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Research Article

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On meromorphic functions for sharing two sets and three sets in m-punctured complex plane

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Abstract: In this article, we study the uniqueness problem of meromorphic functions in m-punctured complex plane Ω and obtain that there exist two sets S_1, S_2 with $\sharp S_1 = 2$ and $\sharp S_2 = 9$, such that any two admissible meromorphic functions f and g in Ω must be identical if f, g share S_1, S_2 IM in Ω.

Keywords: Meromorphic function, m-puncture, Uniqueness

MSC: 30D30, 30D35

1 Introduction

We firstly assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions such as $m(r, f)$, $N(r, f)$, $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevalinna theory (see Hayman [1], Yang [2] and Yi and Yang [3]).

In the past few decades, the uniqueness of meromorphic functions of single connected region attracted many investigations (see [3]) where a number of interesting results were obtained. Around 2000s, Fang, Zheng and Mao investigated the uniqueness of meromorphic functions in the unit disc and some angular domain, and obtained some important results (see [4-9]).

Recently, there were some articles discussing the Nevanlinna theory of meromorphic functions on the annuli (see [10, 11]). In 2004, Korhonen [12] established analogues of Navanlinna's main theorems including the lemma on the logarithmic derivatives on annuli $\mathbb{A} := \{z : R_1 \le |z| \le R_2\}$, by adopting two parameters R_1, R_2. In 2005 and 2006, Khrystiyanyn and Kondratyuk [13, 14] proposed the Nevanlinna theory for meromorphic functions on annuli $\mathbb{A} := \{z : \frac{1}{R} \le |z| \le R\}$(see also [15]) by adopting one parameter R where $1 < R \le +\infty$. Khrystiyanyn and Kondratyuk [13, 14], and Kondratyuk and Laine [15] obtained a series of results of value distribution and uniqueness of meromorphic functions on annuli $\mathbb{A} := \{z : \frac{1}{R} \le |z| \le R\}$ where $1 < R \le +\infty$, including the first and second main theorems, lemma on the logarithmic derivatives on annuli, also including five-values theorem of Nevanlinna on annulus. In 2010, Fernández [16] further investigated the value distribution of meromorphic functions on annulus and gave some extension of some results about meromorphic functions in the plane with finitely many poles. At about the same time, Cao [17, 18] investigated the uniqueness of meromorphic functions on annuli sharing some values

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and some sets, and obtained a number of results which is an improvement of the five-values theorem of Nevanlinna on annulus given by [15]. In 2012, Cao and Deng [19] investigated the uniqueness of meromorphic functions that share three or two finite sets on annulus, and obtained that there exist three sets S_1, S_2, S_3 with $\sharp S_1 = \sharp S_2 = 1$ and $\sharp S_3 = 5$, such that any two admissible meromorphic functions f and g must be identical if f, g share S_1, S_2, S_3 CM on annuli $\mathbb{A}$. In the same year, Xu and Xuan [20] further investigated the problem of meromorphic functions sharing four values on annulus, and gave a theorem which is also an improvement of the five-values theorem of Nevanlinna on annuli given by [15].

As we all know, annulus is a double connected region, can be regarded as a special multiply connected region. Thus, it is natural to ask: *what results can we get when meromorphic functions f, g share some values or finite sets on the multiply connected region?* However, there is no paper discussing uniqueness for meromorphic functions in the multiply connected region. *The main purpose of this article is to investigate the uniqueness of meromorphic functions in a special multiply connected region — m-punctured complex plane.*

The structure of this paper is as follows. *In Section 2, we introduce the basic notations and fundamental theorems of meromorphic functions m-punctured complex plane. Section 3 is devoted to study the uniqueness of meromorphic functions that share three finite sets IM in m-punctured complex planes. Section 4 is devoted to give the uniqueness theorem for meromorphic functions sharing two finite sets IM in m-punctured complex planes.*

2 Nevanlinna theory in m-punctured complex planes

We call that $\Omega = \mathbb{C} \setminus \bigcup_{j=1}^{m}\{c_j\}$ is an m-punctured complex plane, where $c_j \in \mathbb{C}$, $j \in \{1, 2, \ldots, m\}, m \in \mathbb{N}_+$ are distinct points. The annulus is regarded as a special m-punctured plane if $m = 1$ which is studied by [13, 14]. The main purpose of this article is to study meromorphic functions of those m-punctured planes for which $m \geq 2$.

Denote $d = \frac{1}{2}\min\{|c_k - c_j| : j \neq k\}$ and $r_0 = \frac{1}{d} + \max\{|c_j| : j \in \{1, 2, \ldots, m\}\}$. Then

$$\frac{1}{r_0} < \frac{1}{r_0 - \max\{|c_j| : j \in \{1, 2, \ldots, m\}\}} = d,$$

$\overline{D}_{1/r_0}(c_j) \bigcap \overline{D}_{1/r_0}(c_k) = \emptyset$ for $j \neq k$ and $\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0)$ for $j \in \{1, 2, \ldots, m\}$, where $D_\delta(c) = \{z : |z - c| < \delta\}$ and $\overline{D}_\delta(c) = \{z : |z - c| \leq \delta\}$. For an arbitrary $r \geq r_0$, we define

$$\Omega_r = D_r(0) \setminus \bigcup_{j=1}^{m} \overline{D}_{1/r}(c_j).$$

Thus, it follows that $\Omega_r \supset \Omega_{r_0}$ for $r_0 < r \leq +\infty$. It is easy to see that Ω_r is $m + 1$ connected region.

