Open Mathematics Open Access

Research Article

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An incremental approach to obtaining attribute reduction for dynamic decision systems

DOI 10.1515/math-2016-0077
Received June 2, 2016; accepted August 25, 2016.

Abstract: In the 1960s Professor Hu Guoding proposed a method of measuring information based on the idea that connotation and denotation of a concept satisfies inverse ratio rule. According to this information measure, firstly we put forward the information quantity for information systems and decision systems; then, we discuss the updating mechanism of information quantity for decision systems; finally, we give an attribute reduction algorithm for decision tables with dynamically varying attribute values.

Keywords: Rough set, Conditional information quantity, Dynamic decision system, Attribute reduction

MSC: 03E99

1 Introduction

In recent years a major challenge has been created due to increasing data volumes. The prevalence of continuously collected data has led to an increasing interest in the field of data streams. For example, Internet traffic generates large streams that cannot even be stored effectively unless significant resources are spent on storage. As data sets change with time, it is very time-consuming or even infeasible to run a knowledge acquisition algorithm repeatedly. To overcome this deficiency, the researchers have recently proposed many new analytic techniques. These techniques mainly address knowledge updating from three aspects: the expansion of data [1–7], the increasing number of attributes [8–11] and the variation of data values [12, 13]. For the first two aspects, a number of incremental techniques have been developed to acquire new knowledge without recomputation. However, little research has been done on the third aspect in knowledge acquisition, which motivates this study. This paper concerns attribute reduction for data sets with dynamically varying data values.

Feature selection, a common technique for data preprocessing in many areas including machine learning, pattern recognition and data mining, has hold great significance. Among various approaches to select useful features, a special theoretical framework is Pawlak's rough set model [14, 15]. One can use rough set theory to select a subset of features that is most suitable for a given recognition problem [16–21]. Rough feature selection is also called attribute reduction, which aims to select those features that keep the discernibility ability of the original ones [22–26]. The feature subset generated by an attribute reduction algorithm is called a reduct. In the last two decades, researchers have proposed many reduction algorithms [27–32]. However, most of these algorithms can only be applicable to static data sets. In paper [33–40], several algorithms have been proposed for dynamic data sets. Here, we continue the research on the attribute reduction algorithm of dynamic data sets.

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The remainder of this paper is organized as follows. Some preliminaries about rough set theory are reviewed in Section 2. In Section 3, a new form of conditional information quantity for decision systems is introduced, the properties of this information quantity are discussed. In Section 4, the updating mechanism of information quantity for decision systems are researched. Based on the conditional information quantity, an attribute reduction algorithm for decision systems with dynamically varying attribute values is constructed in Section 5.

2 Preliminaries

In this section, we first review some basic concepts in rough set theory, which can also be referred to [14, 15]. Throughout this paper, the universe U is assumed a finite nonempty set.

In rough set theory, knowledge is regarded as the classification ability of objects. Suppose we are given a finite set $U \neq \phi$ of objects we are interested in. Any subset $X \subseteq U$ will be called a concept or a category in U and any family of concepts in U will be referred to as abstract knowledge about U. We will be mainly interested in the concepts which form a partition and often use equivalence relations instead of classifications, since these two concepts are mutually interchangeable and relations are easier to deal with. Suppose R is an equivalence relation over U, then by U/R we mean the family of all equivalence classes of R, and $[x]_R$ denotes an equivalence class of R containing the element $x \in U$. With each subset $X \subseteq U$, we associate two subsets:

$$\underline{R}X = \cup\{Y \in U/R | Y \subseteq X\}, \overline{R}X = \cup\{Y \in U/R | Y \cap X \neq \phi\}$$

called the $R-$lower and $R-$upper approximations of X respectively. When $\underline{R}X = \overline{R}X$, then X is called $R-$definable; otherwise X is called $R-$undefinable.

An information system, as a basic concept in rough set theory, provides a convenient framework for the representation of objects in terms of their attribute values.

