{i \neq j} \{a_{ii} \beta_{ii} + a_{jj} \beta_{jj} + \Delta_{ij}\}, \tag{6}$$

where $\Delta_{ij} = [(a_{ii} \beta_{ii} - a_{jj} \beta_{jj})^2 + 4 w_i w_j \rho^2(\mathcal{J}_A) d_{ii} d_{jj} \beta_{ii} \beta_{jj}]^{\frac{1}{2}}$.

Proof. It is evident that the result holds with equality for $n = 1$.

We next assume that $n \geq 2$. For convenience and without loss of generality, we assume that B is a strictly row diagonally dominant matrix.

(i) First, we assume that A and B are irreducible matrices. Since A is irreducible and so is $\mathcal{J}_{A^T}$. Then there exists a positive vector $y = (y_i)$ such that $\mathcal{J}_{A^T} y = \rho(\mathcal{J}_{A^T}) y = \rho(\mathcal{J}_A) y$, thus, we obtain

$$\sum_{k \neq i} a_{ki} y_k = \rho(\mathcal{J}_A) d_{ii} y_i, \quad \sum_{k \neq j} a_{kj} y_k = \rho(\mathcal{J}_A) d_{jj} y_j.$$

Let $\widehat{A} = (\hat{a}_{ij}) = Y A Y^{-1}$ in which Y is the positive matrix $Y = diag(y_1, y_2, \ldots, y_n)$. Then, we have

$$\widehat{A} = (\hat{a}_{ij}) = Y A Y^{-1} = \begin{pmatrix} a_{11} & \frac{a_{12} y_1}{y_2} & \cdots & \frac{a_{1n} y_1}{y_n} \\ \frac{a_{21} y_2}{y_1} & a_{22} & \cdots & \frac{a_{2n} y_2}{y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1} y_n}{y_1} & \frac{a_{n2} y_n}{y_2} & \cdots & a_{nn} \end{pmatrix}.$$

From Lemma 2.3, we get

$$\widehat{A} \circ B^{-1} = (Y A Y^{-1}) \circ B^{-1} = Y (A \circ B^{-1}) Y^{-1}.$$

Thus, we obtain $\rho(\widehat{A} \circ B^{-1}) = \rho(A \circ B^{-1})$. Let $\lambda = \rho(\widehat{A} \circ B^{-1})$, so that $\lambda \geq a_{ii}\beta_{ii} (\forall i \in N)$. By Lemma 2.2, there exist $i_0, j_0 \in N, i_0 \neq j_0$ such that

$$|\lambda - a_{i_0 i_0}\beta_{i_0 i_0}||\lambda - a_{j_0 j_0}\beta_{j_0 j_0}| \leq \left(w_{i_0} \sum_{k \neq i_0} \frac{1}{w_k}\hat{a}_{k i_0}\beta_{k i_0}\right)\left(w_{j_0} \sum_{k \neq j_0} \frac{1}{w_k}\hat{a}_{k j_0}\beta_{k j_0}\right).$$

Note that

$$w_{i_0} \sum_{k \neq i_0} \frac{1}{w_k}\hat{a}_{k i_0}\beta_{k i_0} \leq w_{i_0} \sum_{k \neq i_0} \frac{1}{w_k}\hat{a}_{k i_0}w_{k i_0}\beta_{i_0 i_0} \leq w_{i_0}\beta_{i_0 i_0} \sum_{k \neq i_0} \hat{a}_{k i_0} = w_{i_0}\beta_{i_0 i_0}\rho(\mathcal{J}_A)d_{i_0 i_0},$$

$$w_{j_0} \sum_{k \neq j_0} \frac{1}{w_k}\hat{a}_{k j_0}\beta_{k j_0} \leq w_{j_0} \sum_{k \neq j_0} \frac{1}{w_k}\hat{a}_{k j_0}w_{k j_0}\beta_{j_0 j_0} \leq w_{j_0}\beta_{j_0 j_0} \sum_{k \neq j_0} \hat{a}_{k j_0} = w_{j_0}\beta_{j_0 j_0}\rho(\mathcal{J}_A)d_{j_0 j_0}.$$

Hence, we obtain

$$\lambda \leq \frac{1}{2}\left(a_{i_0 i_0}\beta_{i_0 i_0} + a_{j_0 j_0}\beta_{j_0 j_0} + \Delta_{i_0 j_0}\right),$$

i.e.,

$$\rho(A \circ B^{-1}) \leq \frac{1}{2}\left(a_{i_0 i_0}\beta_{i_0 i_0} + a_{j_0 j_0}\beta_{j_0 j_0} + \Delta_{i_0 j_0}\right) \leq \frac{1}{2} \max_{i \neq j}\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} + \Delta_{ij}\},$$

where $\Delta_{ij} = [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4w_i w_j \rho^2(\mathcal{J}_A)d_{ii}d_{jj}\beta_{ii}\beta_{jj}]^{\frac{1}{2}}$.