In 2007, Hanyak and Kondratyuk [21] proposed the Nevanlinna value distribution theory for meromorphic functions in m-punctured complex planes and proved a number of theorems which are analog of those results on the whole plane $\mathbb{C}$.

Let f be a meromorphic function in an m-punctured plane Ω, we use $n_0(r, f)$ to denote the counting function of its poles in $\overline{\Omega}_r$, $r_0 \leq r < +\infty$ and

$$N_0(r, f) = \int_{r_0}^{r} \frac{n_0(t, f)}{t} dt,$$

and we also define

$$\begin{aligned} m_0(r, f) =& \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(re^{i\theta})\right| d\theta + \frac{1}{2\pi}\sum_{j=1}^{m}\int_0^{2\pi} \log^+ \left|f(c_j + \frac{1}{r}e^{i\theta})\right| d\theta - \\ &- \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f(r_0e^{i\theta})\right| d\theta - \frac{1}{2\pi}\sum_{j=1}^{m}\int_0^{2\pi} \log^+ \left|f(c_j + \frac{1}{r_0}e^{i\theta})\right| d\theta, \end{aligned}$$

where $\log^+ x = \max\{\log x, 0\}$ and $r_0 \le r < +\infty$, then we call that

$$T_0(r, f) = m_0(r, f) + N_0(r, f)$$

is the Nevanlinna characteristic of f.

Theorem 2.1 (see [21, Theorem 3]). *Let f, f_1, f_2 be meromorphic functions in an m-punctured plane Ω. Then*

(i) *the function $T_0(r, f)$ is non-negative, continuous, non-decreasing and convex with respect to* $\log r$ *on* $[r_0, +\infty)$, $T_0(r_0, f) = 0$;
(ii) *if f identically equals a constant, then $T_0(r, f)$ vanishes identically;*
(iii) *if f is not identically equal to zero, then $T_0(r, f) = T_0(r, 1/f), r_0 \le r < +\infty$;*
(iv) $T_0(r, f_1 f_2) \le T_0(r, f_1) + T_0(r, f_2) + O(1)$ *and* $T_0(r, f_1 + f_2) \le T_0(r, f_1) + T_0(r, f_2) + O(1)$, *for* $r_0 \le r < +\infty$.

Theorem 2.2 (see [21, Theorem 4]). *Let f be a non-constant meromorphic function in an m-punctured plane Ω. Then*

$$T_0(r, \frac{1}{f-a}) = T_0(r, f) + O(1),$$

for any fixed $a \in \mathbb{C}$ and all r, $r_0 \le r < +\infty$.

Let f be a non-constant meromorphic function in an m-punctured plane Ω, for any $a \in \mathbb{C}$, we use $\widetilde{n}_0(r, \frac{1}{f-a})$ to denote the counting function of zeros of $f - a$ with the multiplicities reduced by 1, then it follows that $n_0(r, \frac{1}{f'}) = \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, \frac{1}{f-a})$ for $r_0 \le r < +\infty$, and the equalities

$$\widehat{n}_0(r, f) := \widetilde{n}_0(r, f) + \sum_{a\in\mathbb{C}} \widetilde{n}_0(r, \frac{1}{f-a}) = n_0(r, \frac{1}{f'}) + 2n_0(r, f) - n_0(r, \frac{1}{f'}),$$

and $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, \frac{1}{f'})$, where $\widehat{N}_0(r, f) = \int_1^r \frac{\widehat{n}_0(t,f)}{t} dt$, $r \ge 1$, hold for $r_0 \le r < +\infty$.

Theorem 2.3 (see [21, Theorem 6] (The second fundamental theorem in m-punctured planes)). *Let f be a non-constant meromorphic function in an m-punctured plane Ω, and let $a_1, a_2, \ldots, a_q$ be distinct complex numbers. Then*

$$m_0(r, f) + \sum_{\nu=1}^{q} m_0(r, \frac{1}{f - a_\nu}) \le 2T_0(r, f) - \widehat{N}_0(r, f) + S(r, f), \quad r_0 \le r < +\infty,$$

where $\widehat{N}_0(r, f) = N_0(r, \frac{1}{f'}) + 2N_0(r, f) - N_0(r, \frac{1}{f'})$ and

$$S(r, f) = O(\log T_0(r, f)) + O(\log^+ r), \quad r \to +\infty,$$

outside a set of finite measure.

By [21, Lemma 6] and using the same argument as in [15, Theorem 16.1], we can get the following result.

Theorem 2.4. *Let f be a non-constant meromorphic function in an m-punctured plane Ω, $f^{(k)}$ be its derivative of order k. Then $m_0(r, \frac{f^{(k)}}{f}) \le S(r, f)$, for $r_0 \le r < +\infty$, where $S(r, f)$ is stated as in Theorem 2.3.*

At the end of this section, we introduce other interesting form of the second fundamental theorem in m-punctured planes as follows, which is similar to these on the complex plane $\mathbb{C}$, and play an important role throughout this article.

