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A class of 3-dimensional almost Kenmotsu manifolds with harmonic curvature tensors

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Abstract: Let M^3 be a three-dimensional almost Kenmotsu manifold satisfying $\nabla_{\xi}h = 0$. In this paper, we prove that the curvature tensor of M^3 is harmonic if and only if M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. This generalizes a recent result obtained by [Wang Y., Three-dimensional locally symmetric almost Kenmotsu manifolds, Ann. Polon. Math., 2016, 116, 79-86] and [Cho J.T., Local symmetry on almost Kenmotsu three-manifolds, Hokkaido Math. J., 2016, 45, 435-442].

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MSC: 53D15, 53C25

1 Introduction

Let M^{2n+1} be a $(2n + 1)$ -dimensional smooth differentiable manifold on which there exists an almost contact metric structure (ϕ, ξ, η, g) defined by

$$\begin{aligned}\phi^2 &= -\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y),\end{aligned}$$

where id denotes the identity map, ϕ a $(1, 1)$ -type tensor field, ξ a global vector field tangent to M^{2n+1} , η a global one form and g a Riemannian metric. Generally, ξ is said to be the Reeb or characteristic vector field and η is a contact 1-form. We denote by $(M^{2n+1}, \phi, \xi, \eta, g)$ a smooth manifold furnished with an almost contact metric structure and it is said to be an almost contact metric manifold. The fundamental 2-form Φ of an almost contact metric manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y)$$

for any vector fields X, Y . An almost contact metric manifold turns out to be a contact metric manifold if $d\eta = \Phi$ (see [3]) or an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ (see [4]). On the Riemannian product of an almost contact metric manifold M^{2n+1} and \mathbb{R} , there exists an almost complex structure defined by

$$J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X) \frac{d}{dt}\right),$$

where X denotes a vector field tangent to M^{2n+1} , t is the coordinate of \mathbb{R} and f is a C^∞ -function on $M^{2n+1} \times \mathbb{R}$. We denote by $[\phi, \phi]$ the the Nijenhuis tensor of ϕ , if $[\phi, \phi] = -2d\eta \otimes \xi$ (or equivalently, the almost complex structure J is integrable), then the almost contact metric structure is said to be normal. A normal contact metric manifold and a normal almost Kenmotsu manifold is said to be a Sasakian and a Kenmotsu manifold (see [5]) respectively. We refer the reader to Blair [3] for more details on the geometry of almost contact manifolds.

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In 1972, K. Kenmotsu in [5] proved that a locally symmetric Kenmotsu manifold is of constant sectional curvature -1 . Generalizing Kenmotsu’s results, Dileo and Pastore in [6] proved that a locally symmetric almost Kenmotsu manifold with $R(X, Y)\xi = 0$ for any vector fields X, Y orthogonal to the Reeb vector field and $h \neq 0$ is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Moreover, the present author and X. Liu in [7] proved that a locally symmetric CR -integrable almost Kenmotsu manifold of dimension greater than three is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Recently, the present author [1] and J. T. Cho [2] independently obtained that a three-dimensional locally symmetric almost Kenmotsu manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product $\mathbb{H}^2(-4) \times \mathbb{R}$.

In this paper, we start to study a class of three-dimensional almost Kenmotsu manifolds satisfying a reasonable geometric condition, namely $\nabla_\xi h = 0$, where $2h$ is the Lie differentiation of the contact structure ϕ along the Reeb vector field. We mainly obtain that the Riemannian curvature tensor of such an almost Kenmotsu manifold is harmonic if and only if the manifold is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$. We remark that this generalizes a recent result obtained by Y. Wang [1] and J. T. Cho [2] (see Corollary 4.3 for more details).

The present paper is arranged as follows. In Section 2, we collect some necessary basics and formulas on three-dimensional almost Kenmotsu manifolds. Next, in Section 3, we discuss the relations between three-dimensional Lie groups endowed with a left invariant almost Kenmotsu structure and the generalized (k, μ, ν) -nullity conditions. Using these results, we present a concrete example of a three-dimensional almost Kenmotsu manifold satisfying $\nabla_\xi h = 0$ but $h \neq 0$. Finally, in Section 4, we provide our main results with their proofs.

