

Gumrah Uysal*

A new approach to nonlinear singular integral operators depending on three parameters

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Abstract: In this paper, we present some theorems on weighted approximation by two dimensional nonlinear singular integral operators in the following form:

$$T_{\lambda}(f; x, y) = \iint_{\mathbb{R}^2} K_{\lambda}(t-x, s-y, f(t, s)) ds dt, \quad (x, y) \in \mathbb{R}^2, \lambda \in \Lambda,$$

where Λ is a set of non-negative numbers with accumulation point λ_0 .

Keywords: Nonlinear integral operators, Generalized Lebesgue point, Weighted approximation

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Dedicated to Prof. Rašhid Hamidoglu Mamedov.

1 Introduction

Approximation by singular integral operators is one of the oldest topics of approximation theory. Here, the concept of singular integral operator refers to the integral operator whose kernel shows the behaviour of Dirac's δ function (for the properties of δ -function, see [1]). Also, singular integral operators arise from the Fourier analysis of the functions. It is well known that Fourier analysis is one of the most useful tools of many branches of science. Therefore, indicated integral operators have various applications in many academic disciplines such as physics, engineering and medicine. In fact, magnetic resonance imaging, face recognition, differential equation solving and computer aided geometric design are some of the application areas in which the indicated operators are used. For mentioned applications, we refer the reader to [2, 3].

The convergence of various type linear integral operators have been examined at characteristic points such as continuity point, μ -generalized Lebesgue point, and so on, by many researchers throughout years: one parameter family of singular integral operators [4, 5], a sequence of singular integral operators with general Poisson type kernels [6], a family of singular integral operators depending on two parameters [7-9], a sequence of Gegenbauer singular integrals [10] and a sequence of m -singular integral operators [11]. One may consider the pointwise approximation of singular integral operators in weighted Lebesgue spaces as well as usual Lebesgue spaces. Therefore, for some advanced studies concerning weighted pointwise approximation by singular integral operators, we refer the reader to [11-13].

*Corresponding Author: Gumrah Uysal: Department of Computer Technologies, Division of Technology of Information Security, Karabuk University, 78050, Karabuk, Turkey, E-mail: guysal@karabuk.edu.tr

Musiela [14] studied the convergence of convolution type nonlinear integral operators in the following form:

$$T_\alpha f(y) = \int_G K_\alpha(x-y, f(x)) dx, \quad y \in G, \quad \alpha \in \Lambda, \quad (1)$$

where G is a locally compact Abelian group equipped with Haar measure and $\Lambda \neq \emptyset$ is an index set with any topology, and he extended the concept of the singularity condition via replacing the linearity property of the integral operators by an assumption of Lipschitz condition for K_α with respect to second variable. Therefore, traditional solution technics became applicable to nonlinear problems by the aid of indicated Lipschitz condition. In [15], Musielak advanced his previous analysis [14] by obtaining significant results for generalized Orlicz spaces. After this important study, Swiderski and Wachnicki [16] investigated the pointwise convergence of the operators of type (1) at p -Lebesgue points of functions $f \in L_p(-\pi, \pi)$ ($1 \leq p < \infty$). For further results concerning the convergence of several types of nonlinear singular integral operators in different function spaces, the studies [17, 18] are strongly recommended.

In [19], Taberski studied the pointwise approximation of functions $f \in L_1(\mathbb{R})$ by convolution type two dimensional integral operators in the following form:

$$V_\lambda(f; x, y) = \iint_R f(t, s) K_\lambda(t-x, s-y) ds dt, \quad (x, y) \in R, \quad (2)$$

where R denotes a given rectangle and $K_\lambda(t, s)$ denotes a kernel satisfying suitable conditions with $\lambda \in \Lambda$, where Λ is a given set of non-negative numbers with accumulation point λ_0 . The earlier results concerning the operators of type (2) were obtained by Gahariya [20], i.e., the indicated operators were handled as a sequence of double integral operators in this work. The studies [21-23], which are based on Taberski's study [19], are devoted to the study of pointwise convergence of the operators of type (2) on some planar sets consisting of characteristic points (x_0, y_0) of various types. Later, Musielak [24] investigated the conditions under which the two dimensional counterparts of the operators of type (2) are (σ, I_φ) -conservative, where σ is a modular defined on the space of functions which are Lebesgue measurable on arbitrary closed and bounded subset of \mathbb{R}^2 , and I_φ is a Musielak-Orlicz modular. Recently, Karsli [25] obtained the convergence of convolution type linear singular integral operators depending on three parameters at μ -generalized Lebesgue points of the integrable functions. For some studies concerning double singular integral operators in several settings, we refer the reader to [26-30]. On the other hand, for some other important works related to approximation by linear and nonlinear operators in several function spaces, we refer the reader to [31-37].

