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Research Article

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Fractional multilinear integrals with rough kernels on generalized weighted Morrey spaces

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Abstract: In this paper, we study the boundedness of fractional multilinear integral operators with rough kernels $T_{\Omega,\alpha}^{A_1,A_2,\dots,A_k}$, which is a generalization of the higher-order commutator of the rough fractional integral on the generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1,φ_2) with $w \in A_{p,q}$ which ensures the boundedness of the operators $T_{\Omega,\alpha}^{A_1,A_2,\dots,A_k}$ from $M_{p,\varphi_1}(w^p)$ to $M_{p,\varphi_2}(w^q)$ for $1 < p < q < \infty$. In all cases the conditions for the boundedness of the operator $T_{\Omega,\alpha}^{A_1,A_2,\dots,A_k}$ are given in terms of Zygmund-type integral inequalities on (φ_1,φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

Keywords: Fractional multilinear integral, Rough kernel, BMO, Generalized weighted Morrey space

MSC: 42B20, 42B35, 47G10

1 Introduction and results

Multilinear harmonic analysis is an active area of research that is still developing. Multilinear operators appear also as technical tools in the study of linear singular integral (through the method of rotations), the analysis of nonlinear operators (through power series and similar expansions), and the resolution of many linear and nonlinear partial differential equations [4, 5, 15–17, 41].

It is well known that in 1967, Bajsanski and Coifman [3] proved the boundedness of the multilinear operator associated with the commutators of singular integrals considered by Calderon. In 1981, Cohen [9] studied the L_p boundedness of the multilinear integral operator T^A defined by

$$T^A f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n+m-1}} R_m(A;x,\ y) f(y)dy,$$

where Ω is homogeneous of degree zero on $\mathbb{R}^n$ with mean value zero on S^{n-1}. Moreover, $R_m(A;x,\ y)$ denotes the m-th ($m \geq 2$) remainder of the Taylor series of A at x about y; more precisely,

$$R_m(A;x,\ y) \ = A(x) - \sum_{|\gamma|<m} \frac{1}{\gamma!} D^\gamma A(y)(x-y)^\gamma.$$

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Using the method of good $-\lambda$ inequality, in 1986, Cohen and Gosselin [10] proved that if $\Omega \in Lip_1(S^{n-1})$ and $D^\gamma A \in BMO(\mathbb{R}^n)$, then

$$\|T^A f\|_{L_p} \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{BMO} \|f\|_{L_p},\ 1 < p < \infty,$$

where the constant $C > 0$ is independent of f and A.

In 1994, for $m = 2$, Hofmann [32] proved that the multilinear operator T^A is a bounded operator on $L_{p,w}$ when $\Omega \in L_\infty(S^{n-1})$ and $w \in A_p$.

It is natural to ask whether the multilinear fractional integral operator with a rough kernel has the mapping properties similar to those of T_Ω^A. The purpose of [12] is to study this problem. Let us give the definition of the multilinear fractional integral operator as follows:

$$T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+N}} \prod_{j=1}^k R_{m_j}(A_j; x,\ y) f(y) dy,$$

where $0 < \alpha < n$, $N = \sum_{j=1}^k (m_j - 1)$, $\min_{1\le j\le k} m_j \ge 2$, Ω is homogeneous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$, $R_{m_j}(A_j; x,\ y)$ is as above.

When $k = 1$ and $m = 1$, then $T_{\Omega,\alpha}^A$ is just the commutator of the fractional integral $T_{\Omega,\alpha} f$ with function A,

$$T_{\Omega,\alpha}^A f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y)) f(y) dy.$$

When $m_j = 1$ and $A_j = A$ for $j = 1, \ldots, k$, then $T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k}$ is just the higher-order commutator $T_{\Omega,\alpha}^{A,k} f$ given in [11],

$$T_{\Omega,\alpha}^{A,k} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} (A(x) - A(y))^k f(y) dy.$$

When $m_j \ge 2$, $T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f$ is a non-trivial generalization of the above commutator.

The classical Morrey spaces were originally introduced by Morrey in [36] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [17, 18, 23, 36, 41]. Mizuhara [35] introduced generalized Morrey spaces. Later, in [23] Guliyev defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [34] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev [24] introduced the generalized weighted Morrey spaces $M_{p,\varphi}(w)$ and studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also [24, 29, 33, 40]. In [24] the author gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$.

The weighted (L_p, L_q)-boundedness of such a commutator is given by Ding [13] and Lu in [14].

The following theorem was proved by Ding and Lu in [12].

Theorem 1.1 ([12]). *Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$, $1 \le s' < p < n/\alpha$, $w^{s'} \in A(p/s', q/s')$ and let Ω be homogeneous of degree zero with $\Omega \in L_s(S^{n-1})$. Moreover, for $1 \le j \le k$, $|\gamma_j| = m_j - 1$, $m_j \ge 2$ and $D^{\gamma_j} A_j \in BMO(\mathbb{R}^n)$. Then there exists a constant C, independent of A_j, $1 \le j \le k$ and f, such that*

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{L_{q,w^q}(\mathbb{R}^n)} \le C \prod_{j=1}^k \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f\|_{L_{p,w^p}(\mathbb{R}^n)}.$$

Here and in the sequel, we always denote by p' the conjugate index of any $p > 1$, that is $1/p + 1/p' = 1$, and by C a constant which is independent of the main parameters and may vary from line to line.

We define the generalized weighed Morrey spaces as follows.

Definition 1.2. *Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} = \left(\int\limits_{B(x,r)} |f(y)|^p w(y)dy \right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} = \sup_{t>0} t \left(\int\limits_{\{y\in B(x,r):\,|f(y)|>t\}} w(y)dy \right)^{\frac{1}{p}}.$$

Remark 1.3.