2 Three-dimensional Almost Kenmotsu manifolds

Let $(M^3, \phi, \xi, \eta, g)$ be a 3-dimensional almost Kenmotsu manifold. In what follows, we denote by $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$, where \mathcal{L} denotes the Lie differentiation and R is the Riemannian curvature tensor. From Dileo and Pastore [6, 8], we see that both h and h' are symmetric operators and we recall some properties of almost Kenmotsu manifolds as follows:

$$h\xi = l\xi = 0, \text{ tr}h = \text{tr}(h') = 0, h\phi + \phi h = 0, \tag{1}$$

$$\nabla\xi = h' + \text{id} - \eta \otimes \xi, \tag{2}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{3}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{4}$$

$$\text{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \text{tr}h^2, \tag{5}$$

where S denotes the Ricci curvature tensor and Q the associated Ricci operator with respect to the metric g . Throughout this paper, we denote by \mathcal{D} the distribution $\mathcal{D} = \ker \eta$ which is of dimension $2n$. It is easy to check that each integral manifold of \mathcal{D} , with the restriction of ϕ , admits an almost Kähler structure. If the associated almost Kähler structure is integrable, or equivalently (see [8]),

$$(\nabla_X\phi)Y = g(\phi X + hX, Y)\xi - \eta(Y)(\phi X + hX) \tag{6}$$

for any vector fields X, Y , then we say that M^{2n+1} is CR -integrable. Following [4, Theorem 2.1], we see that an almost Kenmotsu manifold is Kenmotsu if and only if

$$(\nabla_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \tag{7}$$

for any vector fields X, Y . Notice that a three-dimensional almost Kenmotsu manifold is always CR -integrable. Then the following result follows immediately from (6) and (7).

Proposition 2.1. Any 3-dimensional almost Kenmotsu manifold is Kenmotsu if and only if h vanishes.

Let \mathcal{U}_1 be the open subset of a 3-dimensional almost Kenmotsu manifold M^3 such that $h \neq 0$ and \mathcal{U}_2 the open subset of M^3 which is defined by $\mathcal{U}_2 = \{p \in M^3 : h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \phi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On \mathcal{U}_1 , we may set $he = \lambda e$ and hence $h\phi e = -\lambda\phi e$, where λ is a positive function on \mathcal{U}_1 . Note that the eigenvalue function λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 2.2 ([9, Lemma 6]). On \mathcal{U}_1 we have

$$\begin{aligned} \nabla_\xi \xi &= 0, \quad \nabla_\xi e = a\phi e, \quad \nabla_\xi \phi e = -ae, \\ \nabla_e \xi &= e - \lambda\phi e, \quad \nabla_e e = -\xi - b\phi e, \quad \nabla_e \phi e = \lambda\xi + be, \\ \nabla_{\phi e} \xi &= -\lambda e + \phi e, \quad \nabla_{\phi e} e = \lambda\xi + c\phi e, \quad \nabla_{\phi e} \phi e = -\xi - ce, \end{aligned} \tag{8}$$

where a, b, c are smooth functions.

Applying Lemma 2.2 in the following Jacobi identity

$$[[\xi, e], \phi e] + [[e, \phi e], \xi] + [[\phi e, \xi], e] = 0 \tag{9}$$

yields that

$$\begin{cases} e(\lambda) - \xi(b) - e(a) + c(\lambda - a) - b = 0, \\ \phi e(\lambda) - \xi(c) + \phi e(a) + b(\lambda + a) - c = 0. \end{cases} \tag{10}$$

Moreover, applying Lemma 2.2 we have (see also [9]) the following:

Lemma 2.3. On \mathcal{U}_1 , the Ricci operator can be written as

$$\begin{aligned} Q\xi &= -2(\lambda^2 + 1)\xi - (\phi e(\lambda) + 2\lambda b)e - (e(\lambda) + 2\lambda c)\phi e, \\ Qe &= -(\phi e(\lambda) + 2\lambda b)\xi - (e(c) + \phi e(b) + b^2 + c^2 + 2\lambda a + 2)e + (\xi(\lambda) + 2\lambda)\phi e, \\ Q\phi e &= -(e(\lambda) + 2\lambda c)\xi + (\xi(\lambda) + 2\lambda)e - (e(c) + \phi e(b) + b^2 + c^2 - 2\lambda a + 2)\phi e, \end{aligned}$$

with respect to the local basis $\{\xi, e, \phi e\}$.