Theorem 2.5. *Let f be a non-constant meromorphic function in an m-punctured plane Ω, and let $a_1, a_2, \ldots, a_q$ be distinct complex numbers in the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then for $r_0 \le r < +\infty$,*

$$(i) \ \ (q-2)T_0(r, f) \le \sum_{\nu=1}^{q} N_0\left(r, \frac{1}{f - a_\nu}\right) - N_0(r, \frac{1}{f'}) + S(r, f),$$

$$(ii)\ \ (q-2)T_0(r,f) \le \sum_{\nu=1}^{q} \widetilde{N}_0\left(r, \frac{1}{f-a_\nu}\right) + S(r,f),$$

where $\widetilde{N}_0(r,\frac{1}{f-a_\nu}) = \int_1^r \frac{\widetilde{n}_0(t,\frac{1}{f-a_\nu})}{t}dt$, $r \ge 1$ *and* $S(r,f)$ *is stated as in Theorem 2.3.*

Proof. If z_0 is a pole of f in m-punctured plane Ω_r with multiply k, then $\widetilde{n}_0(r,f)$ counts $k-1$ times at z_0, and if z_0 is a zero of $f-a$ in Ω_r with multiply k, then $\widetilde{n}_0(r,f)$ also counts $k-1$ times at z_0. Then we have

$$\sum_{\nu=1}^{q} N_0(r,\frac{1}{f-a_\nu}) - \widehat{N}_0(r,f) \le \sum_{\nu=1}^{q} \widetilde{N}_0(r,\frac{1}{f-a_\nu}),\quad r_0 \le r < +\infty. \tag{1}$$

By Theorem 2.2, for any $a \in \widehat{\mathbb{C}}$ and $r_0 \le r < +\infty$, we have

$$m_0(r,\frac{1}{f-a}) = T_0(r,f) - N_0(r,\frac{1}{f-a}) + O(1), \tag{2}$$

where $m_0(r,\frac{1}{f-a}) = m_0(r,f)$ and $N_0(r,\frac{1}{f-a}) = N_0(r,f)$ as $a=\infty$. From (1), (2) and Theorem 2.3, we can get Theorem 2.5 (ii). Noting that $2N_0(r,f) - N_0(r,\frac{1}{f'}) \ge 0$, from (2) and Theorem 2.3, we can get Theorem 2.5 (i) easily.

Thus, this completes the proof of Theorem 2.5. □

3 Meromorphic functions share two sets IM

In this section, we will discuss the uniqueness of meromorphic functions in m-punctured planes that shared two sets with finite elements IM. Some basic notations of uniqueness of meromorphic functions would be introduced as follows.

Let S be a set of distinct elements in $\widehat{\mathbb{C}}$ and $\Omega \subseteq \mathbb{C}$. Define

$$E_\Omega(S,f) = \bigcup_{a\in S}\{z \in \Omega | f_a(z) = 0,\quad counting\quad multiplicities\},$$

$$\overline{E}_\Omega(S,f) = \bigcup_{a\in S}\{z \in \Omega | f_a(z) = 0,\quad ignoring\quad multiplicities\},$$

where $f_a(z) = f(z) - a$ if $a \in \mathbb{C}$ and $f_\infty(z) = 1/f(z)$.

Let f and g be two non-constant meromorphic functions in $\mathbb{C}$, we say f and g share the set S CM(counting multiplicities) in Ω if $E_\Omega(S,f) = E_\Omega(S,g)$, we say f and g share the set S IM(ignoring multiplicities) in Ω if $\overline{E}_\Omega(S,f) = \overline{E}_\Omega(S,g)$. In particular, when $S = \{a\}$, where $a \in \widehat{\mathbb{C}}$, we say f and g share the value a CM in Ω if $E_\Omega(S,f) = E_\Omega(S,g)$, and we say f and g share the value a IM in Ω if $\overline{E}_\Omega(S,f) = \overline{E}_\Omega(S,g)$.

Definition 3.1. *Let f be a nonconstant meromorphic function in m-punctured plane Ω. The function f is called transcendental in m-punctured plane Ω provided that*

$$\limsup_{r\to+\infty} \frac{T_0(r,f)}{\log r} = +\infty,\quad r_0 \le r < +\infty.$$

Now, we will show my first main theorem of this article as follows.

Theorem 3.2. *Let f and g be two transcendental meromorphic functions in Ω, and let $S_1 = \{0,1\}$, $S_2 = \{w : P_1(w) = 0\}$, where*

$$P_1(w) = \frac{w^9}{9} - \frac{4w^8}{8} + \frac{15w^7}{7} - \frac{4w^6}{6} + \frac{w^5}{5} + 1.$$

If $\overline{E}_\Omega(S_j,f) = \overline{E}_\Omega(S_j,g) (j = 1,2)$, then $f(z) \equiv g(z)$.

Corollary 3.3. *There exist two sets S_1, S_2 with $\sharp S_1 = 2$ and $\sharp S_2 = 9$, such that any two transcendental meromorphic functions f and g must be identical if $\overline{E}_\Omega(S_j, f) = \overline{E}_\Omega(S_j, g)(j = 1, 2)$, where $\sharp S$ is to denote the cardinality of a set S.*

To prove this theorem, we require some lemmas as follows.

Lemma 3.4. *Let f, g be two non-constant meromorphic functions in m-punctured plane Ω, and let z_0 be a common pole of f, g in Ω with multiply 1, then z_0 is a zero of $\frac{f''}{f'} - \frac{g''}{g'}$ in Ω with multiply $k \geq 1$.*

Proof. From the assumptions of this lemma, we can set

$$f(z) = \frac{\varphi(z)}{z - z_0}, \quad g(z) = \frac{\psi(z)}{z - z_0},$$

where $\varphi(z), \psi(z)$ are analytic in Ω and $\varphi(z_0)\psi(z_0) \neq 0$, then

$$f'(z) = \frac{\varphi'(z)(z - z_0) - \varphi(z)}{(z - z_0)^2}, \quad f''(z) = \frac{\varphi''(z)(z - z_0)^2 - 2\varphi'(z)(z - z_0) - 2\varphi(z)}{(z - z_0)^3}.$$

It follows that

$$\frac{f''}{f'} = (z - z_0)\frac{\varphi''(z)}{\varphi'(z)(z - z_0) - \varphi(z)} - \frac{2}{z - z_0}.$$

Similarly, we have

$$\frac{g''}{g'} = (z - z_0)\frac{\psi''(z)}{\psi'(z)(z - z_0) - \psi(z)} - \frac{2}{z - z_0}.$$

Thus, it follows that

$$\frac{f''}{f'} - \frac{g''}{g'} = (z - z_0)\zeta(z),$$

where $\zeta(z)$ is analytic at z_0 in Ω. Therefore, we prove the conclusion of this lemma. □

By a similar discussion as in [22], we can obtain a stand and Valiron-Mohon'ko type theorem in Ω as follows.