(ii) Now, assume that one of A and B is reducible. We substitute $A - \varepsilon T$ and $B + \varepsilon T$ for A and B, respectively, in the previous case (as in the proof of Theorem 2.6), and then letting $\varepsilon \to 0$, the result follows by continuity. $\qquad\square$

Theorem 2.9. *Let $B = (b_{ij}) \in M_n$ and $B^{-1} = (\beta_{ij})$. Then*

$$\tau(B) \geq \frac{2}{\max\limits_{i \neq j}\{\beta_{ii} + \beta_{jj} + \Delta_{ij}\}}, \tag{7}$$

where $\Delta_{ij} = [(\beta_{ii} - \beta_{jj})^2 + 4(n-1)^2 w_i w_j \beta_{ii}\beta_{jj}]^{\frac{1}{2}}$.

Proof. Let all entries of A in (6) be 1. Then

$$a_{ii} = 1 (\forall i \in N), \quad \rho(\mathcal{J}_A) = n - 1, \quad \Delta_{ij} = [(\beta_{ii} - \beta_{jj})^2 + 4(n-1)^2 w_i w_j \beta_{ii}\beta_{jj}]^{\frac{1}{2}}.$$

Therefore, by (6), we have

$$\tau(B) = \frac{1}{\rho(B^{-1})} \geq \frac{2}{\max\limits_{i \neq j}\{\beta_{ii} + \beta_{jj} + \Delta_{ij}\}}.$$

The proof is completed. $\qquad\square$

Remark 2.10. *We next give a simple comparison between* (4) *and* (6), (5) *and* (7), *respectively. For convenience and without loss of generality, we assume that for $i, j \in N, i \neq j$,*

$$a_{jj}\beta_{jj} + w_j d_{jj}\beta_{jj}\rho(\mathcal{J}_A) \leq a_{ii}\beta_{ii} + w_i d_{ii}\beta_{ii}\rho(\mathcal{J}_A),$$

i.e.

$$w_j d_{jj}\beta_{jj}\rho(\mathcal{J}_A) \leq a_{ii}\beta_{ii} - a_{jj}\beta_{jj} + w_i d_{ii}\beta_{ii}\rho(\mathcal{J}_A).$$

Hence,

$$\Delta_{ij} = [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4w_i w_j \rho^2(\mathcal{J}_A)d_{ii}d_{jj}\beta_{ii}\beta_{jj}]^{\frac{1}{2}}$$

$$\leq [(a_{ii}\beta_{ii} - a_{jj}\beta_{jj})^2 + 4w_i\rho(\mathcal{J}_A)d_{ii}\beta_{ii}(a_{ii}\beta_{ii} - a_{jj}\beta_{jj} + w_i d_{ii}\beta_{ii}\rho(\mathcal{J}_A))]^{\frac{1}{2}}$$
$$= a_{ii}\beta_{ii} - a_{jj}\beta_{jj} + 2w_i d_{ii}\beta_{ii}\rho(\mathcal{J}_A).$$

Further, we obtain

$$a_{ii}\beta_{ii} + a_{jj}\beta_{jj} + \Delta_{ij} \leq 2a_{ii}\beta_{ii} + 2w_i d_{ii}\beta_{ii}\rho(\mathcal{J}_A),$$

by (6),

$$\rho(A \circ B^{-1}) \leq \frac{1}{2}\max_{i \neq j}\{a_{ii}\beta_{ii} + a_{jj}\beta_{jj} + \Delta_{ij}\} \leq \max_{1 \leq i \leq n}\{(a_{ii} + w_i\rho(\mathcal{J}_A)d_{ii})\beta_{ii}\},$$

So, the bound in (6) *is better than the bound in* (4). *Similarly, we can prove that the bound in* (7) *is better than the bound in* (5).

3 Numerical examples

In this section, we present numerical examples to illustrate the advantages of our derived results.

Example 3.1. *Let*

$$B = \begin{pmatrix} 1.1 & -0.4 & -0.4 \\ -0.3 & 1 & -0.5 \\ -0.3 & -0.2 & 0.6 \end{pmatrix}.$$

It is easy to see that B is an M-matrix. By calculations with Matlab 7.1, we have

$\tau(B) \geq 0.10000000$ (by (1)), $\tau(B) \geq 0.09701726$ (by (2)), $\tau(B) \geq 0.12770275$ (by (3)),
$\tau(B) \geq 0.10389610$ (by (5)), $\tau(B) \geq 0.13733592$ (by (7)),

respectively. In fact, $\tau(B) = 0.15010895$. It is obvious that the bound in (7) is the best result.

Example 3.2. *Let*

$$B = \begin{pmatrix} 1 & -0.2 & -0.1 & -0.3 & -0.1 \\ -0.4 & 1 & -0.2 & -0.1 & -0.1 \\ -0.3 & -0.2 & 1.2 & -0.1 & -0.2 \\ -0.2 & -0.3 & -0.3 & 1 & -0.1 \\ -0.1 & -0.3 & -0.1 & -0.2 & 1 \end{pmatrix}.$$

It is easy to see that B is an M-matrix. By calculations with Matlab 7.1, we have

$\tau(B) \geq 0.10000000$ (by (1)), $\tau(B) \geq 0.15469345$ (by (2)), $\tau(B) \geq 0.15975146$ (by (3)),
$\tau(B) \geq 0.18290441$ (by (5)), $\tau(B) \geq 0.24600529$ (by (7)),

respectively. In fact, $\tau(B) = 0.25174938$. It is obvious that the bound in (7) is the best result.

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