3 Three-dimensional Lie group and some nullity conditions

Let us first recall the following definition.

Definition 3.1. A 3-dimensional almost Kenmotsu manifold is called a (k, μ, ν) -almost Kenmotsu manifold if the Reeb vector field satisfies the (k, μ, ν) -nullity condition, that is,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)h'X - \eta(X)h'Y) \tag{11}$$

for any vector fields X, Y , where k, μ and ν are smooth functions.

In the framework of almost Kenmotsu manifolds, some classes of nullity conditions were studied by many authors. We observe that a (k, μ, ν) -nullity condition becomes a

- k -nullity condition if k is a constant and $\mu = \nu = 0$ (see [10]);
- generalized k -nullity condition if k is a function and $\mu = \nu = 0$ (see [11]);
- (k, μ) -nullity condition if k and μ are constants and $\nu = 0$ (see [8]);
- generalized (k, μ) -nullity condition if k and μ are functions and $\nu = 0$ (see [12] and [11]);
- $(k, \nu)'$ -nullity condition if k and ν are constants and $\mu = 0$ (see [8]);
- generalized $(k, \nu)'$ -nullity condition if k and ν are functions and $\mu = 0$ (see [12] and [11]).

Using the above definitions and some results shown in [8] we have

Theorem 3.2. Any 3-dimensional non-unimodular Lie group admits a left invariant almost Kenmotsu structure for which the Reeb vector field satisfies the (k, μ, ν) -nullity condition with k, μ and ν being constants.

Proof. By [8, Theorem 5.2] we know that on any 3-dimensional non-unimodular Lie group there exists an almost Kenmotsu structure. Next, we recall the proof of this result shown in [8]. Let G be a 3-dimensional non-unimodular Lie group, then there exists a left invariant local orthonormal frame fields $\{e_1, e_2, e_3\}$ satisfying

$$[e_1, e_2] = \alpha e_2 + \beta e_3, [e_2, e_3] = 0, [e_1, e_3] = \gamma e_2 + \delta e_3 \tag{12}$$

and $\alpha + \delta = 2$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We define a metric g on G by $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$. Also, we denote by $\xi = -e_1$ and denote by η the dual 1-form of ξ . Thus, we may define a $(1, 1)$ -type tensor field ϕ by $\phi(\xi) = 0, \phi(e_2) = e_3$ and $\phi(e_3) = -e_2$. Then, one can check that (G, ϕ, ξ, η, g) admits a left invariant almost Kenmotsu structure. Next, we prove that the Reeb vector field of this almost Kenmotsu structure satisfies the (k, μ, ν) -nullity condition with k, μ, ν being constants. Firstly, using the Levi-Civita equation and (12) we obtain

$$\begin{aligned} \nabla_{\xi}\xi &= 0, \nabla_{e_2}\xi = \alpha e_2 + \frac{1}{2}(\beta + \gamma)e_3, \nabla_{e_3}\xi = \frac{1}{2}(\beta + \gamma)e_2 + (2 - \alpha)e_3, \\ \nabla_{\xi}e_2 &= \frac{1}{2}(\gamma - \beta)e_3, \nabla_{e_2}e_2 = -\alpha\xi, \nabla_{e_3}e_2 = -\frac{1}{2}(\beta + \gamma)\xi, \\ \nabla_{\xi}e_3 &= \frac{1}{2}(\beta - \gamma)e_2, \nabla_{e_2}e_3 = -\frac{1}{2}(\beta + \gamma)\xi, \nabla_{e_3}e_3 = (\alpha - 2)\xi. \end{aligned} \tag{13}$$