Lemma 3.5. *Let f be a nonconstant meromorphic function in m-punctured plane Ω, and let*

$$R(f) = \sum_{k=0}^{n} a_k f^k / \sum_{j=0}^{m} b_j f^j$$

be an irreducible rational function in f with coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T_0(r, R(f)) = dT_0(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

Lemma 3.6. *Suppose that f is a transcendental meromorphic function in m-punctured plane Ω. Let $Q(f) = a_0 f^p + a_1 f^{p-1} + \cdots + a_p (a_0 \neq 0)$ be a polynomial of f with degree p, where the coefficients $a_j\,(j = 0, 1, \ldots, p)$ are constants, and let $b_j\,(j = 1, 2, \ldots, q)$ be $q(q \geq p + 1)$ distinct finite complex numbers. Then*

$$m_0\left(r, \frac{Q(f) \cdot f'}{(f - b_1)(f - b_2) \cdots (f - b_q)}\right) = S(r, f),$$

where $S(r, f)$ is stated as in Theorem 2.3.

Proof. Since

$$\frac{Q(f) \cdot f'}{(f - b_1)(f - b_2) \cdots (f - b_q)} = \sum_{j=1}^{q} \frac{\phi_j}{f - b_j},$$

where ϕ_j are non-zero constants. Then, it follows from Theorem 2.4 that

$$\begin{aligned} m_0\left(r, \frac{Q(f)\cdot f'}{(f-b_1)(f-b_2)\cdots(f-b_q)}\right) &= m_0(r, \sum_{j=1}^{q}\frac{\phi_j f'}{f-b_j}) \\ &\leq \sum_{j=1}^{q} m_0(r, \frac{\phi_j f'}{f-b_j}) + O(1) \\ &\leq S(r,f). \end{aligned}$$

Thus, this completes the proof of Lemma 3.6. □

Definition 3.7 ([23]). *We also call $P(w)$ a uniqueness polynomial in a broad sense if $P(f) = P(g)$ implies $f = g$ for any nonconstant meromorphic functions f, g.*

Lemma 3.8 (see [23]). *Let $S = \{a_1, a_2, \ldots, a_q\}$, $a_1, a_2, \ldots, a_q$ be q distinct complex constants, $P(w)$ be a monic polynomial of the form $P(w) = (w-a_1)(w-a_2)\cdots(w-a_q)$. If $P'(w)$ has mutually distinct k zeros $e_1, e_2, \ldots, e_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively, and satisfies*

$$P(e_\ell) \neq P(e_m), \quad for \;\; 1 \leq \ell < m \leq k.$$

Then $P(w)$ is a uniqueness polynomial in a broad sense if and only if

$$\sum_{1\leq \ell < m \leq k} q_\ell q_m > \sum_{\ell=1}^{k} q_\ell. \tag{3}$$

Proof of Theorem 3.2. Set $F = P_1(f)$ and $G = P_1(g)$. Since $\overline{E}_\Omega(S_j, f) = \overline{E}_\Omega(S_j, g)$, then we have that F, G share 0, 1 IM in Ω and $F' = P_1'(f) = f^4(f-1)^4 f', G' = g^4(g-1)^4 g'$. From Lemma 3.6, we have $T_0(r,F) = 9T_0(r,f) + S(r,f), T_0(r,G) = 9T_0(r,g) + S(r,g)$ and $S(r,F) = S(r,f), S(r,G) = S(r,g)$.

Next, the following two cases will be discussed.

Case 1: Suppose that there exist a constant $\lambda(>\frac{1}{2})$ and a set $I \subset [r_0, +\infty)(mesI = +\infty)$ such that

$$\widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \geq \lambda(T_0(r,f) + T_0(r,g)) + S(r,f) + S(r,g), \;\; (r \to +\infty, r \in I). \tag{4}$$

Set $U = \frac{F'}{F} - \frac{G'}{G}$, from Theorem 2.4 we have $m_0(r,U) = S(r,F) + S(r,G) = S(r,f) + S(r,g)$. Suppose that $U \not\equiv 0$, since f, g share 0,1 IM in Ω, we can see that the common zeros of f, g are the zero of U in Ω, and the common zeros of $f-1, g-1$ are also the zero of U in Ω. Thus, we have

$$4\widetilde{N}_0(r, \frac{1}{f}) + 4\widetilde{N}_0(r, \frac{1}{f-1}) \leq N_0(r, \frac{1}{U}). \tag{5}$$

On the other hand, it is easy to see that the pole of U in Ω may occur at the poles of F, G or the zeros of F, G in Ω. Then it follows that

$$N_0(r,U) \leq \widetilde{N}_0(r,f) + \widetilde{N}_0(r,g) + \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G}). \tag{6}$$

Hence,

$$T_0(r,U) \leq \widetilde{N}_0(r,f) + \widetilde{N}_0(r,g) + \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G}) + S(r,f) + S(r,g). \tag{7}$$