(1) *If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.*
(2) *If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.*
(3) *If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.*
(4) *If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0<\lambda<n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.*
(5) *If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.*

The commutators are useful in many nondivergence elliptic equations with discontinuous coefficients, [15–17, 26, 27, 41]. In the recent development of commutators, Pérez and Trujillo-González [42] generalized these multilinear commutators and proved the weighted Lebesgue estimates. Ye and Zhu in [45] obtained the boundedness of the multilinear commutators in weighted Morrey spaces $L_{p,\kappa}(w)$ for $1<p<\infty$ and $0<\kappa<1$, where the symbol $\vec{b}$ belongs to bounded mean oscillation $(BMO)^n$. Furthermore, the weighted weak type estimate of these operators in weighted Morrey spaces is $L_{p,\kappa}(w)$ for $p=1$ and $0<\kappa<1$. The following statement was proved by Guliyev in [24].

Theorem 1.4 ([24]). *Let $0<\alpha<n$, $1<p<n/\alpha$ and $1/q = 1/p - \alpha/n$, $\Omega \in L_\infty(\mathbb{S}^{n-1})$, $w \in A_{p,q}$, $A \in BMO(\mathbb{R}^n)$, and (φ_1,φ_2) satisfies the condition*

$$\int\limits_r^\infty \ln^k\left(e+\frac{t}{r}\right) \frac{\operatorname*{ess\,sup}_{t<s<\infty} \varphi_1(x,s) w^p(B(x,s))^{\frac{1}{p}}}{w^q(B(x,t))^{\frac{1}{q}}} \frac{dt}{t} \le C\,\varphi_2(x,r), \tag{1}$$

where C does not depend on x and r. Then the operator $T^{A,k}_{\Omega,\alpha}$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$.

It has been proved by many authors that most of the operators which are bounded on a weighted (unweighted) Lebesgue space are also bounded in an appropriate weighted (unweighted) Morrey space, see [8, 44]. As far as we know, there is no research regarding boundedness of the fractional multilinear integral operator on Morrey space. In this paper, we are going to prove that these results are valid for the rough fractional multilinear integral operator $T^{A_1,A_2,\ldots,A_k}_{\Omega,\alpha}$ on generalized weighted Morrey space. Our main results can be formulated as follows.

Theorem 1.5. *Let $0 < \alpha < n$, $1 \le s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that Ω is homogeneous of degree zero with $\Omega \in L_s(S^{n-1})$ and (φ_1, φ_2) satisfy the condition* (1). *Let also, for $1 \le j \le k$, $|\gamma_j| = m_j - 1$, $m_j \ge 2$ and $D^{\gamma_j} A_j \in BMO(\mathbb{R}^n)$. Suppose $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, then the operator $T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k}$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$. Moreover, then there is a constant $C > 0$ independent of f and $A_1, A_2, \ldots, A_k$ such that*

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{M_{q,\varphi_2}(w^q)} \le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f\|_{M_{p,\varphi_1}(w^p)}.$$

In the case $m_j = 1$ and $A_j = A$ for $j = 1, \ldots, k$ from the Theorem 1.5 we get the Theorem 1.4. Also, in the case $\omega \equiv 1$ we get the following corollary, which was proved in [1].

Corollary 1.6 ([1]). *Let $0 < \alpha < n$, $1 \le s' < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that Ω is homogeneous of degree zero with $\Omega \in L_s(S^{n-1})$, and (φ_1, φ_2) satisfy the condition*

$$\int_r^\infty \ln^k\left(e + \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C_0\, \varphi_2(x,r),$$

where C_0 does not depend on x and r. Let also, for $1 \le j \le k$, $|\gamma_j| = m_j - 1$, $m_j \ge 2$ and $D^{\gamma_j} A_j \in BMO(\mathbb{R}^n)$. Then the operator $T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k}$ is bounded from $M_{p,\varphi_1}(\mathbb{R}^n)$ to $M_{q,\varphi_2}(\mathbb{R}^n)$. Moreover, there is a constant $C > 0$ independent of f such that

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{M_{q,\varphi_2}} \le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f\|_{M_{p,\varphi_1}}.$$

Example 1.7. *Let $\varphi_1(x,t) = (w^q(B(x,t)))^{\frac{k}{p}} (w^p(B(x,t)))^{-\frac{1}{p}}$, $\varphi_2(x,t) = (w^q(B(x,t)))^{\frac{k}{p}-\frac{1}{q}}$, $0 < k < p/q$ and $w^q \in A_\infty(\mathbb{R}^n)$. Then (φ_1, φ_2) satisfies the condition* (1).

In fact, from (2.8) in Section 2 we have constant $\delta > 0$ such that

$$w^q(B(x, 2^j r)) \ge C 2^{\delta j} w^q(B(x,r)).$$

Since $0 < \kappa < p/q$, then $\frac{\kappa}{p} - \frac{1}{q} < 0$. Thus

$$\begin{aligned}
&\int_r^\infty \left(1 + \ln\frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau)(w^p(B(x,\tau)))^{\frac{1}{p}}}{(w^q(B(x,t)))^{\frac{1}{q}}} \frac{dt}{t} \\
&= \int_r^\infty \left(1 + \ln\frac{t}{r}\right) (w^q(B(x,t)))^{\frac{\kappa}{p}-\frac{1}{q}} \frac{dt}{t} \\
&\le \sum_{j=0}^\infty (1+j) \int_{2^j r}^{2^{j+1} r} (w^q(B(x,t)))^{\frac{\kappa}{p}-\frac{1}{q}} \frac{dt}{t} \\
&\le C \sum_{j=0}^\infty (1+j)(w^q(B(x,2^j r)))^{\frac{\kappa}{p}-\frac{1}{q}} \\
&\le C \sum_{j=0}^\infty (1+j) 2^{\delta j(\frac{\kappa}{p}-\frac{1}{q})} (w^q(B(x,r)))^{\frac{\kappa}{p}-\frac{1}{q}} \\
&\le C(w^q(B(x,r)))^{\frac{\kappa}{p}-\frac{1}{q}} = C\varphi_2(x,r).
\end{aligned}$$

If $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, then by Lemma 2.2 in Section 2 we know $w^q \in A_{1+q/p'}(\mathbb{R}^n)$. Therefore, we have the following corollaries.