Using (13), by a straightforward calculation we obtain

$$\begin{aligned} R(e_2, e_3)\xi &= 0, \\ R(e_2, \xi)\xi &= -(\alpha^2 + \frac{1}{4}(\beta + \gamma)(3\beta - \gamma))e_2 + (\beta(\alpha - 2) - \alpha\gamma)e_3, \\ R(e_3, \xi)\xi &= (\beta(\alpha - 2) - \alpha\gamma)e_2 - ((\alpha - 2)^2 + \frac{1}{4}(\beta + \gamma)(3\gamma - \beta))e_3. \end{aligned} \tag{14}$$

In view of (2), it follows from (13) that

$$he_2 = (\alpha - 1)e_3 - \frac{1}{2}(\beta + \gamma)e_2 \text{ and } he_3 = \frac{1}{2}(\beta + \gamma)e_3 + (\alpha - 1)e_2. \tag{15}$$

If ξ satisfies the (k, μ, ν) -nullity condition, it follows from (11) and (15) that

$$\begin{aligned} R(e_2, e_3)\xi &= 0, \\ R(e_2, \xi)\xi &= (k - \frac{1}{2}\mu(\beta + \gamma) + \nu(\alpha - 1))e_2 + (\mu(\alpha - 1) + \frac{1}{2}\nu(\beta + \gamma))e_3, \\ R(e_3, \xi)\xi &= (\mu(\alpha - 1) + \frac{1}{2}\nu(\beta + \gamma))e_2 + (k + \frac{1}{2}\mu(\beta + \gamma) - \nu(\alpha - 1))e_3. \end{aligned} \tag{16}$$

Comparing (14) with (16) we state that there exists a unique solution for k, μ and ν provided that either $\beta + \gamma \neq 0$ or $\alpha \neq 1$, namely,

$$k = -\alpha^2 + 2\alpha - \frac{1}{4}(\beta + \gamma)^2 - 2, \mu = \beta - \gamma, \nu = -2. \tag{17}$$

Notice that the $(k, \mu, -2)$ -nullity condition defined by relation (17) implies that G has a non-Kenmotsu almost Kenmotsu structure if we assume that $h \neq 0$ (or equivalently, either $\beta + \gamma \neq 0$ or $\alpha \neq 1$).

Otherwise, if $h = 0$, taking into account (15) we observe that the condition $\beta + \gamma = 0$ and $\alpha = 1$ holds. Using $h = 0$ in (2) gives that $\nabla\xi = \text{id} - \eta \otimes \xi$ and hence $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$ for any vector fields X, Y . This implies that ξ satisfies the $(-1, 0, 0)$ -nullity condition and by Proposition 2.1 we see that in this case G has a Kenmotsu structure. This completes the proof. \square

The following proposition follows directly from (15) and (17).

Proposition 3.3. *The Reeb vector field of the non-Kenmotsu almost Kenmotsu structure defined in Theorem 3.2 satisfies the (k, ν) -nullity condition if and only if $\beta = \gamma$ and either $\alpha \neq 1$ or $\beta \neq 0$.*

From (13) and (15) we obtain the following:

Proposition 3.4. *On the non-Kenmotsu almost Kenmotsu structure defined in Theorem 3.2 there holds $\nabla_{\xi}h = (\beta - \gamma)h'$.*

Next, we show that under certain restrictions of k and μ the converse of the above Theorem 3.2 is true.

Theorem 3.5. *Any 3-dimensional non-Kenmotsu (k, μ, ν) -almost Kenmotsu manifold with k a constant and μ invariant along the Reeb vector field is locally isometric to a 3-dimensional non-unimodular Lie group.*