An information system is a quadruple $IS = (U, A, V, f)$,where: U is a set of finite and nonempty objects, called the universe; A is a nonempty finite set of attributes; V is the value domain of attributes; f is an information function which assigns particular values from domains of attributes to objects, such as $\forall a \in A, x \in U$, $f(a, x) \in V$, where $f(a, x)$ denotes the value of attribute a on object x.

With every subset of attributes $B \subseteq A$, there is an associated equivalence relation $ind(B) = \{(x, y) \in U^2 | \forall a \in B, f(a, x) = f(a, y)\}$. This equivalence relation ind(B) divides the universe U into a family of disjoint classes, the approximation space determined by the B-equivalence relation, denoted by π_B, is defined as : $\pi_B = \{X | X \in U/ind(B)\}$, where X is called a B-equivalence block and depicts the collection of objects that are indiscernible from each other with respect to B.

One type of special information system is called a decision system, which is denoted as $DS = (U, C \cup \{d\}, V, f)$, where d is the decision attribute, C is the conditional attribute set. The positive region of d with respect to C is defined as $POS_C(d) = \bigcup_{X \in \pi_d} \underline{C}X$. $DS = (U, C \cup \{d\}, V, f)$ is called a consistent decision system, if $POS_C(d) = U$, else it is called an inconsistent decision system.

The consistent degree of a decision system $DS = (U, C \cup d, V, f)$ is defined as $\gamma = \dfrac{|POS_C(d)|}{|U|}$. (1)

Obviously, a decision table is consistent if and only if its consistent degree γ is 1.

3 The information quantity for information systems and decision systems

In this section, we will use a new form of condition information quantity in decision system based on the equivalence relation. Some properties of the conditional information quantity will be given.

Definition 3.1. *Given an information system* $IS = (U, A, V, f)$ *and* $P, Q \subseteq A$, $\pi_P = \{P_1, P_2, \cdots, P_s\}$ *is finer than* $\pi_Q = \{Q_1, Q_2, \cdots, Q_t\}$ *is defined as: for every* $P_i \in \pi_P$, *there exists* $Q_j \in \pi_Q$, *such that* $P_i \subseteq Q_j$, *denotes* $\pi_P \preceq \pi_Q$. *In this case, we also say that* π_Q *is coarser than* π_P. *If* $\pi_P \preceq \pi_Q$ *and* $\pi_P \neq \pi_Q$, *we say* π_P *is strictly finer than* π_Q, *denotes* $\pi_P \prec \pi_Q$.

Obviously, if $B \subseteq A$, then $\pi_A \preceq \pi_B$.

Definition 3.2. *Let* $IS = (U, A, V, f)$ *be an information system, if* $\pi_A = \{X_1, X_2, \cdots, X_n\}$, *the information quantity of block* X_i *is defined as* $I(X_i) = p(X_i)(1 - p(X_i))$; *the information quantity of* π_A *is defined as* $I(\pi_A) = \sum_{i=1}^{n} p(X_i)(1 - p(X_i))$, *where* $p(X_i) = \frac{|X_i|}{|U|}$, $i = 1, 2, \cdots, n$.

Theorem 3.3. *Let* $IS = (U, A, V, f)$ *be an information system, if* $\pi_A = \{X_1, X_2, \cdots, X_n\}$, *the information quantity of* π_A *satisfies the following properties:*

(1) $0 \leq I(\pi_A) \leq 1 - \frac{1}{n}$.

(2) $I(\pi_A) = 1 - \frac{1}{n}$ *if and only if* $p(X_i) = \frac{1}{n} \quad (i = 1, 2, \cdots, n)$.

(3) $I(\pi_A) = 0$ *if and only if* $\pi_A = \{U\}$.

(4) *For each* $X_i, X_j \in \pi_A$, $I(X_i) + I(X_j) \geq I(X_i \cup X_j)$.

