Open Math. 2016; 14: 832–840 **DE GRUYTER** OPEN

Open Mathematics **Open Access**

Research Article

Zhibin Du and Carlos M. da Fonseca*

The Bruhat rank of a binary symmetric staircase pattern

DOI 10.1515/math-2016-0079
Received February 29, 2016; accepted October 10, 2016.

Abstract: In this work we show that the Bruhat rank of a symmetric (0, 1)-matrix of order n with a staircase pattern, total support, and containing I_n, is at most 2. Several other related questions are also discussed. Some illustrative examples are presented.

Keywords: Permutation matrix, Bruhat order, Bruhat shadow, Bruhat rank, Inversion

MSC: 05B20, 06A07, 15A36

1 The Bruhat shadow

Setting

$$L_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the standard inversion-reducing interchange process applied to a permutation matrix P, replaces a 2×2 submatrix equals to L_2 by I_2, for short,

$$L_2 \to I_2 .$$

Given two permutation matrices P and Q of the same order, Q is below P in the *Bruhat order* and written as $Q \preccurlyeq_B P$, if Q can be obtained from P by a sequence of $L_2 \to I_2$ interchanges. The Bruhat order in terms of permutation matrices has attracted considerable attentions recently [2–6].

If S_n denotes the set of all permutation matrices of order n, a nonempty subset $\mathcal{I}$ of S_n is called a *Bruhat ideal* if $P \in \mathcal{I}$ and $Q \preccurlyeq_B P$ imply that $Q \in \mathcal{I}$. A *principal Bruhat ideal* $\langle P \rangle$ is an ideal generated by a single permutation matrix P. Denoting the Boolean sum of two (0, 1)-matrices A and B by $A +_b B$, the *Bruhat shadow* of $\mathcal{I}$ is the matrix $\mathcal{S}(\mathcal{I}) = +_b\{Q \in \mathcal{I}\}$.

As an example, setting

$$P = \begin{pmatrix} 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \tag{1}$$

Zhibin Du: School of Mathematics and Statistics, Zhaoqing University, Zhaoqing 526061, Guangdong, China,
E-mail: zhibindu@126.com
***Corresponding Author: Carlos M. da Fonseca:** Department of Mathematics, Kuwait University, Safat 13060, Kuwait,
E-mail: carlos@sci.kuniv.edu.kw

we have

$$\mathcal{S}(\langle P, Q \rangle) = \begin{pmatrix} 1&1&1&0&0&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&1&0 \\ 0&1&1&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&0&0&1&1 \end{pmatrix}.$$

If $\mathcal{I} = \langle P \rangle$, then we simply write $\mathcal{S}(P)$: the Bruhat shadow of P.

Observe that $\mathcal{S}(\langle P, Q \rangle)$ has a *staircase pattern* with $I_6, P, Q \leqslant \mathcal{S}(\langle P, Q \rangle)$ (see [2, Theorem 2.4]). We recall that a $(0,1)$-matrix $A = (a_{ij})$ has a staircase pattern provided the 1's in each row and each column occur consecutively and the leftmost (resp., rightmost) 1 in a row occurs above or to the left of the leftmost (resp., rightmost) 1 in the next row. Here, without loss of generality, we confine our attention to indecomposable staircase patterns, i.e., our matrices cannot be represented as the direct sum of two matrices.

Suppose now that we have a set of indices $1 = i_1 < i_2 < \cdots < i_p \leqslant n$, such that

$$r_{i_1} = r_{i_1+1} = \cdots = r_{i_2-1} < r_{i_2} = r_{i_2+1} = \cdots = r_{i_3-1} < \cdots < r_{i_p} = r_{i_p+1} = \cdots = r_n \tag{2}$$

is a sequence with integers in the set $\{1, \ldots, n\}$, and $r_i \geqslant i$. This sequence $r = r_1, \ldots, r_n$ is called a *right-sequence* or, for short, r-*sequence*. Analogously, for a given set of indices $1 \leqslant j_1 < j_2 < \cdots < j_q = n$, the sequence $\ell = \ell_1, \ldots, \ell_n$ with integers in the set $\{1, \ldots, n\}$, and $\ell_i \leqslant i$, such that

$$\ell_1 = \ell_2 = \cdots = \ell_{j_1} < \ell_{j_1+1} = \cdots = \ell_{j_2} < \cdots < \ell_{j_{q-1}+1} = \cdots = \ell_{j_q} \tag{3}$$

is called a *left-sequence* or, briefly, ℓ-*sequence*.