Corollary 1.8. *Let $0 < \alpha < n$, let $1 \le s' < p < n/\alpha$, and let $1/q = 1/p - \alpha/n$. Let also, for $1 \le j \le k$, $|\gamma_j| = m_j - 1$, $m_j \ge 2$ and $D^{\gamma_j} A_j \in BMO(\mathbb{R}^n)$. Suppose $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, then $T^{A_1,A_2,\ldots,A_k}_{\Omega,\alpha}$ is bounded from $L_{p,\kappa}(w^p, w^q)(\mathbb{R}^n)$ to $L_{q,\kappa q/p}(w^q, \mathbb{R}^n)$ and*

$$\|T^{A_1,A_2,\ldots,A_k}_{\Omega,\alpha} f\|_{L_{q,\kappa q/p}(w^q,\mathbb{R}^n)} \le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f\|_{L_{p,k}(w^p,w^q)(\mathbb{R}^n)},$$

where the constant $C > 0$ is independent of f and $A_1, A_2, \ldots, A_k$.

Remark 1.9. *Note that, in [2] the Nikolskii-Morrey type spaces were introduced and the authors studied some embedding theorems. In the next paper, we shall introduce the generalized weighted Nikolskii-Morrey spaces and will study some embedding theorems. We will also investigate the boundedness of fractional multilinear integral operators with rough kernels $T^{A_1,A_2,\ldots,A_k}_{\Omega,\alpha}$ on the generalized weighted Nikolskii-Morrey spaces, see for example, [30]. These results may be applicable to some problems of partial differential equations; see for example [6, 7, 19, 20, 26, 28, 30, 43].*

2 Some preliminaries

We begin with some properties of A_p weights which play a great role in the proofs of our main results. A weight w is a nonnegative, locally integrable function on $\mathbb{R}^n$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given weight function w and a measurable set E, we also denote the Lebesgue measure of E by $|E|$ and set weighted measure $w(E) = \int_E w(x)dx$. For any given weight function w on $\mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_{p,w}(\Omega)$ the space of all function f satisfying

$$\|f\|_{L_{p,w}(\Omega)} = \left(\int_{\Omega} |f(x)|^p w(x)dx \right)^{\frac{1}{p}} < \infty.$$

A weight w is said to belong to A_p for $1 < p < \infty$, if there exists a constant

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'}dx \right)^{p-1} \le C,$$

where p' is the dual of p such that $\frac{1}{p} + \frac{1}{p'} = 1$. The class A_1 is defined by

$$\frac{1}{|B|} \int_B w(y)dy \le C \cdot \operatorname*{ess\,inf}_{x \in B} w(x) \quad \text{for every ball} \quad B \subset \mathbb{R}^n.$$

A weight w is said to belong to $A_\infty(\mathbb{R}^n)$ if there are positive numbers C and δ so that

$$\frac{w(E)}{w(B)} \le C \left(\frac{|E|}{|B|} \right)^{\delta}$$

for all balls B and all measurable $E \subset B$. It is well known that

$$A_\infty = \bigcup_{1 \le p < \infty} A_p.$$

The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L_p-boundedness of Hardy-Littlewood maximal function in [37].

Lemma 2.1 ([21, 37]). *Suppose $w \in A_p$ and the following statements hold.*

(i) *For any* $1 \le p < \infty$, *there is a positive number* C *such that*

$$\frac{w(B_k)}{w(B_j)} \le C2^{np(k-j)} \quad \text{for} \quad k > j$$

(ii) *For any* $1 \le p < \infty$, *there is a positive number* C *and* δ *such that*

$$\frac{w(B_k)}{w(B_j)} \ge C2^{\delta(k-j)} \quad \text{for} \quad k > j \tag{2}$$

(iii) *For any* $1 < p < \infty$, *one has* $w^{1-p'} \in A_{p'}$.

We also need another weight class $A_{p,q}$ introduced by Muckenhoupt and Wheeden in [38] to study weighted boundedness of fractional integral operators.

Given $1 \le p \le q < \infty$. We say that $\omega \in A_{p,q}$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$, the inequality

$$\Big(\frac{1}{|B|}\int\limits_B w(y)^{-p'}dy\Big)^{1/p'}\Big(\frac{1}{|B|}\int\limits_B w(y)^{q}dy\Big)^{1/q} \le C \tag{3}$$

holds when $1 < p < \infty$, and for every ball $B \subset \mathbb{R}^n$ the inequality

$$\Big(\frac{1}{|B|}\int\limits_B w(y)^{q}dy\Big)^{1/q} \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x)$$

holds when $p = 1$.

By (3), we have

$$\Big(\int\limits_B w(y)^{-p'}dy\Big)^{1/p'}\Big(\int\limits_B w(y)^{q}dy\Big)^{1/q} \le C|B|^{1/p'+1/q}. \tag{4}$$

We summarize some properties about weights $A_{p,q}$; see [21, 38].