Proof. Let M^3 be a 3-dimensional (k, μ, ν) -almost Kenmotsu manifold with $h \neq 0$ and k a constant, then \mathcal{U}_1 is a non-empty subset. It follows from (11) that the Reeb vector field ξ is an eigenvector field of the Ricci operator, i.e. $Q\xi = 2k\xi$. Using this in Lemma 2.3 we have $\lambda^2 = -k - 1 \neq 0$ and hence we get

$$b = c = 0. \tag{18}$$

In view of (8), by a simple computation we obtain that

$$R(e, \xi)\xi = -(\lambda^2 + 2\lambda a + 1)e + 2\lambda\phi e. \tag{19}$$

Also, it follows from (11) that

$$R(e, \xi)\xi = (k + \lambda\mu)e - \lambda\nu\phi e. \tag{20}$$

Obviously, comparing (19) with (20) we obtain $\mu = -2a$ and $\nu = -2$. Using (18) in (10) we obtain that $e(a) = \phi e(a) = 0$. In view of the assumption μ invariant along ξ , we conclude that a is a constant. In this context, it follows from (8) that

$$[\xi, e] = (\lambda + a)\phi e - e, [e, \phi e] = 0, [\phi e, \xi] = (a - \lambda)e + \phi e. \tag{21}$$

A Lie group G is said to be unimodular if its left-invariant Haar measure is also right-invariant. It is well-known a Lie group G is unimodular if and only if the endomorphism $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\text{ad}_X(Y) = [X, Y]$ has trace equal to zero for any $X \in \mathfrak{g}$, where \mathfrak{g} denotes the Lie algebra associated to G . Following Milnor [13], we state that M^3 is locally isometric to a 3-dimensional non-unimodular Lie group. In fact, from (21) we see that its unimodular kernel $\{X \in \mathfrak{g} : \text{trace}(\text{ad}_X) = 0\}$ is commutative and of 2-dimension and $\text{trace}(\text{ad}_{\xi}) = -2$. This completes the proof. \square

If the Reeb vector field ξ satisfies the (k, μ, ν) -nullity condition, putting $Y = \xi$ into (11) gives that

$$l = -k\phi^2 + \mu h + \nu h'. \tag{22}$$

Using (22) in (3) and (4) we obtain $h^2 = (k + 1)\phi^2$ and hence the following proposition is true.

Proposition 3.6. *On any 3-dimensional (k, μ, ν) -almost Kenmotsu manifold there holds that $\nabla_{\xi}h = \mu h' - (\nu + 2)h$.*

Note that the above proposition is in fact a generalization of Proposition 3.4.

4 Almost Kenmotsu manifolds with harmonic curvature tensors

A Riemannian manifold M is said to have harmonic curvature tensor if $\text{div}R = 0$, where R denotes the Riemannian curvature tensor. As is well known, the curvature tensor R is harmonic if and only if the associated Ricci tensor Q is of Codazzi-type, namely,

$$(\nabla_X Q)Y = (\nabla_Y Q)X \tag{23}$$

for any vector fields X, Y on M .

Almost Kenmotsu manifolds with the Reeb vector field belonging to $(k, \mu)'$ -nullity distribution and harmonic curvature tensor were studied by the present author and X. Liu [14]. In this section, we aim to obtain a classification of three-dimensional almost Kenmotsu manifolds satisfying $\nabla_{\xi}h = 0$ whose curvature tensors are harmonic.

By Proposition 3.6, we see that any 3-dimensional $(k, 0, -2)$ -almost Kenmotsu manifold with k a function satisfies $\nabla_{\xi}h = 0$. As a special case of this result, from Propositions 3.3 and 3.4 or [8, Theorem 4.1] we see that if a 3-dimensional non-unimodular Lie group G with a left invariant local orthonormal frame fields satisfies

$$[e_1, e_2] = \alpha e_2 + \beta e_3, [e_2, e_3] = 0, [e_1, e_3] = \beta e_2 + (2 - \alpha)e_3 \tag{24}$$

and either $\alpha \neq 1$ or $\beta \neq 0$, then $\nabla_{\xi}h = 0$ holds on the almost Kenmotsu structure defined by (24). In fact, the Reeb vector field ξ of the almost Kenmotsu structure satisfies the $(k, 0, -2)$ -nullity condition with k a constant.

Notice that the condition $\nabla_{\xi}h = 0$ was also used by Dileo and Pastore [15] in the study of almost Kenmotsu manifolds with η -parallel tensor field h' .

Using the well-known formula $\operatorname{div} Q = \frac{1}{2} \operatorname{grad}(r)$, we obtain from relation (23) that the following proposition is true.