From (5)-(7), it follows that for $r_0 \leq r < +\infty$

$$\begin{aligned} 4\widetilde{N}_0(r, \frac{1}{f}) + 4\widetilde{N}_0(r, \frac{1}{f-1}) &\leq N_0(r, \frac{1}{U}) \leq T_0(r, \frac{1}{U}) + S(r,f) \\ &\leq \widetilde{N}_0(r,f) + \widetilde{N}_0(r,g) + \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G}) + S(r,f) + S(r,g). \end{aligned} \tag{8}$$

By adding $\widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G})$ into both sides of (8), and from $\overline{E}_\Omega(f, S_1) = \overline{E}_\Omega(f, S_1)$, for $r_0 \le r < +\infty$ we have

$$\widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, \frac{1}{G}) + \widetilde{N}_0(r, \frac{1}{g}) + \widetilde{N}_0(r, \frac{1}{g-1}) +$$
$$+ 2\widetilde{N}_0(r, \frac{1}{f}) + 2\widetilde{N}_0(r, \frac{1}{f-1})$$
$$\le 2\widetilde{N}_0(r, \frac{1}{F}) + 2\widetilde{N}_0(r, \frac{1}{G}) + \widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + S(r, f) + S(r, g).$$

Thus, we can deduce by applying Theorem 2.5 and (4) that

$$9\{T_0(r, f) + T_0(r, g)\} + 2\lambda(T_0(r, f) + T_0(r, g)) + S(r, f) + S(r, g)$$
$$\le 10\{T_0(r, f) + T_0(r, g)\} + S(r, f) + S(r, g), \ r \to +\infty, r \in I. \tag{9}$$

Since $\lambda > 0$ and f, g are admissible functions in Ω, we can get a contradiction. Thus, it follows that $U \equiv 0$, by integration, we have

$$F \equiv KG, \tag{10}$$

where K a non-zero constant. From Lemma 3.5, we have

$$T_0(r, f) = T_0(r, g) + S(r, g), \ \ r_0 \le r < +\infty. \tag{11}$$

The three following subcases will be considered.

Subcase 1.1. Suppose that $K = 1$. Thus, if follows from (10) that $F \equiv G$, that is,

$$P_1(f) \equiv P_1(g). \tag{12}$$

From the form of $P_1(w)$, we can see that there exist nine distinct complex constants α_j ($j = 1, 2, \dots, 9$) such that

$$P_1(w) = (w - \alpha_1)(w - \alpha_2) \cdots (w - \alpha_9).$$

Moreover, we have $P_1'(w) = w^4(w-1)^4$, that is, $P_1'(w)$ has mutually distinct two zeros 0, 1 with multiplicities 4,4, respectively, and satisfying $4 \times 4 = 16 > 8 = 4 + 4$. Thus, $P_1(w)$ is a uniqueness polynomial in a broad sense. From Lemma 3.8, we can get that $f \equiv g$.

Subcase 1.2. Suppose that $K = \zeta_1$, where $\zeta_1 = \frac{1}{9} - \frac{1}{2} + \frac{15}{7} - \frac{2}{3} + \frac{1}{5} + 1$. Obviously, $\zeta_1 \ne 0, 1$. Then from (10) we have $F \equiv \zeta_1 G$, that is,

$$F - 1 \equiv \zeta_1 G - 1. \tag{13}$$

It follows that 0,1 is a Picard exceptional value of f, g in Ω. In fact, if there exists $z_0 \in \Omega$ such that $f(z_0) = 1$, since $\overline{E}_\Omega(S_1, f) = \overline{E}_\Omega(S_1, g)$, then $g(z_0) = 1$. Thus from (13), we have that $\zeta_1 - 1 = \zeta_1^2 - 1$, which implies $\zeta_1 = 0$ or $\zeta_1 = 1$, a contradition. Similarly, we can get that 0 is a Picard exceptional value of f, g in Ω.

Let $\beta_v (v = 1, 2, \dots, 9)$ be nine distinct roots of equation $\zeta_1 P_1(w) - 1$, obviously, $\beta_v \ne 0, 1$. It is easy to find that $P_1(w) - 1$ have one root 0 with order 5 and four distinct roots, say $\alpha_t (t = 1, 2, 3, 4)$. Thus, we can deduce from (11) that

$$\sum_{v=1}^{9} \widetilde{N}_0(r, \frac{1}{g - \beta_v}) = \widetilde{N}_0(r, \frac{1}{f}) + \sum_{t=1}^{4} \widetilde{N}_0(r, \frac{1}{f - \alpha_t}), \ \ r_0 \le r < +\infty.$$

Since 0 is a Picard exceptional of f in Ω, by applying Theorem 2.4 for above equality, it follows that

$$7T_0(r, g) + S(r, g) \le 4T_0(r, f) + S(r, f), \ \ r_0 \le r < +\infty,$$

which is a contradiction with (11).

Subcase 1.3. Suppose that $K \ne 1$ and $K \ne \zeta_1$. From (10), we have

$$F - K \equiv K(G - 1). \tag{14}$$

It is easy to see that 0 is a Picard exceptional value of f, g in Ω. In fact, if there exists $z_0 \in \Omega$ such that $f(z_0) = 0$, since f, g share 0 IM in Ω, then $F(z_0) = G(z_0) = 1$. Thus, we can deduce from (14) that $1 - K \equiv 0$, a contradiction. Similarly, we can prove that 0 is a Picard exceptional value of g in Ω.