Lemma 2.2. *Given* $1 \le p \le q < \infty$.

(i) $w \in A_{p,q}$ *if and only if* $w^q \in A_{1+q/p'}$;

(ii) $w \in A_{p,q}$ *if and only if* $w^{-p'} \in A_{1+p'/q}$;

(iii) $w \in A_{p,p}$ *if and only if* $w^p \in A_p$;

(iv) *If* $p_1 < p_2$ *and* $q_2 > q_1$, *then* $A_{p_1,q_1} \subset A_{p_2,q_2}$.

In this paper, we need the following statement on the boundedness of the Hardy type operator

$$(H_1 g)(t) := \frac{1}{t}\int\limits_0^t \ln\Big(e+\frac{t}{r}\Big)g(r)d\mu(r),\ 0 < t < \infty,$$

where μ is a non-negative Borel measure on $(0,\infty)$.

Theorem 2.3. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)H_1 g(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g *on* $(0,\infty)$ *if and only if*

$$A_1 := \sup_{t>0}\frac{w(t)}{t}\int\limits_0^t \ln\Big(e+\frac{t}{r}\Big)\frac{d\mu(r)}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A_1$.

Note that Theorem 2.3 is proved analogously to Theorem 4.3 in [24, 25].

Lemma 2.4 ([39, Theorem 5, p. 236]). *Let $w \in A_\infty$. Then the norm of $BMO(\mathbb{R}^n)$ is equivalent to the norm of $BMO(w)$, where*

$$BMO(w) = \{b \,:\, \|b\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(B(x,r))} \int\limits_{B(x,r)} |b(y) - b_{B(x,r),w}| w(y) dy < \infty\}$$

and

$$b_{B(x,r),w} = \frac{1}{w(B(x,r))} \int\limits_{B(x,r)} b(y) w(y) dy.$$

Remark 2.5 ([24]).

(1) *The John-Nirenberg inequality : there are constants $C_1, C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$*

$$|\{x \in B \,:\, |b(x) - b_B| > \beta\}| \le C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) *For $1 < p < \infty$ the John-Nirenberg inequality implies that*

$$\|b\|_* \approx \sup_B \left(\frac{1}{|B|} \int\limits_B |b(y) - b_B|^p dy \right)^{\frac{1}{p}}$$

and for $1 \le p < \infty$ and $w \in A_\infty$

$$\|b\|_* \approx \sup_B \left(\frac{1}{w(B)} \int\limits_B |b(y) - b_B|^p w(y) dy \right)^{\frac{1}{p}}.$$

The following lemma was proved by Guliyev in [24].

Lemma 2.6 ([24]).

i) *Let $w \in A_\infty$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 \le p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w(B(x,r_1))} \int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^p w(y) dy \Big)^{\frac{1}{p}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

ii) *Let $w \in A_p$ and b be a function in $BMO(\mathbb{R}^n)$. Let also $1 < p < \infty$, $x \in \mathbb{R}^n$, and $r_1, r_2 > 0$. Then*

$$\Big(\frac{1}{w^{1-p'}(B(x,r_1))} \int\limits_{B(x,r_1)} |b(y) - b_{B(x,r_2),w}|^{p'} w(y)^{1-p'} dy \Big)^{\frac{1}{p'}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_*,$$

where $C > 0$ is independent of f, x, r_1 and r_2.

Below we present some conclusions about $R_m(A; x, y)$.

Lemma 2.7 ([22]). *Suppose b is a function on $\mathbb{R}^n$ with the m-th derivatives in $L_q(\mathbb{R}^n)$, $q > n$. Then*

$$|R_m(b; x, y)| \le C|x - y|^m \sum_{|\gamma|=m} \left(\frac{1}{B(x, 5\sqrt{n}|x-y|)} \int\limits_{B(x,5\sqrt{n}|x-y|)} |D^\gamma b(z)| dz \right)^{1/q}.$$

The following property is valid.

Lemma 2.8. *Let $x \in B(x_0, r)$, $y \in B(x_0, 2^{j+1} r) \backslash B(x_0, 2^j r)$. Assume that A has derivatives of order $m-1$ in $BMO(\mathbb{R}^n)$. Then there exists a constant C, independent of A, such that*

$$|R_m(A; x, y)|$$

$$\leq C|x-y|^{m-1}\left(\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{*}+\sum_{|\gamma|=m-1}|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,r)}|\right). \tag{5}$$

Proof. For fixed $x \in \mathbb{R}^n$, let

$$\bar{A}(x)=A(x)-\sum_{|\gamma|=m-1}\frac{1}{\gamma!}(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}x^{\gamma}.$$

Then

$$|R_m(A;x,y)|=|R_m(\bar{A};x,y)|\leq|R_{m-1}(\bar{A};x,y)|+\sum_{|\gamma|=m-1}\frac{1}{\gamma!}|(D^{\gamma}\bar{A}(y))||x-y|^{m-1}. \tag{6}$$

From Lemma 2.7 we have,

$$|R_{m-1}(\bar{A};x,y)|\leq C|x-y|^{m-1}\sum_{|\gamma|=m-1}\|D^{\gamma}A\|_{*}. \tag{7}$$

When $x\in B(x_0,r)$, $y\in B(x_0,2^{j+1}r)\backslash B(x_0,2^{j}r)$, then $2^{j-1}r\leq|x-y|\leq 2^{j+2}r$. Thus, we have

$$B(x_0,2^{j-1}r)\subset B(x,5\sqrt{n}|x-y|)\subset 100\sqrt{n}B(x_0,2^{j}r).$$

Then

$$\frac{|100\sqrt{n}B(x_0,2^{j}r)|}{|B(x,5\sqrt{n}|x-y|)|}\leq\frac{|100\sqrt{n}B(x_0,2^{j}r)|}{|B(x_0,2^{j-1}r)|}\leq C.$$