Let $f \in L_p^\varphi(\mathbb{R}^2)$, where $L_p^\varphi(\mathbb{R}^2)$ ($1 \leq p < \infty$) is the space of all measurable functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ for which $\left| \frac{f}{\varphi} \right|^p$ is integrable on \mathbb{R}^2 . Here, $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ is a weight function satisfying suitable conditions. The norm formula for the space $L_p^\varphi(\mathbb{R}^2)$ (see, e.g., [11, 13]) is given by

$$\|f\|_{L_p^\varphi(\mathbb{R}^2)} = \left(\iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} \right|^p ds dt \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

The main aim of this paper is to investigate both the weighted pointwise convergence and the rate of weighted pointwise convergence of nonlinear double singular integral operators of the form as such:

$$T_\lambda(f; x, y) = \iint_{\mathbb{R}^2} K_\lambda(t-x, s-y, f(t, s)) ds dt, \quad (x, y) \in \mathbb{R}^2, \quad \lambda \in \Lambda, \quad (3)$$

where Λ is a set of non-negative numbers with accumulation point λ_0 .

The paper is organized as follows: In Section 2, we give some preliminary concepts. In Section 3, main result is presented. In Section 4, the rate of pointwise convergence of the operators of type (3) is established.

2 Preliminaries

In this section, basic concepts used in this paper are introduced.

Definition 2.1. Let $1 \leq p < \infty$, and $\delta_0, \delta_1 \in \mathbb{R}^+$ be fixed numbers. A point $(x_0, y_0) \in \mathbb{R}^2$ at which the following relations

$$\lim_{h \rightarrow 0} \left(\frac{1}{\mu_1(h)} \int_{x_0}^{x_0+h} |g(t, y_0) - g(x_0, y_0)|^p dt \right)^{\frac{1}{p}} = 0, \tag{4}$$

and

$$\lim_{k \rightarrow 0} \left(\frac{1}{\mu_2(k)} \int_{y_0}^{y_0+k} |g(t, s) - g(t, y_0)|^p ds \right)^{\frac{1}{p}} = 0, \tag{5}$$

hold uniformly with respect to almost every $t \in \mathbb{R}$ is called a $\mu - p$ -generalized Lebesgue point of locally p -integrable function (i.e., a function whose $p - th$ power is locally integrable) $g : \mathbb{R}^2 \rightarrow \mathbb{R}$. Here, $\mu_1 : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and absolutely continuous on $0 < h \leq \delta_0$ and $\mu_1(0) = 0$ and also, $\mu_2 : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and absolutely continuous on $0 < k \leq \delta_1$ and $\mu_2(0) = 0$.

Remark 2.2. Basically, Definition 2.1 is obtained by combining the characterization of the function $\mu(t)$ presented by Gadjiev [9] with the definition of d -point given by Siudut [21]. Also, some different modifications are done according to our problem’s needs, such as predisposing the definition to L_p^φ space. On the other hand, for some other μ -generalized Lebesgue point definitions, we refer the reader to [8, 25] and [28].

Definition 2.3. Let λ_0 be an accumulation point of the non-negative set of numbers Λ or $\lambda_0 = \infty$, and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ be a locally bounded weight function such that the following inequality

$$\varphi(t + x, s + y) \leq \varphi(t, s)\varphi(x, y) \tag{6}$$

holds for every $(t, s) \in \mathbb{R}^2$ and $(x, y) \in \mathbb{R}^2$.