Proposition 4.1. *The scalar curvature of a Riemannian manifold with harmonic curvature tensor is a constant.*

Applying Proposition 4.1 we obtain the following main result.

Theorem 4.2. *A 3-dimensional almost Kenmotsu manifold satisfying $\nabla_{\xi}h = 0$ and having a harmonic curvature tensor is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

Proof. Let M^3 be three-dimensional almost Kenmotsu manifold. We shall first consider the case $h = 0$, then by Proposition 2.1 we see that M^3 is Kenmotsu and $\nabla_{\xi}h = 0$ holds trivially. If the curvature tensor of M^3 is harmonic, following Proposition 4.1 we observe that the scalar curvature of M^3 is a constant. Notice that Inoguchi in [16, Proposition 3.1] proved that a three-dimensional Kenmotsu manifold of constant scalar curvature is of constant sectional curvature -1 . This yields that a three-dimensional Kenmotsu manifolds with harmonic curvature tensor is of constant sectional curvature -1 .

Now let us consider a three-dimensional almost Kenmotsu manifold M^3 satisfying $h \neq 0$, then \mathcal{U}_1 is a non-empty subset. By applying Lemma 2.2 and a direct calculation we obtain

$$(\nabla_{\xi}h)e = \xi(\lambda)e + 2a\lambda\phi e \text{ and } (\nabla_{\xi}h)\phi e = -\xi(\lambda)\phi e + 2a\lambda e. \tag{25}$$

By the assumption condition $\nabla_{\xi}h = 0$ and (25) we have

$$\xi(\lambda) = a = 0, \tag{26}$$

where we have used that λ is positive on \mathcal{U}_1 . In what follows, we denote f by

$$f = e(c) + \phi e(b) + b^2 + c^2 + 2. \tag{27}$$

Then, using (26) and (27) we obtain from Lemmas 2.2 and 2.3 that

$$(\nabla_{\xi}Q)\xi = -\xi(\phi e(\lambda) + 2\lambda b)e - \xi(e(\lambda) + 2\lambda c)\phi e. \tag{28}$$

$$(\nabla_{\xi}Q)e = -\xi(\phi e(\lambda) + 2\lambda b)\xi - \xi(f)e. \tag{29}$$

$$(\nabla_{\xi}Q)\phi e = -\xi(e(\lambda) + 2\lambda c)\xi - \xi(f)\phi e. \tag{30}$$

$$\begin{aligned} (\nabla_e Q)\xi &= 2(\phi e(\lambda) - 3\lambda e(\lambda) + 2\lambda b - 2\lambda^2 c)\xi \\ &\quad + (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))e \\ &\quad + (2\lambda^3 + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c) - \lambda f)\phi e. \end{aligned} \tag{31}$$

$$\begin{aligned}
 (\nabla_e Q)e &= (f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c))\xi \\
 &\quad - (e(f) + 2\phi e(\lambda))e + (e(\lambda) + \lambda\phi e(\lambda) + 2\lambda^2 b - 2\lambda c)\phi e.
 \end{aligned}
 \tag{32}$$

$$\begin{aligned}
 (\nabla_e Q)\phi e &= (2\lambda^3 - f\lambda + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c))\xi \\
 &\quad + (e(\lambda) + \lambda\phi e(\lambda) - 2\lambda c + 2\lambda^2 b)e \\
 &\quad + (2\lambda(e(\lambda) + 2\lambda c) - e(f) - 4\lambda b)\phi e.
 \end{aligned}
 \tag{33}$$

$$\begin{aligned}
 (\nabla_{\phi e} Q)\xi &= 2(e(\lambda) - 3\lambda\phi e(\lambda) + 2\lambda c - 2\lambda^2 b)\xi \\
 &\quad + (2\lambda^3 + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) - \lambda f)e \\
 &\quad + (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\phi e.
 \end{aligned}
 \tag{34}$$

$$\begin{aligned}
 (\nabla_{\phi e} Q)e &= (2\lambda^3 - f\lambda + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b))\xi \\
 &\quad - (\phi e(f) + 4\lambda c - 2\lambda(\phi e(\lambda) + 2\lambda b))e \\
 &\quad + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)\phi e.
 \end{aligned}
 \tag{35}$$

$$\begin{aligned}
 (\nabla_{\phi e} Q)\phi e &= (f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b))\xi \\
 &\quad + (\phi e(\lambda) + \lambda e(\lambda) + 2\lambda^2 c - 2\lambda b)e - (\phi e(f) + 2e(\lambda))\phi e.
 \end{aligned}
 \tag{36}$$