Let $\gamma_v (v = 1, 2, \dots, 9)$ be nine distinct roots of $P_1(w) - K$ in Ω, obviously, $\beta_v \neq 0, 1$. Similar to Subcase 1.2, we have

$$\sum_{v=1}^{9} \widetilde{N}_0(r, \frac{1}{f - \gamma_v}) = \widetilde{N}_0(r, \frac{1}{g}) + \sum_{t=1}^{4} \widetilde{N}_0(r, \frac{1}{g - \alpha_t}), \quad r_0 \le r < +\infty. \tag{15}$$

Since 0 is a Picard exceptional of g in Ω, by applying Theorem 2.4 for above equality, it follows that

$$7T_0(r, f) + S(r, g) \le 4T_0(r, g) + S(r, f), \quad r_0 \le r < +\infty,$$

which is a contradiction with (11).

Case 2. Suppose that there exist a constant $\kappa(\frac{1}{2} \le \kappa < \frac{7}{12})$ and a set $I \subset [r_0, +\infty)$ $(mesI = +\infty)$ such that

$$\widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \le \kappa(T_0(r, f) + T_0(r, g)) + S(r, f) + S(r, g), \tag{16}$$

as $r \to +\infty, r \in I$. Set

$$H = \frac{(\frac{1}{F})''}{(\frac{1}{F})'} - \frac{(\frac{1}{G})''}{(\frac{1}{G})'} = (\frac{F''}{F'} - \frac{2F'}{F}) - (\frac{G''}{G'} - \frac{2G'}{G}). \tag{17}$$

From [21, Lemma 6] we have $m_0(r, H) = S(r, F) + S(r, G) = S(r, f) + S(r, g)$.

Suppose that $H \not\equiv 0$, we know that the pole of H in Ω may occur at the zeros of F', G' in Ω and the poles of F, G in Ω. Then we have

$$\begin{aligned} N_0(r, H) \le & \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + \\ & + \widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}), \quad r_0 \le r < +\infty. \end{aligned} \tag{18}$$

where $\widetilde{N}_0^*(r, \frac{1}{f'})$ is the reduced counting function of those zeros of f' in Ω which are not the zeros of $f(f-1)$ and $\widetilde{N}_0^*(r, \frac{1}{g'})$ is similarly defined. From Lemma 3.4, we have $\widetilde{N}_0^{1)E}(r, \frac{1}{F}) = \widetilde{N}_0^{1)E}(r, \frac{1}{G}) \le N_0(r, \frac{1}{H})$ where $\widetilde{N}_0^{1)E}(r, \frac{1}{F})$ is the counting function of those common zeros of F, G with multiply 1 in Ω. Then it follows from Theorem 2.2 and (18) that

$$\begin{aligned} \widetilde{N}_0^{1)E}(r, \frac{1}{F}) \le & \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + \\ & + \widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) + S(r, f) + S(r, g), \quad r_0 \le r < +\infty. \end{aligned} \tag{19}$$

Let $V = \frac{f'g'}{f(f-1)g(g-1)}$, by Lemma 3.6 we have $m_0(r, V) = S(r, f) + S(r, g)$. Noting that the zeros of f' in Ω which are not the zeros of $f, f-1$ in Ω may be the zeros of V in Ω, and the zeros of g' in Ω which are not the zeros of $g, g-1$ in Ω may also be the zeros of V in Ωthen

$$\widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) \le N_0(r, \frac{1}{V}), \quad r_0 \le r < +\infty. \tag{20}$$

On the other hand, the poles of V in Ω can occur at the zeros of $f, f-1, g$ or $g-1$ in Ω. It follows that

$$N_0(r, V) \le \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0(r, \frac{1}{g}) + \widetilde{N}_0(r, \frac{1}{g-1}), \quad r_0 \le r < +\infty. \tag{21}$$

Since $E_\Omega(S_1, f) = E_\Omega(S_1, g)$, from (20), (21) and Theorem 2.2, we have

$$\widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) \le 2\widetilde{N}_0(r, \frac{1}{f}) + 2\widetilde{N}_0(r, \frac{1}{f-1}) + S(r, f) + S(r, g), \quad r_0 \le r < +\infty. \tag{22}$$

Noting that $\widetilde{N}_0^{[2}(r,\frac{1}{F}) \le N_0^*(r,\frac{1}{f'})$ and $\widetilde{N}_0^{[2}(r,\frac{1}{G}) \le N_0^*(r,\frac{1}{g'})$, then from (19)-(22) we have for $r_0 \le r < +\infty$

$$\begin{aligned}\widetilde{N}_0(r,\frac{1}{F}) &= \widetilde{N}_0^{1)E}(r,\frac{1}{F}) + \widetilde{N}_0^{[2}(r,\frac{1}{F}) + \widetilde{N}_0^{[2}(r,\frac{1}{G}) \le N_0(r,\frac{1}{H}) + \widetilde{N}_0^{[2}(r,\frac{1}{F}) + \widetilde{N}_0^{[2}(r,\frac{1}{G})\\ &\le T_0(r,H) + \widetilde{N}_0^{[2}(r,\frac{1}{F}) + \widetilde{N}_0^{[2}(r,\frac{1}{G}) + O(1)\\ &\le N_0(r,H) + \widetilde{N}_0^{[2}(r,\frac{1}{F}) + \widetilde{N}_0^{[2}(r,\frac{1}{G}) + S(r,f)\\ &\le \widetilde{N}_0(r,f) + \widetilde{N}_0(r,g) + 5\widetilde{N}_0(r,\frac{1}{f}) + 5\widetilde{N}_0(r,\frac{1}{f-1}) + S(r,f) + S(r,g), \end{aligned} \tag{23}$$

where $\widetilde{N}_0^{[2}(r,\frac{1}{F})$ is the reduced counting function of those zeros of F with multiply ≥ 2, and $\widetilde{N}_0^{[2}(r,\frac{1}{G})$ is similarly defined.