Remark 1.1. *We interchange the roles of r and ℓ in [2] since we believe this is a more natural convention.*

When $a_{ij} = 1$, for $1 \leqslant j \leqslant r_i$, where $r_1 \leqslant \cdots \leqslant r_n = n$, and $a_{ij} = 0$, otherwise, we call $A = (a_{ij})$ a *full staircase pattern*.

For any $n \times n$ permutation matrix P, corresponding to a permutation σ of the set $\{1, \ldots, n\}$, we may associate the permutation $\sigma = (\sigma(1) \cdots \sigma(k) \cdots \sigma(n))$ with an ℓ- and r-sequences: ℓ_k is the smallest integer in the set $\{\sigma(k), \ldots, \sigma(n)\}$ and r_k is the largest integer in the set $\{\sigma(1), \ldots, \sigma(k)\}$. For the permutation matrix P in (1), we have $\ell = 1, 1, 1, 2, 2, 5$ and $r = 3, 4, 4, 6, 6, 6$.

If one defines the $(0,1)$-matrix $A(P) = (a_{uv})$ of order n having a staircase pattern such that $a_{uv} = 1$ if $\ell_u \leqslant v \leqslant r_u$, and 0 otherwise, for $1 \leqslant u, v \leqslant n$, then $\mathcal{S}(P) = A(P)$ (see [2, Theorem 2.2]). Thus, the Bruhat shadow of a permutation matrix is determined by the ℓ- and r-sequences. With P as in (1), we have

$$\mathcal{S}(P) = A(P) = \begin{pmatrix} 1&1&1&0&0&0 \\ 1&1&1&1&0&0 \\ 1&1&1&1&0&0 \\ 0&1&1&1&1&1 \\ 0&1&1&1&1&1 \\ 0&0&0&0&1&1 \end{pmatrix}.$$

The remainder of this paper is organized as follows. In Section 2 we give an explicit characterization for the ℓ- and r-sequences of the Bruhat shadow of a permutation matrix. The main section will be devoted to the symmetric permutation matrices, especially for their Bruhat shadows and Bruhat ranks. In the last section we extend the results on the numbers of inversions established in [2]. Several examples illustrate the results. To our knowledge, this is the first attempt to partially answer the questions left open by Brualdi and Dahl in [2].

2 Characterizing the Bruhat shadow

The characterization of the Bruhat shadow of a permutation matrix was first established by Brualdi and Dahl in [2, Theorem 3.5].

Theorem 2.1. *[2] Let A be a* $(0,1)$*-matrix. Then A is the Bruhat shadow of a permutation matrix if and only if A has a staircase pattern and the matrix* A' *of order m obtained from A by striking out the rows and columns of its extreme positions satisfies* $I_m \leqslant A'$.

Taking into account Theorem 2.1, given an r-sequence (2) and an ℓ-sequence (3), we may characterize explicitly these sequences such that both define the Bruhat shadow of an indecomposable permutation matrix.