If we replace X and Y in (23) by e and ξ , respectively, then we have from (29) and (31) that

$$\begin{cases}
 2(\phi e(\lambda) - 3\lambda e(\lambda) + 2\lambda b - 2\lambda^2 c) + \xi(\phi e(\lambda) + 2\lambda b) = 0, \\
 f - 2 - e(\phi e(\lambda) + 2\lambda b) - b(e(\lambda) + 2\lambda c) + \xi(f) = 0, \\
 2\lambda^3 + b(\phi e(\lambda) + 2\lambda b) - e(e(\lambda) + 2\lambda c) - \lambda f = 0.
 \end{cases}
 \tag{37}$$

Similarly, if we replace X and Y in (23) by ϕe and ξ , respectively, we obtain from (30) and (34) that

$$\begin{cases}
 2(e(\lambda) - 3\lambda\phi e(\lambda) + 2\lambda c - 2\lambda^2 b) + \xi(e(\lambda) + 2\lambda c) = 0, \\
 f - 2 - \phi e(e(\lambda) + 2\lambda c) - c(\phi e(\lambda) + 2\lambda b) + \xi(f) = 0, \\
 2\lambda^3 + c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) - \lambda f = 0.
 \end{cases}
 \tag{38}$$

Similarly, if we replace X and Y in (23) by e and ϕe , respectively, we obtain from (33) and (35) that

$$\begin{cases}
 c(e(\lambda) + 2\lambda c) - \phi e(\phi e(\lambda) + 2\lambda b) + e(e(\lambda) + 2\lambda c) = b(\phi e(\lambda) + 2\lambda b), \\
 e(\lambda) - \lambda\phi e(\lambda) + 2\lambda c - 2\lambda^2 b + \phi e(f) = 0, \\
 \lambda e(\lambda) - \phi e(\lambda) + 2\lambda^2 c - 2\lambda b - e(f) = 0.
 \end{cases}
 \tag{39}$$

Moreover, it follows from Lemma 2.3 that

$$r = -2(\lambda^2 + 1) - 2f.
 \tag{40}$$

Applying again the well known formula $\operatorname{div} Q = \frac{1}{2}\operatorname{grad}(r)$ we have from equations (28), (32) and (36) that

$$\begin{cases}
 -\xi(\phi e(\lambda) + 2\lambda b) + 3\lambda e(\lambda) - \phi e(\lambda) + 2\lambda^2 c - 2\lambda b = 0, \\
 -\xi(e(\lambda) + 2\lambda c) + 3\lambda\phi e(\lambda) - e(\lambda) + 2\lambda^2 b - 2\lambda c = 0,
 \end{cases}
 \tag{41}$$

where we have used the scalar curvature $r = \text{constant}$, equation (40), the second terms of relations (37) and (38).

Next, using the first equation of (41) and the second equation of (41) in the first terms of (37) and (38), respectively, we obtain

$$\begin{cases} 3\lambda e(\lambda) - \phi e(\lambda) + 2\lambda^2 c - 2\lambda b = 0, \\ 3\lambda \phi e(\lambda) - e(\lambda) + 2\lambda^2 b - 2\lambda c = 0, \end{cases} \tag{42}$$

and

$$\begin{cases} \xi(\phi e(\lambda)) + 2\lambda \xi(b) = 0, \\ \xi(e(\lambda)) + 2\lambda \xi(c) = 0. \end{cases} \tag{43}$$

Taking the covariant differentiation of relation (42) and using (43) and (26) we have

$$\xi(b) = \xi(c) = 0, \tag{44}$$

where we have used that λ is a positive function. Using (44) and (26) in (10) gives that

$$\begin{cases} e(\lambda) = b - c\lambda, \\ \phi e(\lambda) = c - b\lambda. \end{cases} \tag{45}$$

Putting (45) into (42) yields that

$$\begin{cases} 2b\lambda - c\lambda^2 - c = 0, \\ 2c\lambda - b\lambda^2 - b = 0. \end{cases} \tag{46}$$

It follows from (46) that $(\lambda^2 + 1)(b^2 - c^2) = 0$ and hence we get either $b - c = 0$ or $b + c = 0$. We continue the discussion with the following two cases.