A family $(K_\lambda)_{\lambda \in \Lambda}$ consisting of the functions $K_\lambda : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is called class \mathcal{A} , if the following conditions hold:

- (a) $K_\lambda(t, s, 0) = 0$ for every $(t, s) \in \mathbb{R}^2$ and for each $\lambda \in \Lambda$, and $K_\lambda(\cdot, \cdot, u) \in L_1(\mathbb{R}^2)$ for every $u \in \mathbb{R}$ and for each $\lambda \in \Lambda$.
- (b) There exists a family $(L_\lambda)_{\lambda \in \Lambda}$ consisting of the (globally) integrable functions $L_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that the Lipschitz inequality given by

$$|K_\lambda(t, s, u) - K_\lambda(t, s, v)| \leq L_\lambda(t, s)|u - v|$$

holds for every $(t, s) \in \mathbb{R}^2, u, v \in \mathbb{R}$, and for each fixed $\lambda \in \Lambda$.

- (c) $\lim_{(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)} \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{u}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - u \right| = 0$ for every $u \in \mathbb{R}$ and for any $(x_0, y_0) \in \mathbb{R}^2$.
- (d) For every $\xi > 0, \lim_{\lambda \rightarrow \lambda_0} \left[\sup_{\xi \leq \sqrt{t^2 + s^2}} [\varphi(t, s)L_\lambda(t, s)] \right] = 0$.
- (e) For every $\xi > 0, \lim_{\lambda \rightarrow \lambda_0} \left[\iint_{\xi \leq \sqrt{t^2 + s^2}} \varphi(t, s)L_\lambda(t, s) ds dt \right] = 0$.
- (f) $\|\varphi L_\lambda\|_{L_1(\mathbb{R}^2)} \leq M < \infty$ for every $\lambda \in \Lambda$.
- (g) $L_\lambda(t, s)$ is non-increasing on $[0, \infty)$ and non-decreasing on $(-\infty, 0]$ for each fixed $\lambda \in \Lambda$ as a function of t . Similarly, $L_\lambda(t, s)$ is non-increasing on $[0, \infty)$ and non-decreasing on $(-\infty, 0]$ for each fixed $\lambda \in \Lambda$ as a function of s .

Throughout this paper the kernel function K_λ belongs to class \mathcal{A} .

Remark 2.4. The studies [11, 13, 16, 21] and [18], among others, are used as main reference works in the construction stage of class \mathcal{A} . Therefore, we refer the reader to see the indicated works. On the other hand, we recommend the reader to compare the usage of the inequality (6) used in [11] with the current study. Also, for the Lipschitz inequality included in the definition class \mathcal{A} , we refer the reader to see the works [14, 18].

Remark 2.5. Existence of the operators of type (3) is guaranteed by the conditions of class \mathcal{A} , that is $T_\lambda (f; x, y) \in L_p^\varphi (\mathbb{R}^2)$ whenever $f \in L_p^\varphi (\mathbb{R}^2)$.

Example 2.6. A first example is the linear kernel. Let Λ be a set of non-negative numbers such that $\Lambda = (0, \infty)$ with accumulation point $\lambda_0 = 0$. Now, the definition of the function $K_\lambda : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is as follows:

$$K_\lambda (t, s, u) = \frac{u}{4\pi\lambda} e^{-\frac{(t^2+s^2)}{4\lambda}}, \quad u \in \mathbb{R}.$$

Since

$$\begin{aligned} |K_\lambda (t, s, u) - K_\lambda (t, s, v)| &= \frac{1}{4\pi\lambda} e^{-\frac{(t^2+s^2)}{4\lambda}} |u - v| \\ &= L_\lambda (t, s) |u - v|, \end{aligned}$$

one may easily observe that given function K_λ belongs to class \mathcal{A} . For detailed analysis of the function $L_\lambda (t, s)$, we recommend the reader to see [21].

Example 2.7. Define the kernel function such that

$$K_\lambda (t, s, u) = \begin{cases} \frac{\lambda u}{2} + \sin \frac{\lambda u}{2}, & \text{if } (t, s) \in \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right] \times \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right], \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right] \times \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right], \end{cases}$$

where $\lambda \in \mathbb{N}$ and $\lambda_0 = \infty$. This kernel is the two dimensional analogue of the kernel given in [16].