Theorem 2.2. *The two sequences* (2) *and* (3) *are, respectively, the* r*- and* ℓ*-sequences of an indecomposable permutation matrix if and only if*

(i) $\{i_1,\dots,i_p\} \cap \{j_1,\dots,j_q\} = \emptyset$,

(ii) $\{r_{i_1},\dots,r_{i_p}\} \cap \{\ell_{j_1},\dots,\ell_{j_q}\} = \emptyset$, *and*

(iii) *if* κ_x *is the* x*-th lowest integer in* $\{1,\dots,n\}$ *not in* $\{r_{i_1},\dots,r_{i_p},\ell_{j_1},\dots,\ell_{j_q}\}$, *and* τ_x *is the* x*-th lowest integer in* $\{1,\dots,n\}$ *not in* $\{i_1,\dots,i_p,j_1,\dots,j_q\}$, *then* $\ell_{\tau_x} < \kappa_x < r_{\tau_x}$.

Proof. Let us start by assuming that (2) and (3) are, respectively, the r- and ℓ-sequences of a permutation matrix P. Since P is indecomposable and contains exactly one entry 1 in each row and each column and 0's elsewhere, the assertion (i) and (ii) are immediate. According to Theorem 2.1, $m = n - p - q$, the diagonal entries of A' are equal to 1. Note that the diagonal entries of A' are the (τ_t, κ_t)-entry in A, for $t = 1,\dots,m$, thus the inequalities in (iii) follow clearly.

Conversely, condition (i) guarantees that there exists exactly one 1 in each row, while condition (ii) says that there is exactly one 1 in each column. If $\{r_{i_1},\dots,r_{i_p},\ell_{j_1},\dots,\ell_{j_q}\} = \{1,\dots,n\}$, then the result comes immediately. Otherwise, let κ_1 be the lowest positive integer not in $\{r_{i_1},\dots,r_{i_p},\ell_{j_1},\dots,\ell_{j_q}\}$ and let τ_1 be the lowest positive integer not in $\{i_1,\dots,i_p,j_1,\dots,j_q\}$. From condition (iii), we have $\ell_{\tau_1} < \kappa_1 < r_{\tau_1}$, which means that the (τ_1,κ_1)-entry of A is 1. If $\{r_{i_1},\dots,r_{i_p},\ell_{j_1},\dots,\ell_{j_q},\kappa_1\} = \{1,\dots,n\}$, then in Theorem 2.1, $m = 1$ and $A' = A[\tau_1;\kappa_1]$, the submatrix of A resulting from the retention of the row and column indexed by τ_1 and κ_1, respectively. Otherwise, we proceed analogously to choose κ_2 as the lowest positive integer not in $\{r_{i_1},\dots,r_{i_p},\ell_{j_1},\dots,\ell_{j_q},\kappa_1\}$, and let τ_2 be the lowest positive integer not in $\{i_1,\dots,i_p,j_1,\dots,j_q,\tau_1\}$. From condition (iii), we have $\ell_{\tau_2} < \kappa_2 < r_{\tau_2}$, i.e., the (τ_2,κ_2)-entry of A is 1. In the end we would get the submatrix $A' = A[\tau_1,\dots,\tau_m;\kappa_1,\dots,\kappa_m]$ with $I_m \leqslant A'$. □

We remark that in the condition (iii) of the previous theorem, we are considering the integers in $\{1,\dots,n\}$ ordered increasingly.

Example 2.3. *Let us consider the* r*-sequence* $r = 5,7,7,7,7,7,7$ *and the* ℓ*-sequence* $\ell = 1,1,1,2,2,4,4$. *Then* $p = 2$, $q = 3$, $i_1 = 1$, $i_2 = 2$, $j_1 = 3$, $j_2 = 5$, *and* $j_3 = 7$. *Now, we have* $\{1,2\} \cap \{3,5,7\} = \emptyset$ *and* $\{5,7\} \cap \{1,2,4\} = \emptyset$. *Moreover,* $\kappa_1 = 3$ *and* $\kappa_2 = 6$. *On the other hand,* $\tau_1 = 4$ *and* $\tau_2 = 6$. *It is straightforward to see now that* $\ell_4 < 3 < r_4$ *and* $\ell_6 < 6 < r_6$. *In another words, following the notation of [2], interchanging the roles of r and* ℓ, *as we pointed out previously, we have*