Hence

$$\begin{aligned}&|(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}-(D^{\gamma}A)_{B(x_0,2^{j}r)}|\\&\leq\frac{1}{|B(x,5\sqrt{n}|x-y|)|}\int_{B(x,5\sqrt{n}|x-y|)}|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,2^{j}r)}|dy\\&\leq\frac{1}{|100\sqrt{n}B(x_0,2^{j}r)|}\int_{100\sqrt{n}B(x_0,2^{j}r)}|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,2^{j}r)}|dy\\&\leq C\|D^{\gamma}A\|_{*}.\end{aligned}$$

Note that

$$|(D^{\gamma}A)_{B(x_0,2^{j}r)}-(D^{\gamma}A)_{B(x_0,r)}|\leq\sum_{k=1}^{j}|(D^{\gamma}A)_{B(x_0,2^{k}r)}-(D^{\gamma}A)_{B(x_0,2^{k-1}r)}|\leq 2^{n}j\|D^{\gamma}A\|_{*}.$$

Then

$$\begin{aligned}&|(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}-(D^{\gamma}A)_{B(x_0,r)}|\\&\leq|(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}-(D^{\gamma}A)_{B(x_0,2^{j}r)}|+|(D^{\gamma}A)_{B(x_0,2^{j}r)}-(D^{\gamma}A)_{B(x_0,r)}|\\&\leq C_j\|D^{\gamma}A\|_{*}.\end{aligned}$$

Thus

$$\begin{aligned}|D^{\gamma}\bar{A}(y)|&=|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}|\\&\leq|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,r)}|+|(D^{\gamma}A)_{B(x,5\sqrt{n}|x-y|)}-(D^{\gamma}A)_{B(x_0,r)}|\\&\leq|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,r)}|+C_j\|D^{\gamma}A\|_{*}.\end{aligned} \tag{8}$$

Combining with (6), (7) and (8), then (5) is proved. □

Finally, we present a relationship between essential supremum and essential infimum.

Lemma 2.9 ([10]). *Let f be a real-valued nonnegative function and measurable on E. Then*

$$\left(\operatorname*{ess\,inf}_{x\in E}f(x)\right)^{-1}=\operatorname*{ess\,sup}_{x\in E}\frac{1}{f(x)}.$$

3 A local weighted Guliyev type estimates

In the following theorem we get local weighted Guliyev type estimate (see, for example, [22, 23] in the case $w = 1$, $m = 1$ and [24] in the case $w \in A_p$, $m = 1$) for the operator $T^{A,m}_{\Omega,\alpha}$.

Theorem 3.1. *Let $1 \le s' < p < n/\alpha$, and let $1/q = 1/p - \alpha/n$. Let also, for $1 \le j \le k$, $|\gamma_j| = m_j - 1$, $m_j \ge 2$ and $D^{\gamma_j} A_j \in BMO(\mathbb{R}^n)$. Suppose that Ω is homogeneous of degree zero with $\Omega \in L_s(S^{n-1})$, $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, then for any $r > 0$, there is a constant C independent of f such that*

$$\|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f\|_{L_{q,w^q}(B(x_0,r))} \le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* (w^q(B(x_0,r)))^{\frac{1}{q}}$$
$$\times \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big)^k \|f\|_{L_{p,w^p}(B(x_0,t))} (w^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}. \tag{9}$$

Proof. We write f as $f = f_1 + f_2$, where $f_1(y) = f(y)\chi_{B(x_0,2r)}(y)$, $\chi_{B(x_0,2r)}$ denotes the characteristic function of $B(x_0, 2r)$. Then

$$\|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f\|_{L_{q,w^q}(B(x_0,r))} \le \|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_1\|_{L_{q,w^q}(B(x_0,r))} + \|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_2\|_{L_{q,w^q}(B(x_0,r))}.$$

Since $f_1 \in L_{p,w^p}(\mathbb{R}^n)$, by the boundedness of $T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha}$ from $L_{p,w^p}(\mathbb{R}^n)$ to $L_{q,w^q}(\mathbb{R}^n)$ (Theorem 1.1) we get

$$\begin{aligned}
\|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_1\|_{L_{q,w^q}(B(x_0,r))} &\le \|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_1\|_{L_{q,w^q}(\mathbb{R}^n)} \\
&\le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f_1\|_{L_{p,w^p}(\mathbb{R}^n)} \\
&= C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \|f\|_{L_{p,w^p}(B(x_0,2r))}.
\end{aligned}$$

Note that $q > p > 1$ and $\frac{s'p}{p'(p-s')} \ge 1$, then by Hölder's inequality,

$$1 \le \left(\frac{1}{|B|}\int_B w(y)^p dy\right)^{\frac{1}{p}} \left(\frac{1}{|B|}\int_B w(y)^{-p'} dy\right)^{\frac{1}{p'}} \le \left(\frac{1}{|B|}\int_B w(y)^q dy\right)^{\frac{1}{q}} \left(\frac{1}{|B|}\int_B w(y)^{-\frac{s'p}{p-s'}} dy\right)^{\frac{p-s'}{s'p}}.$$

This means

$$r^{\frac{n}{s'}-\alpha} \le (w^q(B(x_0,r)))^{\frac{1}{q}} \|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))}.$$