It is easy to see that the conditions of class \mathcal{A} are satisfied. Observe that one may take the desired linear kernel as

$$L_\lambda (t, s) = \begin{cases} \lambda, & \text{if } (t, s) \in \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right] \times \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right], \\ 0, & \text{if } (t, s) \in \mathbb{R}^2 \setminus \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right] \times \left[\frac{-1}{\sqrt{2\lambda}}, \frac{1}{\sqrt{2\lambda}} \right]. \end{cases}$$

Example 2.8. The appropriate weight functions, which are defined on \mathbb{R}^2 , may be given by $\varphi_1 (t, s) = e^{t+s}$ and $\varphi_2 (t, s) = (1 + |t|) (1 + |s|)$.

3 Convergence at characteristic points

Theorem 3.1. If $(x_0, y_0) \in \mathbb{R}^2$ is a common μ -generalized Lebesgue point of the functions $f \in L_p^\varphi (\mathbb{R}^2)$ ($1 \leq p < \infty$) and φ , then

$$\lim_{(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)} T_\lambda (f; x, y) = f (x_0, y_0),$$

on any set Z consisting of the points (x, y, λ) on which the functions

$$\int_{y_0-\delta}^{y_0+\delta} \int_{x_0-\delta}^{x_0+\delta} L_\lambda (t-x, s-y) |\{\mu_1 (|t-x_0|)\}'_t| dt ds + 2\mu_1 (|x-x_0|) \int_{y_0-\delta}^{y_0+\delta} L_\lambda (0, s-y) ds, \tag{7}$$

and

$$\int_{x_0-\delta}^{x_0+\delta} \int_{y_0-\delta}^{y_0+\delta} L_\lambda (t-x, s-y) |\{\mu_2 (|s-y_0|)\}'_s| ds dt + 2\mu_2 (|y-y_0|) \int_{x_0-\delta}^{x_0+\delta} L_\lambda (t-x, 0) dt, \tag{8}$$

where $0 < \delta < \min \{\delta_0, \delta_1\}$, are bounded as (x, y, λ) tends to (x_0, y_0, λ_0) .

Proof. Let $0 < |x_0 - x| < \frac{\delta}{2}$ and $0 < |y_0 - y| < \frac{\delta}{2}$. Further, let $0 < x_0 - x < \frac{\delta}{2}$ and $0 < y_0 - y < \frac{\delta}{2}$, and (x_0, y_0) be a common μ -generalized Lebesgue point of the functions $f \in L_p^\varphi(\mathbb{R}^2)$ ($1 \leq p < \infty$) and φ .

The proof of theorem will be given for the case $1 < p < \infty$. The proof for the case $p = 1$ is similar.

Now, set $I(x, y, \lambda) = |T_\lambda(f; x, y) - f(x_0, y_0)|$. Using condition (c), we obtain

$$\begin{aligned}
 I(x, y, \lambda) &= \left| \iint_{\mathbb{R}^2} K_\lambda(t-x, s-y, f(t, s)) dsdt - f(x_0, y_0) \right| \\
 &= \left| \iint_{\mathbb{R}^2} K_\lambda(t-x, s-y, f(t, s)) dsdt - \iint_{\mathbb{R}^2} K_\lambda\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt \right. \\
 &\quad \left. + \iint_{\mathbb{R}^2} K_\lambda\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|.
 \end{aligned}$$

Using condition (b), it is easy to see that the following inequality holds:

$$\begin{aligned}
 I(x, y, \lambda) &\leq \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) L_\lambda(t-x, s-y) dsdt \\
 &\quad + \left| \iint_{\mathbb{R}^2} K_\lambda\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|.
 \end{aligned}$$

Since whenever m, n being positive numbers the inequality $(m + n)^p \leq 2^p(m^p + n^p)$ holds (see, e.g., [38]), we have

$$\begin{aligned}
 [I(x, y, \lambda)]^p &\leq 2^p \left(\iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right| \varphi(t, s) L_\lambda(t-x, s-y) dsdt \right)^p \\
 &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\lambda\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|^p.
 \end{aligned}$$

Now, applying Hölder’s inequality (see, e.g., [38]) to the first integral of the resulting inequality, we have

$$\begin{aligned}
 [I(x, y, \lambda)]^p &\leq 2^p \beta(x, y, \lambda) \iint_{\mathbb{R}^2} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t-x, s-y) dsdt \\
 &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\lambda\left(t-x, s-y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)}\varphi(t, s)\right) dsdt - f(x_0, y_0) \right|^p,
 \end{aligned}$$

where

$$\beta(x, y, \lambda) = \left(\iint_{\mathbb{R}^2} \varphi(t, s) L_\lambda(t-x, s-y) dsdt \right)^{\frac{p}{q}}.$$