ℓ	1	1	1	2	2	4	4
σ	5	7	1	3	2	6	4
r	5	7	7	7	7	7	7

and σ gives rise to the permutation matrix whose Bruhat shadow is

$$\begin{pmatrix} 1&1&1&1&1&0&0\\ 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1 \end{pmatrix}.$$

Example 2.4. *Let us consider now the r-sequence $r = 6,6,7,7,7,7,7$ and the ℓ-sequence $\ell = 1,1,2,2,4,5,5$. Here $p = 2$, $q = 4$, $i_1 = 1$, $i_2 = 3$, $j_1 = 2$, $j_2 = 4$, $j_3 = 5$, and $j_4 = 7$. We have $\{1,3\} \cap \{2,4,5,7\} = \emptyset$ and $\{6,7\} \cap \{1,2,4,5\} = \emptyset$. Moreover, $\kappa_1 = 3$ and $\tau_1 = 6$. However, $\ell_6 < 3 < r_6$ is obviously false.*

Example 2.4 shows that the matrix defined in [2, pp.31]

$$\begin{pmatrix} 1&1&1&1&1&1&0\\ 1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1\\ 0&1&1&1&1&1&1\\ 0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1\\ 0&0&0&0&1&1&1 \end{pmatrix}$$

is not the Bruhat shadow of any permutation matrix.

When a permutation matrix is symmetric, its Bruhat shadow will clearly be symmetric. In this case, any characterization depends on one of the sequences defined previously. We will consider, for example, the integral sequence r introduced in (2). Then, the ℓ-sequence of the symmetric permutation matrix is

$$\underbrace{i_1,\ldots,i_1}_{r_{i_1}},\underbrace{i_2,\ldots,i_2}_{r_{i_2}-r_{i_1}},\ldots,\underbrace{i_p,\ldots,i_p}_{r_{i_p}-r_{i_{p-1}}}.$$

Note that, in this case, $j_t = r_{i_t}$ and $\ell_{j_t} = i_t$, for $t = 1,\ldots,p$.

As a consequence of Theorem 2.2, we have the following:

Corollary 2.5. *The integral sequence* (2) *is the r-sequence of a symmetric permutation matrix if and only if $\{i_1,\ldots,i_p\} \cap \{r_{i_1},\ldots,r_{i_p}\} = \emptyset$.*

Proof. We only need to observe that $\ell_\kappa < \kappa < r_\kappa$, in particular for each $\kappa \notin \{i_1,\ldots,i_p,r_{i_1},\ldots,r_{i_p}\}$. □

This corollary considerably simplifies [2, Theorem 3.6].

3 Bruhat rank: the symmetric case

The *term rank* $\rho(A)$ of a $(0,1)$-matrix A is defined to be the maximum number of 1's of A with no two of the 1's in the same row or column of A. By the Kőnig-Egérvàry Theorem [8, pp.55-56], $\rho(A)$ is equal to the minimum number of rows and columns of A which together contain all the 1's of A. The term rank is a well-studied quantity associated with any $(0,1)$-matrix. The matrix A is said to have *total support* provided $\rho(A(i,j)) = n-1$, for each pair of i,j with $a_{ij} = 1$, where $A(i,j)$ denotes the $(n-1)\times(n-1)$ submatrix obtained from A by deleting row i and column j [1]. Mirsky and Perfect [7] showed that a matrix has total support if and only if there exists a doubly stochastic matrix having the same zero pattern. Equivalently, A has total support if every nonzero element of A occurs in the positive main diagonal under a column permutation.