(5) *If* $B \subseteq A$, *then* $I(\pi_B) \leq I(\pi_A)$, *that is, the finer the partition, the bigger information quantity of it.*

Proof. (1) $I(\pi_A) = \sum_{i=1}^{n} p(X_i)(1 - p(X_i)) = 1 - \sum_{i=1}^{n} p^2(X_i)$, where $\sum_{i=1}^{n} p(X_i) = 1$.

Now, we discuss the extreme value of $\sum_{i=1}^{n} p^2(X_i)$ under restrained condition $\sum_{i=1}^{n} p(X_i) = 1$, let $H(\lambda) = \sum_{i=1}^{n} p^2(X_i) + \lambda(\sum_{i=1}^{n} p(X_i) - 1)$. Since

$$\begin{cases} H'_{\lambda}(\lambda) &= \sum_{i=1}^{n} p(X_i) - 1 = 0 \\ H'_{p(X_i)}(\lambda) &= 2p(X_i) + \lambda = 0 \end{cases}$$

We have $p(X_i) = \frac{1}{n}$, that is, when $p(X_i) = \frac{1}{n}$, $\sum_{i=1}^{n} p^2(X_i)$ gets to its minimum value $\frac{1}{n}$, so $I(\pi_A)$ gets to its maximum value $1 - \frac{1}{n}$, obviously, $I(\pi_A) \geq 0$. So (1) and (2) hold, (3) is obvious.

(4) Since $X_i \cap X_j = \phi$, so $p(X_i \cup X_j) = p(X_i) + p(X_j)$, $I(X_i) + I(X_j) - I(X_i \cup X_j) = p(X_i)(1 - p(X_i)) + p(X_j)(1 - p(X_j)) - p(X_i \cup X_j)(1 - p(X_i \cup X_j)) = 2p(X_i)p(X_j) \geq 0$.

(5) If $B \subseteq A$, then $\pi_A \preceq \pi_B$, so each equivalence class of π_B is made up of one or more equivalence classes of π_A. Obviously, we can get π_B through combining two equivalence classes of π_A each time. From (4), we can get $I(\pi_A) \geq I(\pi_B)$. □

Theorem 3.4. *Let* $IS = (U, A, V, f)$ *be an information system, if* $X, Y \subseteq U$, *then* $I(X) + I(Y) \geq I(X \cup Y)$.

Proof. Let $\Delta = I(X) + I(Y) - I(X \cup Y)$, then

$$\begin{aligned}\Delta &= p(X)[1-p(X)] + p(Y)[1-p(Y)] - p(X\cup Y)[1-p(X\cup Y)]\\ &= p(X)+p(Y)-p(X\cup Y)+[p(X\cup Y)]^2-[p(X)]^2-[p(Y)]^2\\ &= p(X\cap Y)+[p(X)+p(Y)-p(X\cap Y)]^2-[p(X)]^2-[p(Y)]^2\\ &= p(X\cap Y)+2p(X)p(Y)-2p(X)p(X\cap Y)\\ &\quad -2p(Y)p(X\cap Y)+[p(X\cap Y)]^2\\ &= [p(X\cap Y)-p(X)p(X\cap Y)-p(Y)p(X\cap Y)+p^2(X\cap Y)]\\ &\quad +2p(X)p(Y)-p(X)p(X\cap Y)-p(Y)p(X\cap Y)\\ &= p(X\cap Y)[1-p(X)-p(Y)+p(X\cap Y)]\\ &\quad +p(X)p(Y)-p(X)p(X\cap Y)+p(X)p(Y)-p(Y)p(X\cap Y)\\ &= p(X\cap Y)[1-p(X\cup Y)]+p(X)[p(Y)-p(X\cap Y)]\\ &\quad +p(Y)[p(X)-p(X\cap Y)]\end{aligned}$$

Since $0 \le p(X) \le 1$, $0 \le p(Y) \le 1$, $0 \le p(X\cup Y) \le 1$, and $p(X) \ge p(X\cap Y)$,$p(Y) \ge p(X\cap Y)$, so $\Delta \ge 0$, that is $I(X)+I(Y) \ge I(X\cup Y)$. □

This theorem demonstrates that if we combine two blocks, their information quality is lesser.