Then

$$\begin{aligned}
\|f\|_{L^p(w^p,B(x_0,2r))} &\le C r^{\frac{n}{s'}-\alpha} \|f\|_{L_{p,w^p}(B(x_0,2r))} \int_{2r}^{\infty} t^{\alpha-\frac{n}{s'}-1} dt \\
&\le C (w^q(B(x_0,r)))^{\frac{1}{q}} \|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1} dt \\
&\le C (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x_0,t))} \|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1} dt.
\end{aligned}$$

Since $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$, by (4), for all $r > 0$ we get

$$(w^q(B(x_0,r)))^{\frac{1}{q}} \|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \le C r^{\frac{n}{s'}-\alpha}. \tag{10}$$

Then

$$\|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_1\|_{L_{q,w^q}(B(x_0,r))} \le C \prod_{j=1}^{k} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* (w^q(B(x_0,r)))^{\frac{1}{q}} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x_0,t))} (w^q(B(x_0,t)))^{-\frac{1}{q}} \frac{dt}{t}.$$

To simplify process of Theorem 3.1, in the following discussion we consider only the case $k = 2$. The method can be used to deal with the case $k > 2$ without any essential difficulty.

Let $N = m_1 + m_2 - 2$, $\triangle_i = (B(x_0, 2^{i+1}r)) \setminus (B(x_0, 2^i r))$, and let $x \in B(x_0, r)$. By Lemma 2.8,

$$\begin{aligned}
|T^{A_1,A_2,\dots,A_k}_{\Omega,\alpha} f_2(x)| &\le \left| \int_{(B(x_0,2r))^c} \frac{\Omega(x-y)}{|x-y|^{n-\alpha+m-1}} R_{m_1}(A_1;x,y) R_{m_2}(A_2;x,y) f(y) dy \right| \\
&\le C \sum_{i=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} \prod_{j=1}^{2} \left(j + \sum_{|\gamma_j|=m_j-1} |D^{\gamma_j} A_j(y) - (D^{\gamma_j} A_j)_{B(x_0,r)}| \right) dy \\
&\le C \prod_{j=1}^{2} j \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \sum_{i=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} dy \\
&+ C \sum_{|\gamma_1|=m_1-1} \|D^{\gamma_1} A_1\|_* \sum_{|\gamma_2|=m_2-1} \sum_{i=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} |D^{\gamma_2} A_2(y) - (D^{\gamma_2} A_2)_{B(x_0,r)}| dy \\
&+ C \sum_{|\gamma_2|=m_2-1} \|D^{\gamma_2} A_2\|_* \sum_{|\gamma_1|=m_1-1} \sum_{i=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} |D^{\gamma_1} A_1(y) - (D^{\gamma_1} A_1)_{B(x_0,r)}| dy \\
&+ C \sum_{|\gamma_1|=m_1-1} \sum_{|\gamma_2|=m_2-1} \sum_{i=1}^{\infty} \int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} \prod_{j=1}^{2} |D^{\gamma_j} A_j(y) - (D^{\gamma_j} A_j)_{B(x_0,r)}| dy \\
&\le C\big(I_1 + I_2 + I_3 + I_4\big).
\end{aligned}$$

By Hölder's inequalities,

$$\int_{\triangle_i} \frac{|\Omega(x-y) f(y)|}{|x-y|^{n-\alpha}} dy \le \Big(\int_{\triangle_i} |\Omega(x-y)|^s dy \Big)^{\frac{1}{s}} \Big(\int_{\triangle_i} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}} dy \Big)^{\frac{1}{s'}}.$$

When $x \in B(x_0, s)$ and $y \in \triangle_i$, then by a direct calculation, we can see that $2^{i-1} r \le |y - x| < 2^{i+1} r$. Hence

$$\Big(\int_{\triangle_i} |\Omega(x-y)|^s dy \Big)^{\frac{1}{s}} \le C \|\Omega\|_{L^s(\mathbb{S}^{n-1})} |B(x_0, 2^{i+1} r)|^{\frac{1}{s}}. \tag{11}$$

We also note that if $x \in B(x_0, r)$, $y \in B(x_0, 2r)^c$, then $|y - x| \approx |y - x_0|$. Consequently

$$\Big(\int_{\triangle_i} \frac{|f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}} dy \Big)^{\frac{1}{s'}} \le \frac{1}{|B(x_0, 2^{i+1} r)|^{1-\alpha/n}} \Big(\int_{B(x_0,2^{i+1}r)} |f(y)|^{s'} dy \Big)^{\frac{1}{s'}}. \tag{12}$$

Then

$$I_1 \le C \prod_{j=1}^{2} \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j} A_j\|_* \sum_{i=1}^{\infty} j (2^{i+1} r)^{\alpha - \frac{n}{s'}} \Big(\int_{B(x_0,2^{i+1}r)} |f(y)|^{s'} dy \Big)^{\frac{1}{s'}}.$$

Since $s' < p$, it follows from Hölder's inequality that

$$\Big(\int_{B(x_0,2^{i+1}r)} |f(y)|^{s'} dy\Big)^{\frac{1}{s'}} \le C\|f\|_{L_{p,w^p}(B(x_0,2^{i+1}r))}\|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{i+1}r))}.$$

Then

$$I_1 \le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\sum_{i=1}^{\infty} j(2^{i+1}r)^{\alpha-\frac{n}{s'}}\Big(\int_{B(x_0,2^{i+1}s)} |f(y)|^{q'}dy\Big)^{\frac{1}{q'}}.$$

$$\le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\sum_{i=1}^{\infty}\Big(1+\ln\frac{2^{i+1}r}{r}\Big)(2^{i+1}r)^{\alpha-\frac{n}{s'}}\|f\|_{L_{p,w^p}(B(x_0,2^{i+1}r))}\|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,2^{i+1}r))}$$

$$\le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\sum_{i=1}^{\infty}\int_{2^{i+1}r}^{2^{i+2}r}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(x_0,t))}\|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))}t^{\alpha-\frac{n}{s'}-1}dt$$

$$\le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(x_0,t))}\|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,t))}t^{\alpha-\frac{n}{s'}-1}dt.$$