Case i: Using $b = c$ in (46) we have either $b = c = 0$ or $\lambda = 1$. Now we assume that $b = c = 0$ holds and using this in (10) we see that λ is a positive constant. By applying $a = b = c = 0$ in (8) we obtain the following

$$[\xi, e] = \lambda \phi e - e, [e, \phi e] = 0, [\phi e, \xi] = -\lambda e + \phi e.$$

According to J. Milnor [13], we now conclude that M^3 is locally isometric to a three-dimensional non-unimodular Lie group equipped with a left invariant non-Kenmotsu almost Kenmotsu structure. Moreover, using $a = b = 0$ in equations (28)-(36) we get

$$\nabla Q = 0.$$

Notice that the above relation holds on a three-dimensional Riemannian manifold if and only if the curvature tensor is parallel, i.e. the manifold is locally symmetric. We also observe that Wang [1, Theorem 3.4] and [2, Theorem 5] proved that a locally symmetric three-dimensional non-Kenmotsu almost Kenmotsu manifold is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

Otherwise, if $\lambda = 1$ we obtain from (46) again $b = c$ and using this in the last two equations of (37) we obtain

$$e(b) = 0.$$

Moreover, using $\lambda = 1$ and $b = c$ in the last two equations of (38) we obtain

$$\phi e(b) = 0.$$

In view of (44) we observe that both b and c are constants. Then it follows from the second equation of (37) that $f = 2 + 2b^2$. Finally, using this in (28)-(36) gives that $\nabla Q = 0$ and this is equivalent to the local symmetry. Then the proof follows from Wang [1, Theorem 3.4] or [2, Theorem 5].

Case ii: Now we consider the other case: $b + c = 0$. Using this in (46) gives that $b = c = 0$, where we have used that λ is positive. Moreover, putting $b = c = 0$ in (45) and applying (26) we see that λ is a constant. Therefore, the proof follows from **Case i**. □

On a three-dimensional locally symmetric almost Kenmotsu manifold, applying the local symmetry condition we obtain that $\nabla_{\xi}l = 0$. Substituting X with ξ in (6) implies that $\nabla_{\xi}\phi = 0$. Then, by taking the covariant differentiation of (3) along ξ we obtain $\nabla_{\xi}h^2 = \nabla_{\xi}h \circ h + h \circ \nabla_{\xi}h = 0$. It follows directly that $(\nabla_{\xi}\nabla_{\xi}h) \circ h + 2(\nabla_{\xi}h)^2 + h \circ (\nabla_{\xi}\nabla_{\xi}h) = 0$. Also, using $\nabla_{\xi}l = \nabla_{\xi}h = 0$ and taking the covariant differentiation of (4) along ξ we obtain $\nabla_{\xi}\nabla_{\xi}h = -2\nabla_{\xi}h$. Therefore, it is easily seen that $(\nabla_{\xi}h)^2 = 0$ and hence we get $\nabla_{\xi}h = 0$. Then the following corollary follows directly from Theorem 4.2 and can also be regarded as a generalization of Wang [1, Theorem 3.4] and [2, Theorem 5].

Corollary 4.3. *Let M^3 be a three-dimensional almost Kenmotsu manifold, then the following three statements are equivalent:*

- (1) M^3 is locally symmetric.
- (2) M^3 satisfies $\nabla_{\xi}h = 0$ and the curvature tensor is harmonic.
- (3) M^3 is locally isometric to either the hyperbolic space $\mathbb{H}^3(-1)$ or the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.

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