Brualdi and Dahl [2] defined the *Bruhat rank* of an $n \times n$ $(0,1)$-matrix M, denoted by $r_B(M)$, with a staircase pattern and having total support with $I_n \leqslant M$, as the smallest number of permutation matrices P, such that if $\mathcal{I}$ is the Bruhat ideal generated by the P's, then $\mathcal{S}(\mathcal{I}) = M$. For example, a full staircase pattern has Bruhat rank 1. Another more elaborated example is the following staircase pattern

$$M = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \end{pmatrix}.$$

It is not hard to check that the Bruhat rank for this matrix is 3, since a possible minimal decomposition of M is

$$M = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & 1 & 1 & & \\ & & 1 & 1 & 1 & \\ & & 1 & 1 & 1 & \\ & & & & & 1 \end{pmatrix} +_b \begin{pmatrix} 1 & & & & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & 1 & 1 & 1 & & \\ & & & & 1 & 1 \\ & & & & 1 & 1 \end{pmatrix} +_b \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 \end{pmatrix},$$

a decomposition as a Boolean sum of the Bruhat shadows of three permutation matrices.

In [2], the authors proposed the following general problem:

Question 3.1. *Determine either a formula for the Bruhat rank of M or an algorithm for evaluating it.*

In this work, we will answer Question 3.1, in the symmetric case. First, we observe that Corollary 2.5 provides a characterization of Bruhat rank 1 matrices. Interestingly, for the remaining cases the Bruhat rank is always 2.

Theorem 3.2. *The Bruhat rank of any symmetric $(0,1)$-matrix of order n, with a staircase pattern and $I_n \leqslant M$, is at most 2.*

Proof. Let us denote by M such a matrix. Let A be the largest upper left principal submatrix of M, such that A is the Bruhat shadow of some permutation matrix. Assume that n_1 is the order of A. If $M = A$, then $r_B(M) = 1$. Otherwise, we have the following block decomposition for M:

$$M = \left(\begin{array}{c|c} A & C \\ \hline C^t & B \end{array}\right)$$

where C is the matrix of the form

$$\left(\begin{array}{c|c} 0_{\tilde{n}_1 \times m_1} & 0_{\tilde{n}_1 \times m_2} \\ \hline F & 0 \end{array}\right),$$

with $\tilde{n}_1 \geqslant 1$ and F has an $(n_1 - \tilde{n}_1) \times m_1$ full staircase pattern. Consequently, F has no zero columns.

If $m_2 = 0$, then $B = J_{n-n_1}$ and

$$M = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) +_b \left(\begin{array}{c|c|c} I_{\tilde{n}_1} & 0 & 0 \\ \hline 0 & J_{n_1 - \tilde{n}_1} & F \\ \hline 0 & F^t & J_{n-n_1} \end{array}\right),$$

where J_k denotes the $k \times k$ all 1's matrix. Therefore $r_B(M) = 2$. Actually, in this case, one of the matrices is a block decomposition of two principal submatrices with Bruhat rank 1 and the other is a off-diagonal blocks decomposition, with 1's in the extreme positions not included in the diagonal blocks.

Let us assume now that $m_2 > 0$. Note that B has a symmetric staircase pattern. Suppose that (k, r_k) is the entry of the first 1 in the block B verifying $k > n_1$ and $r_k > n_1 + m_1$. Then we partition B into 2×2 blocks:

$$B = \left(\begin{array}{c|c} J_* & 0 \\ \hline 0 & B_1 \end{array}\right),$$

where J_* is of order $k - n_1 - 1$. If B_1 has Bruhat rank 1, then we get the decomposition

$$M = \left(\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & J_* & 0 \\ \hline 0 & 0 & B_1 \end{array}\right) +_b \left(\begin{array}{c|c|c|c} I_* & 0 & 0 & 0 \\ \hline 0 & J_* & F & 0 \\ \hline 0 & F^t & J_* & 0 \\ \hline 0 & 0 & 0 & I_* \end{array}\right).$$

Otherwise, the process now follows on B_1 in the same manner until the order n is exhausted. □

We provide next three illustrative examples of Theorem 3.2.