From (10) we know

$$\|w^{-1}\|_{L^{\frac{s'p}{p-s'}}(B(x_0,r))} \le Cr^{\frac{n}{s'}-\alpha}(w^q(B(x_0,r)))^{-\frac{1}{q}}. \tag{13}$$

Then

$$I_1 \le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

On the other hand, by Hölder's inequality and (11), (12), we have

$$\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}|D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}|dy$$
$$\le\Big(\int_{\triangle_i}|\Omega(x-y)|^s dy\Big)^{\frac{1}{s}}\Big(\int_{\triangle_i}\frac{|D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}f(y)|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}$$
$$\le C\sum_{i=1}^{\infty}(2^{i+1}r)^{\alpha-\frac{n}{s'}}\left(\int_{B(x_0,2^{i+1}r)}|D^{\gamma}A(y)-(D^{\gamma}A)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Applying Hölder's inequality we get

$$\left(\int_{B(x_0,2^{i+1}r)}|D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}$$
$$\le C\|f\|_{L_{p,w^p}(B(x_0,2^{i+1}r))}\|\big(D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big)w(\cdot)^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,2^{i+1}r))}.$$

Consequently,

$$I_2 \le C\sum_{|\gamma_1|=m_1-1}\|D^{\gamma_1}A_1\|_*\sum_{|\gamma_2|=m_2-1}\sum_{i=1}^{\infty}\int_{2^{i+1}r}^{2^{i+2}r}(2^{i+1}r)^{\alpha-\frac{n}{s'}}\|f\|_{L_{p,w^p}(B(x_0,t))}$$

$$\times \|\big(D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big)\omega(\cdot)^{-1}\|_{L_{\frac{ps'}{p-s'}}(B(x_0,t))} dt$$

$$\le C \sum_{|\gamma_1|=m_1-1} \|D^{\gamma_1}A_1\|_* \sum_{|\gamma_2|=m_2-1} \int_{2r}^{\infty} \|f\|_{L_{p,w^p}(B(x_0,t))}$$

$$\times \|\big(D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big)w(\cdot)^{-1}\|_{L_{\frac{ps'}{p-s'}}(B(x_0,t))} t^{\alpha-\frac{n}{s'}-1}\, dt.$$

By $w^{s'} \in A_{\frac{p}{s'},\frac{q}{s'}}$ and (ii) of Lemma 2.2 we know $w^{-\frac{s'p}{p-s'}} \in A_{1+\frac{ps'}{(p-s')q}}$. Then it follows from the Lemma 2.6 and the inequality (13) that

$$\|\big(D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big)\omega(\cdot)^{-1}\|_{L_{\frac{ps'}{p-s'}}(B(x_0,t))}$$

$$\le \Big(\int_{B(x_0,t)} \big|D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big|^{\frac{ps'}{p-s'}} w^{-\frac{ps'}{p-s'}}(y)dy\Big)^{\frac{p-s'}{ps'}}$$

$$\le C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)(w^{-\frac{ps'}{p-s'}}(B(x_0,r)))^{\frac{p-s'}{ps'}}$$

$$= C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)\|w^{-1}\|_{L_{\frac{ps'}{p-s'}}(B(x_0,r))}$$

$$\le C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)r^{\frac{n}{s'}-\alpha}(w^q(B(x_0,r)))^{-\frac{1}{q}}.$$

Then

$$I_2 \le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

Similarly to the estimates for I_2, we have

$$I_3 \le C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

Finally, we come to estimate I_4.

By Hölder's inequality and (11), (12), we have

$$\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|dy$$

$$\le\Big(\int_{\triangle_i}|\Omega(x-y)|^s dy\Big)^{\frac{1}{s}}\Big(\int_{\triangle_i}\frac{\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|^{s'}}{|x-y|^{(n-\alpha)s'}}dy\Big)^{\frac{1}{s'}}$$

$$\le C\sum_{i=1}^{\infty}(2^{i+1}r)^{\alpha-\frac{n}{s'}}\left(\int_{B(x_0,2^{i+1}r)}\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}.$$

Applying Hölder's inequality we get

$$\left(\int_{B(x_0,2^{i+1}r)}\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|^{s'}|f(y)|^{s'}dy\right)^{\frac{1}{s'}}$$

$$\leq C\|f\|_{L_{p,w^p}(B(x_0,2^{i+1}r))}\Big\|\prod_{j=1}^{2}\big(D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}\big)w(\cdot)^{-1}\Big\|_{L^{\frac{ps'}{p-s'}}(B(x_0,2^{i+1}r))}$$

$$\leq C\|f\|_{L_{p,w^p}(B(x_0,2^{i+1}r))}\prod_{j=1}^{2}\big\|\big(D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}\big)w(\cdot)^{-1/2}\big\|_{L^{\frac{2ps'}{p-s'}}(B(x_0,2^{i+1}r))}.$$

Then

$$\sum_{i=1}^{\infty}\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|dy$$

$$\leq C\int_{2r}^{\infty}\|f\|_{L_{p,w^p}(B(x_0,t))}\prod_{j=1}^{2}\big\|\big(D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}\big)w(\cdot)^{-1/2}\big\|_{L^{\frac{2ps'}{p-s'}}(B(x_0,2^{i+1}r))}.$$