Moreover, the following inequality holds:

$$[I(x, y, \lambda)]^p \leq 2^p \beta(x, y, \lambda) \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t-x, s-y) dsdt$$

$$\begin{aligned}
 &+ 2^p \beta(x, y, \lambda) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &+ 2^p \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - f(x_0, y_0) \right|^p,
 \end{aligned}$$

where $B_\delta := \{(t, s) \in \mathbb{R}^2 : (t - x_0)^2 + (s - y_0)^2 < \delta^2\}$.

Now, applying the inequality given by $(m + n)^p \leq 2^p(m^p + n^p)$ once more to the right hand side of the resulting inequality, we obtain

$$\begin{aligned}
 [I(x, y, \lambda)]^p &\leq 2^{2p} \beta(x, y, \lambda) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\mathbb{R}^2 \setminus B_\delta} \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &\quad + 2^{2p} \beta(x, y, \lambda) \iint_{\mathbb{R}^2 \setminus B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &\quad + 2^p \beta(x, y, \lambda) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - f(x_0, y_0) \right|^p.
 \end{aligned}$$

Recalling the initial assumptions $0 < |x_0 - x| < \frac{\delta}{2}$ and $0 < |y_0 - y| < \frac{\delta}{2}$, we may define the following set as follows:

$$N_\delta = \left\{ (x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < \frac{\delta^2}{2} \right\}.$$

Comparing geometric representations of the sets B_δ and N_δ gives the inclusion relation such that $\mathbb{R}^2 \setminus B_\delta \subseteq \mathbb{R}^2 \setminus A_\delta$, where

$$A_\delta = \left\{ (t, s) \in B_\delta : (t - x)^2 + (s - y)^2 < \frac{\delta^2}{2}, (x, y) \in N_\delta \right\}.$$

In the light of these relations, we may write

$$\begin{aligned}
 [I(x, y, \lambda)]^p &\leq 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\mathbb{R}^2 \setminus A_\delta} \varphi(t - x, s - y) L_\lambda(t - x, s - y) ds dt \\
 &\quad + 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \sup_{(t, s) \in \mathbb{R}^2 \setminus A_\delta} [\varphi(t - x, s - y) L_\lambda(t - x, s - y)] \|f\|_{L^p_\varphi(\mathbb{R}^2)}^p \\
 &\quad + 2^p \beta(x, y, \lambda) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - f(x_0, y_0) \right|^p.
 \end{aligned}$$

Rearranging and rewriting the last inequality, we obtain

$$[I(x, y, \lambda)]^p \leq 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\sqrt{u^2 + v^2} \geq \frac{\delta^2}{2}} \varphi(u, v) L_\lambda(u, v) dv du$$

$$\begin{aligned}
 &+ 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \sup_{\sqrt{u^2+v^2} \geq \frac{\delta^2}{2}} [\varphi(u, v) L_\lambda(u, v)] \|f\|_{L^p_\varphi(\mathbb{R}^2)}^p \\
 &+ 2^p \beta(x, y, \lambda) \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &+ 2^p \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) ds dt - f(x_0, y_0) \right|^p \\
 &= I_1(x, y, \lambda) + I_2(x, y, \lambda) + 2^p \beta(x, y, \lambda) I_3(x, y, \lambda) + I_4(x, y, \lambda).
 \end{aligned}$$

Boundedness of the term $\varphi(x, y) \beta(x, y, \lambda)$ as (x, y, λ) tends to (x_0, y_0, λ_0) , which follows from condition (f). On the other hand, $I_4(x, y, \lambda) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ by condition (c). Lastly, $I_1(x, y, \lambda) \rightarrow 0$ and $I_2(x, y, \lambda) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$ by conditions (d) and (e), respectively.