Example 3.3. *Let us consider the matrix*

$$M = \begin{pmatrix}
1&1&1&&&&&&& \\
1&1&1&1&1&1&&&& \\
1&1&1&1&1&1&1&&& \\
&1&1&1&1&1&1&&& \\
&1&1&1&1&1&1&1&& \\
&1&1&1&1&1&1&1&1& \\
&&1&1&1&1&1&1&1& \\
&&&&1&1&1&1&1&1 \\
&&&&&1&1&1&1&1 \\
&&&&&&&1&1&1
\end{pmatrix}.$$

Then

$$M = \left(\begin{array}{cccccc|c|ccc}
1&1&1&&&&&&& \\
1&1&1&1&1&1&&&& \\
1&1&1&1&1&1&&&& \\
&1&1&1&1&1&&&& \\
&1&1&1&1&1&&&& \\
&1&1&1&1&1&&&& \\
\hline
&&&&&&1&&& \\
\hline
&&&&&&&1&1&1 \\
&&&&&&&1&1&1 \\
&&&&&&&1&1&1
\end{array}\right) +_b \left(\begin{array}{cc|cccc|ccc|c}
1&&&&&&&&& \\
&1&&&&&&&& \\
\hline
&&1&1&1&1&1&&& \\
&&1&1&1&1&1&&& \\
&&1&1&1&1&1&1&& \\
&&1&1&1&1&1&1&1& \\
\hline
&&1&1&1&1&1&1&1& \\
&&&&1&1&1&1&1& \\
&&&&&1&1&1&1& \\
\hline
&&&&&&&&&1
\end{array}\right).$$

Example 3.4. *Let us consider now the matrix*

$$M = \begin{pmatrix}
1&1&1&&&&&&&&&&&& \\
1&1&1&1&1&1&&&&&&&&& \\
1&1&1&1&1&1&1&&&&&&&& \\
&1&1&1&1&1&1&&&&&&&& \\
&1&1&1&1&1&1&1&&&&&&& \\
&1&1&1&1&1&1&1&1&&&&&& \\
&&1&1&1&1&1&1&1&&&&&& \\
&&&&1&1&1&1&1&1&&&&& \\
&&&&&1&1&1&1&1&1&&&& \\
&&&&&&&1&1&1&1&1&&& \\
&&&&&&&&1&1&1&1&1&1& \\
&&&&&&&&&1&1&1&1&1& \\
&&&&&&&&&&1&1&1&1& \\
&&&&&&&&&&1&1&1&1&1 \\
&&&&&&&&&&&&&1&1
\end{pmatrix}.$$

Then

$$M=\left(\begin{array}{cccccc|c|cccc|cc|cc}
1&1&1&&&&&&&&&&&&\\
1&1&1&1&1&1&&&&&&&&&\\
1&1&1&1&1&1&&&&&&&&&\\
&1&1&1&1&1&&&&&&&&&\\
&1&1&1&1&1&&&&&&&&&\\
&1&1&1&1&1&&&&&&&&&\\
\hline
&&&&&&1&&&&&&&&\\
\hline
&&&&&&&1&1&1&&&&&\\
&&&&&&&1&1&1&1&&&&\\
&&&&&&&1&1&1&1&&&&\\
&&&&&&&&1&1&1&&&&\\
\hline
&&&&&&&&&&&1&1&&\\
&&&&&&&&&&&1&1&&\\
\hline
&&&&&&&&&&&&&1&1\\
&&&&&&&&&&&&&1&1
\end{array}\right)+_b$$

$$\left(\begin{array}{cc|cccc|ccc|cc|ccc|c}
1&&&&&&&&&&&&&&\\
&1&&&&&&&&&&&&&\\
\hline
&&1&1&1&1&1&&&&&&&&\\
&&1&1&1&1&1&&&&&&&&\\
&&1&1&1&1&1&1&&&&&&&\\
&&1&1&1&1&1&1&1&&&&&&\\
\hline
&&1&1&1&1&1&1&1&&&&&&\\
&&&&1&1&1&1&1&&&&&&\\
&&&&&1&1&1&1&&&&&&\\
\hline
&&&&&&&&&1&1&1&&&\\
&&&&&&&&&1&1&1&1&1&\\
\hline
&&&&&&&&&1&1&1&1&1&\\
&&&&&&&&&&1&1&1&1&\\
&&&&&&&&&&1&1&1&1&\\
\hline
&&&&&&&&&&&&&&1
\end{array}\right).$$