Similarly, for $r_0 \le r < +\infty$ we have

$$\widetilde{N}_0(r,\frac{1}{G}) \le \widetilde{N}_0(r,f) + \widetilde{N}_0(r,g) + 5\widetilde{N}_0(r,\frac{1}{g}) + 5\widetilde{N}_0(r,\frac{1}{g-1}) + S(r,f) + S(r,g), \tag{24}$$

as $r_0 \le r < +\infty$. By applying Theorem 2.4 and from (23) and (24), we have

$$\begin{aligned}9\{T_0(r,f) + T_0(r,g)\} &\le \widetilde{N}_0(r,\frac{1}{F}) + \widetilde{N}_0(r,\frac{1}{G}) + 2\{\widetilde{N}_0(r,\frac{1}{f}) + \widetilde{N}_0(r,\frac{1}{f-1})\}\\ &\quad + S(r,f) + S(r,g)\\ &\le 2N_0(r,f) + 2N_0(r,g) + 12\{\widetilde{N}_0(r,\frac{1}{f}) + \widetilde{N}_0(r,\frac{1}{f-1})\}+\\ &\quad + S(r,f) + S(r,g)\\ &\le (2+12\kappa)\{T_0(r,f) + T_0(r,g)\} + S(r,f) + S(r,g),\quad r_0 \le r < +\infty, \end{aligned} \tag{25}$$

which is a contradiction with $\kappa < \frac{7}{12}$ and f, g are transcendental in Ω.

Thus, $H \equiv 0$, i.e.,

$$\frac{F''}{F'} - \frac{2F'}{F} \equiv \frac{G''}{G'} - \frac{2G'}{G}. \tag{26}$$

By integration, we have from (22) that $\frac{1}{F} = \frac{A}{G} + B$ where A, B are constants which are not equal to zero at the same time.

Suppose that $B \ne 0$. Thus, $\frac{1}{F} = \frac{A+BG}{G}$. From Lemma 3.5, we have $T_0(r,f) + S(r,f) = T_0(r,g) + S(r,g)$ for $r_0 \le r < +\infty$. Moreover, it follows from Theorem 2.4 that

$$\widetilde{N}_0(r,f) = \widetilde{N}_0(r,F) = \widetilde{N}_0(r,\frac{1}{G-\frac{A}{B}}) \ge 3T_0(r,g) + S(r,g),\quad r_0 \le r < +\infty,$$

which is a contradiction with f, g are transcendental in Ω.

Suppose that $B \equiv 0$. Then $G = AF$ where A is a non-zero constant. Similarly to the same argument as in Case 1, we can get that $A \equiv 1$. By Lemma 3.8, we can get $f \equiv g$ easily.

From Case 1 and Case 2, we can get the conclusion of Theorem 3.2. □

4 Meromorphic functions in m-punctured plane share three sets IM

In this section, we will investigate the uniqueness of meromorphic functions in m- punctured plane sharing three sets with finite elements IM. The main result of this chapter is showed as follows.

Theorem 4.1. *Let f and g be two transcendental meromorphic functions in Ω, and let $S_1 = \{0, 1\}$, $S_2 = \{\infty\}$, and $S_3 = \{w : P_2(w) = 0\}$, where*

$$P_2(w) = \frac{w^7}{7} - \frac{3w^6}{6} + \frac{3w^5}{5} - \frac{3w^4}{4} + 1.$$

If $\overline{E}_\Omega(S_j, f) = \overline{E}_\Omega(S_j, g)(j = 1, 2, 3)$, then $f(z) \equiv g(z)$.

Corollary 4.2. *There exist three sets S_1, S_2, S_3 with $\sharp S_1 = 2, \sharp S_2 = 1$ and $\sharp S_2 = 7$, such that any two transcendental meromorphic functions f and g must be identical if $\overline{E}_\Omega(S_j, f) = \overline{E}_\Omega(S_j, g)(j = 1, 2, 3)$.*

Proof of Theorem 4.1. Set $F = P_2(f)$ and $G = P_2(g)$. Since $\overline{E}_\Omega(S_j, f) = \overline{E}_\Omega(S_j, g)$ $(j = 1, 2, 3)$, then we have that F, G share $0, 1, \infty$ IM in Ω and $F' = P_2'(f) = f^3(f-1)^3 f'$, $G' = g^3(g-1)^3 g'$. From Lemma 3.6, we have $T_0(r, F) = 7T_0(r, f) + S(r, f)$, $T_0(r, G) = 7T_0(r, g) + S(r, g)$ and $S(r, F) = S(r, f)$, $S(r, G) = S(r, g)$.

Next, the following two cases will be discussed.

Case 1: Suppose that there exist a constant $\lambda(> 0)$ and a set $I \subset [r_0, +\infty)(mes I = +\infty)$ such that

$$\widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \geq \lambda(T_0(r, f) + T_0(r, g)) + S(r, f) + S(r, g), \quad (r \to +\infty, r \in I). \tag{27}$$

Set $U = \frac{F'}{F} - \frac{G'}{G}$, from Theorem 2.4 we have $m_0(r, U) = S(r, F) + S(r, G) = S(r, f) + S(r, g)$. Suppose that $U \not\equiv 0$, since f, g share 0,1,∞ IM in Ω, we can see that the common zeros of f, g is the zero of U in Ω, and the common zeros of $f - 1, g - 1$ is also the zero of U in Ω. Thus, we have

$$3\widetilde{N}_0(r, \frac{1}{f}) + 3\widetilde{N}_0(r, \frac{1}{f-1}) \leq N_0(r, \frac{1}{U}). \tag{28}$$