Now, we may write the following inequality for the integral $I_3(x, y, \lambda)$

$$\begin{aligned}
 I_3(x, y, \lambda) &= \iint_{B_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \varphi(t, s) L_\lambda(t - x, s - y) ds dt \\
 &\leq \sup_{(t,s) \in Q_\delta} \varphi(t, s) \iint_{Q_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\lambda(t - x, s - y) ds dt \\
 &= \sup_{(t,s) \in Q_\delta} \varphi(t, s) I_{31}(x, y, \lambda),
 \end{aligned}$$

where $Q_\delta = (x_0 - \delta, x_0 + \delta) \times (y_0 - \delta, y_0 + \delta)$.

Observe that $I_{31}(x, y, \lambda)$ may be written in the following form:

$$\begin{aligned}
 I_{31}(x, y, \lambda) &= \iint_{Q_\delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\lambda(t - x, s - y) ds dt \\
 &= \int_{x_0 - \delta}^{x_0 + \delta} \int_{y_0 - \delta}^{y_0 + \delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} + \frac{f(t, y_0)}{\varphi(t, y_0)} - \frac{f(t, y_0)}{\varphi(t, y_0)} \right|^p \\
 &\quad \times L_\lambda(t - x, s - y) ds dt.
 \end{aligned}$$

It is easy to see that the following inequality holds:

$$\begin{aligned}
 I_{31}(x, y, \lambda) &\leq 2^p \int_{x_0 - \delta}^{x_0 + \delta} \left[\int_{y_0 - \delta}^{y_0 + \delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(t, y_0)}{\varphi(t, y_0)} \right|^p L_\lambda(t - x, s - y) ds \right] dt \\
 &\quad + 2^p \int_{y_0 - \delta}^{y_0 + \delta} \left[\int_{x_0 - \delta}^{x_0 + \delta} \left| \frac{f(t, y_0)}{\varphi(t, y_0)} - \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p L_\lambda(t - x, s - y) dt \right] ds \\
 &= 2^p \left\{ \int_{x_0 - \delta}^{x_0 + \delta} i_\lambda(x, y, t) dt + \int_{y_0 - \delta}^{y_0 + \delta} j_\lambda(x, y, s) ds \right\}.
 \end{aligned}$$

Now, we pass to the integral $i_\lambda(x, y, t)$. This integral may be written in the following form:

$$i_\lambda(x, y, t) = \int_{y_0 - \delta}^{y_0 + \delta} \left| \frac{f(t, s)}{\varphi(t, s)} - \frac{f(t, y_0)}{\varphi(t, y_0)} \right|^p L_\lambda(t - x, s - y) ds$$

$$\begin{aligned}
 &= \left\{ \int_{y_0-\delta}^{y_0} + \int_{y_0}^{y_0+\delta} \right\} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(t,y_0)}{\varphi(t,y_0)} \right|^p L_\lambda(t-x,s-y) ds \\
 &= i_\lambda^1(x,y,t) + i_\lambda^2(x,y,t).
 \end{aligned}$$

Now, we consider the integral $i_\lambda^1(x,y,t)$. From relation (5), for every $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that the expression

$$\int_{y_0-k}^{y_0} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(t,y_0)}{\varphi(t,y_0)} \right|^p ds < \varepsilon^p \mu_2(k) \tag{9}$$

holds for every $0 < k \leq \delta < \min\{\delta_0, \delta_1\}$.

Define the function $F(t,s)$ by

$$F(t,s) = \int_s^{y_0} \left| \frac{f(t,w)}{\varphi(t,w)} - \frac{f(t,y_0)}{\varphi(t,y_0)} \right|^p dw. \tag{10}$$

From (10), we have

$$d_s F(t,s) = - \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(t,y_0)}{\varphi(t,y_0)} \right|^p ds. \tag{11}$$

From (9) and (10), for every s satisfying $0 < y_0 - s \leq \delta < \min\{\delta_0, \delta_1\}$, we have

$$|F(t,s)| \leq \varepsilon^p \mu_2(y_0 - s) \tag{12}$$

for any fixed $t \in \mathbb{R}$. By virtue of (10) and (11), we have

$$\begin{aligned}
 i_\lambda^1(x,y,t) &= (L) \int_{y_0-\delta}^{y_0} \left| \frac{f(t,s)}{\varphi(t,s)} - \frac{f(t,y_0)}{\varphi(t,y_0)} \right|^p L_\lambda(t-x,s-y) ds \\
 &= (LS) \int_{y_0-\delta}^{y_0} L_\lambda(t-x,s-y) d_s [-F(t,s)],
 \end{aligned}$$

where (LS) denotes Lebesgue-Stieltjes integral.