Example 3.5. *We analyze a final example, also mentioned in [2, pp.32], considering now the symmetric tridiagonal* $(0,1)$*-matrix:*

$$M=\begin{pmatrix}
1&1&&&&&\\
1&1&1&&&&\\
&1&1&1&&&\\
&&1&1&1&&\\
&&&1&1&1&\\
&&&&1&1&1\\
&&&&&1&1
\end{pmatrix}.$$

In this simple case, we have

$$M=\left(\begin{array}{cc|cc|cc|c}
1&1&&&&&\\
1&1&&&&&\\
\hline
&&1&1&&&\\
&&1&1&&&\\
\hline
&&&&1&1&\\
&&&&1&1&\\
\hline
&&&&&&1
\end{array}\right)+_b\left(\begin{array}{c|cc|cc|cc}
1&&&&&&\\
\hline
&1&1&&&&\\
&1&1&&&&\\
\hline
&&&1&1&&\\
&&&1&1&&\\
\hline
&&&&&1&1\\
&&&&&1&1
\end{array}\right).$$

4 Inversions

The number of *inversions* of a permutation matrix P, $\text{inv}(P)$, can be interpreted as the number of submatrices of P equal to L_2. In [2], for a given (0, 1)-matrix with a full staircase pattern, it was shown that the maximum number of inversions of a permutation matrix $P \leqslant A$ is equal to

$$\beta_A = \sum_{k=1}^{p} n_k[(n_k - 1)/2 + r_{i_k} - i_{k+1} + 1], \tag{4}$$

where the r-sequence is defined as in (2) and $n_k = i_{k+1} - i_k$, for $k = 1, \dots, p$, assuming that $i_{p+1} = n + 1$. Moreover, it was considered a simple algorithm for constructing a permutation matrix $P \leqslant A$. It turned out that such algorithm gave rise to a permutation matrix with the largest possible number of inversions. Following the same technique used in Lemma 4.1 and Theorem 4.2 of [2], we may establish a more general result for Bruhat rank one matrices.

Theorem 4.1. *Let A be a (0, 1)-matrix with $r_B(A) = 1$ defined by the r- and ℓ-sequences in (2) and (3), respectively. The permutation matrix P corresponding to the permutation σ with maximum number of inversions, with $P \leqslant A$, is given by the following algorithm:*

1. *Set $P = 0$.*
2. *For $k = 1, \dots, p$, replace the (i_k, r_{i_k})-entry by 1, i.e., make $\sigma_{i_k} = r_{i_k}$.*
3. *For $k = 1, \dots, q$, replace the (j_k, ℓ_{j_k})-entry by 1, i.e., make $\sigma_{j_k} = \ell_{j_k}$.*
4. *Set $I = \{i_1, \dots, i_p, j_1, \dots, j_q\}$.*
5. *If $I = \{1, \dots, n\}$, then the process stops.*
6. *If $I \neq \{1, \dots, n\}$, let u be the lowest positive integer not in I.*
7. *Let v be the largest integer such that $\ell_u < v < r_u$ and $v \neq \sigma_k$, for $k \in I$.*
8. *Replace the (u, v)-entry by 1, i.e., make $\sigma_u = v$.*
9. *Replace I by $I \cup \{u\}$ in Step 4. and proceed.*

Note that in the above algorithm, the condition $r_B(A) = 1$ is required, since the condition (iii) of Theorem 2.2 would guarantee the existence of integer v in Step 7.

We observe that for the case of a full staircase pattern given by A, i.e., when $\ell = 1, \dots, 1$, we have $r_B(A) = 1$ and, as a consequence, we derive the next corollary.