On the other hand, for the pole of U in Ω we have

$$N_0(r, U) \leq \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + \widetilde{N}_0^{[2}(r, \frac{1}{F}) + \widetilde{N}_0^{[2}(r, \frac{1}{G}). \tag{29}$$

Hence,

$$T_0(r, U) \leq \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + \widetilde{N}_0^{[2}(r, \frac{1}{F}) + \widetilde{N}_0^{[2}(r, \frac{1}{G}) + S(r, f) + S(r, g). \tag{30}$$

Thus, it follows from (27)-(30) that

$$\begin{aligned} 3\widetilde{N}_0(r, \frac{1}{f}) + 3\widetilde{N}_0(r, \frac{1}{f-1}) &\leq \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + \\ &+ \widetilde{N}_0^{[2}(r, \frac{1}{F}) + \widetilde{N}_0^{[2}(r, \frac{1}{G}) + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{31}$$

By adding $\widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G})$ into both sides of (31), and applying Theorem 2.4, we have

$$\begin{aligned} &8\{T_0(r, f) + T_0(r, g)\} + \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \\ &\leq \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + \widetilde{N}_0^{[2}(r, \frac{1}{F}) + \widetilde{N}_0^{[2}(r, \frac{1}{G}) + \widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + \\ &\quad + \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{G}) + S(r, f) + S(r, g) \\ &\leq N_0(r, \frac{1}{F}) + N_0(r, \frac{1}{G}) + N_0(r, f) + N_0(r, g) + S(r, f) + S(r, g) \\ &\leq 8\{T_0(r, f) + T_0(r, g)\} + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{32}$$

Since $\lambda > 0$ and f, g are admissible functions in Ω, we can get a contradiction. Thus, it follows that $U \equiv 0$, by integration, we have

$$F \equiv KG, \tag{33}$$

where K a non-zero constant. By using the same argument as in Case 1 of Theorem 3.2, we can prove that $f \equiv g$.

Case 2. Suppose that there exist a constant $\kappa(0 < \kappa < \frac{1}{2})$ and a set $I \subset [r_0, +\infty)$ $(mesI = +\infty)$ such that

$$\widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \leq \kappa(T_0(r, f) + T_0(r, g)) + S(r, f) + S(r, g), \tag{34}$$

as $r \to +\infty, r \in I$. Set

$$H = \frac{(\frac{1}{F})''}{(\frac{1}{F})'} - \frac{(\frac{1}{G})''}{(\frac{1}{G})'} = (\frac{F''}{F'} - \frac{2F'}{F}) - (\frac{G''}{G'} - \frac{2G'}{G}).$$

From [21, Lemma 6] we have $m_0(r, H) = S(r, F) + S(r, G) = S(r, f) + S(r, g)$.

Suppose that $H \not\equiv 0$. Since F, G share $0, 1, \infty$ IM in Ω, similar to the proof of Theorem 3.2, we have

$$\begin{aligned} \widetilde{N}_0^{1)E}(r, \frac{1}{F}) \leq & \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) + \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + \\ & + \widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{35}$$

By adding $\widetilde{N}_0(r, f) + \widetilde{N}_0(r, g) + \widetilde{N}_0^{[2}(r, \frac{1}{F}) + \widetilde{N}_0^{[2}(r, \frac{1}{G})$ into both sides of (35), and $\widetilde{N}_0^{[2}(r, \frac{1}{F}) \leq N_0^*(r, \frac{1}{f'})$ and $\widetilde{N}_0^{[2}(r, \frac{1}{G}) \leq N_0^*(r, \frac{1}{g'})$, we have

$$\begin{aligned} & \widetilde{N}_0(r, \frac{1}{F}) + \widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1}) \\ & \quad \leq \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) + 2\widetilde{N}_0^*(r, \frac{1}{f'}) + 2\widetilde{N}_0^*(r, \frac{1}{g'}) \\ & \quad + 2\widetilde{N}_0(r, \frac{1}{f}) + 2\widetilde{N}_0(r, \frac{1}{f-1}) + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{36}$$

Let $V = \frac{f'g'}{f(f-1)g(g-1)}$, by Lemma 3.6 we have $m_0(r, V) = S(r, f) + S(r, g)$. Since f, g share ∞ IM in Ω, by using the same discussion as in Case 2 in Theorem 3.2, we have

$$\begin{aligned} & \widetilde{N}_0^*(r, \frac{1}{f'}) + \widetilde{N}_0^*(r, \frac{1}{g'}) + \widetilde{N}_0^{[2}(r, f) + \widetilde{N}_0^{[2}(r, g) \\ & \quad \leq 2\widetilde{N}_0(r, \frac{1}{f}) + 2\widetilde{N}_0(r, \frac{1}{f-1}) + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{37}$$

From (36) and (37), we can deduce by Theorem 2.4 and (34) that

$$\begin{aligned} 7T_0(r, f) & \leq 6\{\widetilde{N}_0(r, \frac{1}{f}) + \widetilde{N}_0(r, \frac{1}{f-1})\} + S(r, f) + S(r, g) \\ & \leq 12\kappa T_0(r, f) + S(r, f) + S(r, g), \quad r_0 \leq r < +\infty. \end{aligned} \tag{38}$$

Since $\kappa < \frac{1}{2}$ and f is a transcendental in Ω, we can get a contradiction from (38) easily. Thus, $H \equiv 0$. By using the same argument as in Case 2 in Theorem 3.2, we can prove that $f \equiv g$. Therefore, we complete the proof of Theorem 4.1 from Case 1 and Case 2. □

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