Using integration by parts and applying (12), we have the following inequality:

$$\begin{aligned}
 |i_\lambda^1(x,y,t)| &\leq \varepsilon^p \mu_2(\delta) L_\lambda(t-x,y_0-\delta-y) \\
 &\quad + \varepsilon^p \int_{y_0-\delta}^{y_0} \mu_2(y_0-s) |d_s L_\lambda(t-x,s-y)|.
 \end{aligned}$$

It is easy to see that (for the similar situation, see [7, 9]) the following inequality holds:

$$\begin{aligned}
 |i_\lambda^1(x,y,t)| &\leq \varepsilon^p \mu_2(\delta) L_\lambda(t-x,y_0-\delta-y) \\
 &\quad + \varepsilon^p \int_{y_0-y-\delta}^{y_0-y} \mu_2(y_0-s-y) \left| d_s \left[\bigvee_{y_0-y-\delta}^s L_\lambda(t-x,u) \right] \right|.
 \end{aligned}$$

Applying integration by parts to the right hand side of the last inequality, we have the following expression:

$$|i_\lambda^1(x,y,t)| \leq -\varepsilon^p \int_{y_0-y-\delta}^{y_0-y} \left[\bigvee_{y_0-y-\delta}^s L_\lambda(t-x,u) \right] \{\mu_2(y_0-s-y)\}'_s ds.$$

Remaining variational operations are evaluated using condition (g) . Hence, we obtain

$$|i_{\lambda}^1(x, y, t)| \leq \varepsilon^p \int_{y_0-\delta}^{y_0} L_{\lambda}(t-x, s-y) |\{\mu_2(y_0-s)\}'_s| ds + 2L_{\lambda}(t-x, 0) \mu_2(y_0-y). \tag{13}$$

Using preceding method, we can estimate the integral $i_{\lambda}^2(x, y, t)$ as follows:

$$|i_{\lambda}^2(x, y, t)| \leq \varepsilon^p \int_{y_0}^{y_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_2(s-y_0)\}'_s| ds. \tag{14}$$

Combining (13) and (14), we obtain

$$|i_{\lambda}(x, y, t)| \leq \varepsilon^p \int_{y_0-\delta}^{y_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_2(|s-y_0|)\}'_t| ds + 2\mu_2(|y-y_0|) L_{\lambda}(t-x, 0).$$

Similar calculations for the integral $j_{\lambda}(x, y, s)$ yield

$$|j_{\lambda}(x, y, s)| \leq \varepsilon^p \int_{x_0-\delta}^{x_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_1(|t-x_0|)\}'_t| dt + 2\mu_1(|x-x_0|) L_{\lambda}(0, s-y).$$

Thus, we have

$$I_{31}(x, y, \lambda) \leq \varepsilon^p 2^p \int_{x_0-\delta}^{x_0+\delta} \left[\int_{y_0-\delta}^{y_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_2(|s-y_0|)\}'_t| ds + 2\mu_2(|y-y_0|) L_{\lambda}(t-x, 0) \right] dt + \varepsilon^p 2^p \int_{y_0-\delta}^{y_0+\delta} \left[\int_{x_0-\delta}^{x_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_1(|t-x_0|)\}'_t| dt + 2\mu_1(|x-x_0|) L_{\lambda}(0, s-y) \right] ds.$$

Since ε is arbitrary and L_{λ} is integrable with respect to each variable, the desired result follows from hypotheses (8) and (7), i.e., $I_3(x, y, \lambda) \rightarrow 0$ as $(x, y, \lambda) \rightarrow (x_0, y_0, \lambda_0)$. Note that the same conclusion is obtained for the case $0 < x-x_0 < \frac{\delta}{2}$ and $0 < y-y_0 < \frac{\delta}{2}$. Thus the proof is completed. □