Corollary 4.2. *If A has a full staircase pattern, then*

$$\beta_A = \sum_{s=p}^{1} \sum_{i_s < k < i_{s+1}} \sigma_{i_s} - \sigma_k \tag{5}$$

with $i_{p+1} = n + 1$.

The full staircase pattern gave in [2, pp.37] is defined by the r-sequence

$$4, 6, 6, 6, 7, 7, 11, 13, 13, 13, 13, 13, 13\,.$$

Taking into account (4), we have

$$\beta_A = 3 + 9 + 3 + 4 + 15 = 34\,.$$

From (5), we get

$$\beta_A = 1 + 3 + 4 + 5 + 12 + 5 + 1 + 3 = 34\,.$$

In general we have:

Corollary 4.3. *If A has a staircase pattern with $r_B(A) = 1$, then*

$$\beta_A = \sum_{s=p}^{1} \sum_{i_s < k < i_{s+1}} \sigma_{i_s} - \sigma_k - \#\{t \mid t > k \text{ and } \sigma_k < \sigma_t < \sigma_{i_s}\} \tag{6}$$

with $i_{p+1} = n + 1$.

Example 4.4. *Setting*

$$A = \begin{pmatrix} 1\,1\,1\,0\,0\,0\,0 \\ 1\,1\,1\,1\,1\,0\,0 \\ 1\,1\,1\,1\,1\,0\,0 \\ 1\,1\,1\,1\,1\,0\,0 \\ 0\,1\,1\,1\,1\,1\,1 \\ 0\,1\,1\,1\,1\,1\,1 \\ 0\,0\,0\,0\,0\,1\,1 \end{pmatrix},$$

then the optimal permutation matrix is

$$P = \begin{pmatrix} 0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0 \end{pmatrix}.$$

Furthermore, from (6) *we have*

$$\beta_A = (7-2-1) + (7-6) + (5-4) + (5-1-1) = 9\,.$$

In the same fashion as the proof of [2, Corollary 4.3], by a sequence of $L_2 \to I_2$ interchanges, it is possible to provide the range for the number of inversions of permutation matrices $P \leqslant A$, where $r_B(A) = 1$.

Theorem 4.5. *For each integer* $0 \leqslant k \leqslant \beta_A$, *with* $r_B(A) = 1$, *there is a permutation matrix* $P \leqslant A$ *with* $\mathrm{inv}(P) = k$.

Acknowledgement: We would like to thank both anonymous referees for their comments.

References

[1] Brualdi R.A., Matrices of 0's and 1's with total support, J. Combin. Theory Ser. A, 1980, 28, 249-256.
[2] Brualdi R.A., Dahl G., The Bruhat shadow of a permutation matrix, Mathematical papers in honour of E. Marques de Sá, 25-38, Textos Mat. Sér. B, 39, Universidade de Coimbra, 2006.
[3] Brualdi R.A., Deaett L. , More on the Bruhat order for $(0,1)$-matrices, Linear Algebra Appl., 2007, 421, 219-232.
[4] Brualdi R.A., Hwang S.-G., A Bruhat order for the class of $(0,1)$-matrices with row sum vector R and column sum vector S, Electron. J. Linear Algebra, 2004/2005, 12, 6-16.
[5] Conflitti A., da Fonseca C.M., Mamede R., On the largest size of an antichain in the Bruhat order for $\mathscr{A}(2k,k)$, Order 2013, 30, 255-260.
[6] Conflitti A., da Fonseca C.M., Mamede R., The maximal length of a chain in the Bruhat order for a class of binary matrices, Linear Algebra Appl., 2012, 436, 753-757.
[7] Mirsky L., Perfect L., The distribution of positive elements in doubly stochastic matrices, J. London Math. Soc., 1965, 40, 689-698.
[8] Ryser H.J., Combinatorial Mathematics, Carus Mathematical Monographs no. 14, The Mathematical Association of America, 1963.