Since $w^{-\frac{s'p}{p-s'}}\in A_{1+\frac{ps'}{(p-s')q}}$, then from the Lemma 2.6 and the inequality (13) we have

$$\begin{aligned}
&\big\|\big(D^{\gamma_j}A_j(y)-(D^{\gamma_2}A_2)_{B(x_0,r)}\big)\omega(\cdot)^{-1}\big\|_{L^{\frac{ps'}{p-s'}}(B(x_0,t))}\\
&\leq\Big(\int_{B(x_0,t)}|D^{\gamma_2}A_2(y)-(D^{\gamma_2}A_2|^{\frac{ps'}{p-s'}}w^{-\frac{ps'}{p-s'}}(y)dy\Big)^{\frac{p-s'}{ps'}}\\
&\leq C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)(w^{-\frac{ps'}{p-s'}}(B(x_0,r)))^{\frac{p-s'}{ps'}}\\
&=C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)\|w^{-1}\|_{L^{\frac{ps'}{p-s'}}(B(x_0,r))}\\
&\leq C\|D^{\gamma_2}A_2\|_*\Big(1+\ln\frac{t}{r}\Big)r^{\frac{n}{s'}-\alpha}(w^q(B(x_0,r)))^{-\frac{1}{q}}.
\end{aligned}\tag{14}$$

Then from (14) we have

$$\sum_{i=1}^{\infty}\int_{\triangle_i}\frac{|\Omega(x-y)f(y)|}{|x-y|^{n-\alpha}}\prod_{j=1}^{2}|D^{\gamma_j}A_j(y)-(D^{\gamma_j}A_j)_{B(x_0,r)}|dy$$

$$\leq C\prod_{j=1}^{2}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,r)))^{-\frac{1}{q}}\frac{dr}{r}.$$

Combining with the estimates of I_1, I_2, I_3 and I_4, we have

$$\sup_{x\in B(x_0,r)}|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k}f_2(x)|$$

$$\leq C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

Then we get

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k}f_2\|_{L_{q,w^q}(B(x_0,r))}\leq C\prod_{j=1}^{2}\sum_{|\gamma_j|=m_j-1}\|D^{\gamma_j}A_j\|_*(w^q(B(x_0,r)))^{\frac{1}{q}}$$

$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2\|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}.$$

This completes the proof of Theorem 3.1. □

4 Proof of Theorem 1.5

First variant proof of Theorem 1.5

By Theorems 2.3 and 3.1 we have

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{M_{q,\varphi_2}(w^q)} = \sup_{x_0\in\mathbb{R}^n,\, r>0} \varphi_2(x_0,r)^{-1}(w^q(B(x_0,r)))^{-\frac{1}{q}} \|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{L_{q,w^q}(B(x_0,r))}$$
$$\le C \sup_{x_0\in\mathbb{R}^n,\, r>0} \varphi_2(x_0,r)^{-1} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)^k \|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}$$
$$\le C \sup_{x_0\in\mathbb{R}^n, r>0} \varphi_1(x_0,r)^{-1} w^p(B(x,r)^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(B(x,r))}$$
$$= \|f\|_{M_{p,\varphi_1}(w^p)}.$$

Second variant proof of Theorem 1.5

Since $f \in M_{p,\varphi_1}(w^p)$, then by Lemma 2.9 and the fact $\|f\|_{L_{p,w^p}(B(x_0,t))}$ is a non-decreasing function of t , we get

$$\frac{\|f\|_{L_{p,w^p}(B(x_0,t))}}{\operatorname*{ess\,inf}_{0<t<\tau<\infty} \varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \le \operatorname*{ess\,sup}_{0<t<\tau<\infty} \frac{\|f\|_{L_{p,w^p}(B(x_0,t))}}{\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}$$
$$\le \sup_{\tau>0, x_0\in\mathbb{R}^n} \frac{\|f\|_{L_{p,w^p}(B(x_0,\tau))}}{\varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \le \|f\|_{M_{p,\varphi_1}(w^p)}.$$

Since (φ_1,φ_2) satisfies (1), we have

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)^k \|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}$$
$$\le \int_r^\infty \frac{\|f\|_{L_{p,w^p}(B(x_0,t))}}{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}} \Big(1+\ln\frac{t}{r}\Big)^k \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(w^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t}$$
$$\le C\|f\|_{M_{p,\varphi_1}(w^p)} \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)^k \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x_0,\tau)(w^p(B(x_0,\tau)))^{\frac{1}{p}}}{(w^q(B(x_0,t)))^{\frac{1}{q}}}\frac{dt}{t}$$
$$\le C\|f\|_{M_{p,\varphi_1}(w^p)}\, \varphi_2(x_0,t).$$

Then by (9) we get

$$\|T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f\|_{M_{q,\varphi_2}(w^q)}$$
$$\le C \sup_{x_0\in\mathbb{R}^n, t>0} \frac{1}{\varphi_2(x_0,t)}\Big(\frac{1}{w^q(B(x_0,t))}\int_{B(x_0,t)} |T_{\Omega,\alpha}^{A_1,A_2,\ldots,A_k} f(y)|^q w^q(y)dy\Big)^{1/q}$$
$$\le C \prod_{j=1}^k \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j}A_j\|_* \sup_{x_0\in\mathbb{R}^n, t>0}\frac{1}{\varphi_2(x_0,t)}$$
$$\times \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)^k \|f\|_{L_{p,w^p}(B(x_0,t))}(w^q(B(x_0,t)))^{-\frac{1}{q}}\frac{dt}{t}$$
$$\le C \prod_{j=1}^k \sum_{|\gamma_j|=m_j-1} \|D^{\gamma_j}A_j\|_* \|f\|_{M_{p,\varphi_1}(w^p)}.$$

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