Definition 3.5. *Let $DS = (U, C\cup\{d\}, V, f)$,$X \subseteq U$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, the information quantity of block X with respective to π_d is denoted as $I(\pi_d|X) = p(X)\sum_{j=1}^{n} p(Y_j|X)(1-p(Y_j|X))$.*

Definition 3.6. *Let $DS = (U, C\cup\{d\}, V, f)$, if $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, the condition information quantity of π_C with respect to π_d is defined as $I(\pi_d|\pi_C) = \sum_{i=1}^{m} p(X_i)\sum_{j=1}^{n} p(Y_j|X_i)(1-p(Y_j|X_i))$, where $p(X_i) = \frac{|X_i|}{|U|}$, $i = 1, 2, \cdots, m$; $p(Y_j|X_i) = \frac{|Y_j\cap X_i|}{|X_i|}$, $j = 1, 2, \cdots, n$.*

Theorem 3.7. *Let $DS = (U, C\cup\{d\}, V, f)$, if $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, the condition information quantity of π_C with respect to π_d satisfies the following properties:*

(1) $I(\pi_d|X_i) + I(\pi_d|X_j) \le I(\pi_d|(X_i \bigcup X_j))$ $(i, j \in \{1, 2, \cdots, m\})$.

(2) $I(\pi_d|X_i) + I(\pi_d|X_j) = I(\pi_d|(X_i \bigcup X_j))$ *if and only if, for each* $k \in \{1, 2, \cdots, n\}$, $\frac{|Y_k \bigcap X_i|}{|X_i|} = \frac{|Y_k \bigcap X_j|}{|X_j|}$*$(i, j \in \{1, 2, \cdots, m\})$.*

(3) $0 \le I(\pi_d|\pi_C) \le I(\pi_d)$.

(4) $I(\pi_d|\pi_C) = 0$ *if and only if* $\pi_C \preceq \pi_d$.

(5) *if* $\pi_C = \{U\}$, *then* $I(\pi_d|\pi_C) = I(\pi_d)$.

Proof. (1)

$$\begin{aligned}\Delta &= I(\pi_d|(X_i \bigcup X_j)) - (I(\pi_d|X_i) + I(\pi_d|X_j))\\ &= p(X_i \bigcup X_j)\sum_{k=1}^{n} p(Y_k|X_i \bigcup X_j)(1-p(Y_k|X_i \bigcup X_j))\\ &\quad - p(X_i)\sum_{k=1}^{n} p(Y_k|X_i)\big(1-p(Y_k|X_i)\big) - p(X_j)\sum_{k=1}^{n} p(Y_k|X_j)\big(1-p(Y_k|X_j)\big)\end{aligned}$$

$$\begin{aligned}&= \frac{|X_i \bigcup X_j|}{|U|}\sum_{k=1}^{n}\frac{|Y_k \bigcap X_i| + |Y_k \bigcap X_j|}{|X_i \bigcup X_j|}\Big(1-\frac{|Y_k \bigcap X_i| + |Y_k \bigcap X_j|}{|X_i \bigcup X_j|}\Big)\\ &\quad -\frac{|X_i|}{|U|}\sum_{k=1}^{n}\frac{|Y_k \bigcap X_i|}{|X_i|}\Big(1-\frac{|Y_k \bigcap X_i|}{|X_i|}\Big) - \frac{|X_j|}{|U|}\sum_{k=1}^{n}\frac{|Y_k \bigcap X_j|}{|X_j|}\Big(1-\frac{|Y_k \bigcap X_j|}{|X_j|}\Big)\end{aligned}$$