4 Rate of pointwise convergence

Theorem 4.1. *Suppose that the hypotheses of Theorem 3.1 are satisfied. Let*

$$\Delta(x, y, \lambda, \delta) = \int_{x_0-\delta}^{x_0+\delta} \Delta_1(x, y, \lambda, \delta, t) dt + \int_{y_0-\delta}^{y_0+\delta} \Delta_2(x, y, \lambda, \delta, s) ds,$$

where $0 < \delta < \min\{\delta_0, \delta_1\}$,

$$\Delta_1(x, y, \lambda, \delta, t) = \int_{y_0-\delta}^{y_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_2(|s-y_0|)\}'_t| ds + 2\mu_2(|y-y_0|) L_{\lambda}(t-x, 0),$$

and

$$\Delta_2(x, y, \lambda, \delta, s) = \int_{x_0-\delta}^{x_0+\delta} L_{\lambda}(t-x, s-y) |\{\mu_1(|t-x_0|)\}'_t| dt + 2\mu_1(|x-x_0|) L_{\lambda}(0, s-y),$$

and the following conditions are satisfied:

- (i) $\Delta(x, y, \lambda, \delta) \rightarrow 0$ as (x, y, λ) tends to (x_0, y_0, λ_0) for some $\delta > 0$.
- (ii) For every $\xi > 0$, we have $\sup_{\xi \leq \sqrt{t^2+s^2}} [\varphi(t, s)L_\lambda(t, s)] = o(\Delta(x, y, \lambda, \delta))$ as (x, y, λ) tends to (x_0, y_0, λ_0) .
- (iii) For every $u \in \mathbb{R}$, we have $\left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{u}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - u \right| = o(\Delta(x, y, \lambda, \delta))$ as (x, y, λ) tends to (x_0, y_0, λ_0) .
- (iv) For every $\xi > 0$, we have $\iint_{\xi \leq \sqrt{t^2+s^2}} \varphi(t, s)L_\lambda(t, s) dsdt = o(\Delta(x, y, \lambda, \delta))$ as (x, y, λ) tends to (x_0, y_0, λ_0) .

Then, at each common μ -generalized Lebesgue point of the functions $f \in L^{\varphi}_p(\mathbb{R}^2)$ ($1 \leq p < \infty$) and φ we have

$$|T_\lambda(f; x, y) - f(x_0, y_0)| = o\left(\Delta(x, y, \lambda, \delta)^{\frac{1}{p}}\right),$$

as (x, y, λ) tends to (x_0, y_0, λ_0) .

Proof. By the hypotheses of Theorem 3.1, we may write

$$\begin{aligned} |T_\lambda(f; x, y) - f(x_0, y_0)|^p &\leq 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \left| \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \right|^p \iint_{\sqrt{u^2+v^2} \geq \frac{\delta^2}{2}} \varphi(u, v) L_\lambda(u, v) dvdu \\ &\quad + 2^{2p} \varphi(x, y) \beta(x, y, \lambda) \sup_{\sqrt{u^2+v^2} \geq \frac{\delta^2}{2}} [\varphi(u, v) L_\lambda(u, v)] \|f\|_{L^{\varphi}_p(\mathbb{R}^2)}^p \\ &\quad + 2^p \varepsilon^p \beta(x, y, \lambda) \sup_{(t,s) \in Q_\delta} \varphi(t, s) \int_{x_0-\delta}^{x_0+\delta} \Delta_1(x, y, \lambda, \delta, t) dt \\ &\quad + 2^p \varepsilon^p \beta(x, y, \lambda) \sup_{(t,s) \in Q_\delta} \varphi(t, s) \int_{y_0-\delta}^{y_0+\delta} \Delta_2(x, y, \lambda, \delta, s) ds \\ &\quad + 2^p \left| \iint_{\mathbb{R}^2} K_\lambda \left(t - x, s - y, \frac{f(x_0, y_0)}{\varphi(x_0, y_0)} \varphi(t, s) \right) dsdt - f(x_0, y_0) \right|^p. \end{aligned}$$

From (i) – (iv), and using class \mathcal{A} conditions, the assertion follows. Thus, the proof is completed. □

5 Conclusion

In this paper, the pointwise convergence of the convolution type nonlinear double singular integral operators depending on three parameters is investigated. In this work, we proved the theorems by using a specific weighted pointwise convergence method. Therefore, the main result is presented as Theorem 3.1. Also, by using main result, the rate of pointwise convergence of the indicated type operators is computed.

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