$$= \frac{1}{|U|}\sum_{k=1}^{n}\Big((|Y_k \bigcap X_i| + |Y_k \bigcap X_j|)(1 - \frac{|Y_k \bigcap X_i| + |Y_k \bigcap X_j|}{|X_i \bigcup X_j|})$$
$$-|Y_k \bigcap X_i| + \frac{|Y_k \bigcap X_i|^2}{|X_i|} - |Y_k \bigcap X_j| + \frac{|Y_k \bigcap X_j|^2}{|X_j|}\Big)$$
$$= \frac{1}{|U|}\sum_{k=1}^{n}\Big(\frac{|Y_k \bigcap X_i|^2}{|X_i|} + \frac{|Y_k \bigcap X_j|^2}{|X_j|} - \frac{(|Y_k \bigcap X_i| + |Y_k \bigcap X_j|)^2}{|X_i \bigcup X_j|}\Big).$$

Let $|X_i| = x$, $|X_j| = y$, $|Y_k \bigcap X_i| = a$, $|Y_k \bigcap X_j| = b$, denotes

$$f_k = \frac{|Y_k \bigcap X_i|^2}{|X_i|} + \frac{|Y_k \bigcap X_j|^2}{|X_j|} - \frac{(|Y_k \bigcap X_i| + |Y_k \bigcap X_j|)^2}{|X_i \bigcup X_j|},$$

then

$$\begin{aligned} f_k &= \frac{a^2}{x} + \frac{b^2}{y} - \frac{(a+b)^2}{x+y} \\ &= \frac{a^2 y(x+y) + b^2 x(x+y) - (a+b)^2 xy}{xy(x+y)} \\ &= \frac{a^2xy + a^2y^2 + b^2x^2 + b^2xy - a^2xy - b^2xy - 2abxy}{xy(x+y)} \\ &= \frac{(ay-bx)^2}{xy(x+y)} \geq 0. \end{aligned}$$

So $I(\pi_d|X_i) + I(\pi_d|X_j) \leq I(\pi_d|(X_i \bigcup X_j))$.

(2) According to the proof of (1), $I(\pi_d|X_i) + I(\pi_d|X_j) = I(\pi_d|(X_i \bigcup X_j))$if and only if $ay = bx$, that is, if and only if for each $k \in \{1, 2, \cdots, n\}$, $\frac{|Y_k \bigcap X_i|}{|X_i|} = \frac{|Y_k \bigcap X_j|}{|X_j|}$, then $I(\pi_d|X_i) + I(\pi_d|X_j) = I(\pi_d|(X_i \bigcup X_j))$.

(3) Obviously, when $\pi_C \preceq \pi_d$, $I(\pi_d|\pi_C) = 0$. According to (1), if we combine two condition classes, the information of condition class block with respective to π_d is increased, so $I(\pi_d|\pi_C) \leq I(\pi_d|\{U\}) = I(\pi_d)$.

(4) If $I(\pi_d|\pi_C) = 0$, that is $\sum_{i=1}^{n} p(X_i) \sum_{j=1}^{m} p(Y_j|X_i)(1 - p(Y_j|X_i)) = 0$, so for every $X_i \in \pi_C$, we have $p(X_i) \sum_{j=1}^{m} p(Y_j|X_i)(1 - p(Y_j|X_i)) = 0$. Since $p(X_i) > 0$, so for each $Y_j \in \pi_D$, $p(Y_j|X_i)(1 - p(Y_j|X_i)) = 0$, that is, for each $Y_j \in \pi_D$, $p(Y_j|X_i) = 0$ or $p(Y_j|X_i) = 1$, thus $X_i \cap Y_j = \phi$ or $X_i \subseteq Y_j$. i.e., $\pi_C \subseteq \pi_d$. The inverse is obvious.

(5) According to definition of $I(\pi_d|\pi_C)$, we can easily determine if $\pi_C = \{U\}$, then $I(\pi_d|\pi_C) = I(\pi_d)$. □

We must pay attention that the inverse of (5) doesn't hold.

Example 3.8. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $U = \{u_1, u_2, \cdots, u_9\}$, $\pi_d = \{\{u_1, u_4, u_7\}, \{u_2, u_5, u_8\}, \{u_3, u_6, u_9\}\}$, $\pi_C = \{\{u_1, u_2, u_3\}, \{u_4, u_5, u_6\}, \{u_7, u_8, u_9\}\}$, obviously, $I(\pi_d|\{U\}) = I(\pi_d|\pi_C) = I(\pi_d)$, but $\pi_C \neq \{U\}$.*

Corollary 3.9. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system,if $B_1 \subseteq B_2 \subseteq C$, then $I(\pi_d|\pi_{B_2}) \leq (\pi_d|\pi_{B_1})$.*

Corollary 3.10. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $C_1, C_2 \subseteq C$, if $I(\pi_d|\pi_{C_1}) = k_1$, $I(\pi_d|\pi_{C_2}) = k_2$, then (1) $I(\pi_d|\pi_{(C_1 \bigcup C_2)}) \leq \min(k_1, k_2)$; (2) $I(\pi_d|\pi_{(C_1 \bigcap C_2)}) \geq \max(k_1, k_2)$.*

Proof. Since $\pi_{(C_1 \cup C_2)} \preceq \pi_{C_1} \preceq \pi_{(C_1 \cap C_2)}$; $\pi_{(C_1 \cup C_2)} \preceq \pi_{C_2} \preceq \pi_{(C_1 \cap C_2)}$, from Corollary 3.9, we have: $I(\pi_d|\pi_{(C_1 \bigcup C_2)}) \leq \min(k_1, k_2)$; $I(\pi_d|\pi_{(C_1 \bigcap C_2)}) \geq \max(k_1, k_2)$. □

Definition 3.11. *Let $DS = (U, C \cup \{d\}, V, f)$, if $X \in \pi_C$ and $|\lambda(X)| > 1$, then X is said to be an inconsistent equivalence block; otherwise, it is said to be a consistent equivalence block, where $\lambda(X) = \{f(u, d)|u \in X\}$ and $|\lambda(X)|$ is the cardinality of $\lambda(X)$.*

An inconsistent equivalence block describes a group of C -indistinguishable objects that have a divergence in their decision-making, while a consistent equivalence block depicts a collection of C -definable objects that share the same decision-making.

Definition 3.12. *Let $DS = (U, C \cup \{d\}, V, f)$, and $X \in \pi_C$,the inconsistent and consistent block families of π_C are denoted by $\pi_C^{inc} = \{X \in \pi_C ||\lambda(X)| > 1\}$, $\pi_C^{con} = \{X \in \pi_C ||\lambda(X)| = 1\}$ respectively.*

The inconsistent block family collects all of the inconsistent equivalence blocks from π_C, whereas the consistent block family gathers all of the consistent equivalence blocks from π_C . It is evident that $\pi_C^{inc} \cup \pi_C^{con} = \pi_C$ and $\pi_C^{inc} \cap \pi_C^{con} = \phi$.

Theorem 3.13. *Let $DS = (U, C \cup \{d\}, V, f)$, $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, $X_i \in \pi_C$ is a consistence block if and only if the information of block X_i with respective to π_d is zero, that is,$X_i \in \pi_C^{con}$ iff $I(\pi_d|X_i) = p(X_i)\sum_{j=1}^{n} p(Y_j|X_i)(1 - p(Y_j|X_i)) = 0$.*

Proof. $X_i \in \pi_C$ is a consistent block iff exists $Y_j \in \pi_d$, such that $X_i \subseteq Y_j$, for each $Y_k \in \pi_d (k \neq j)$, $X_i \cap Y_k = \phi$. Whereas $X_i \subseteq Y_j$ iff $p(Y_j|X_i) = 1$, and $X_i \cap Y_k = \phi$ iff $p(Y_j|X_i) = 0$. So $X_i \in \pi_C^{con}$ iff $I(\pi_d|X_i) = p(X_i)\sum_{j=1}^{n} p(Y_j|X_i)(1 - p(Y_j|X_i)) = 0$. □

Corollary 3.14. *Let $DS = (U, C \cup \{d\}, V, f)$, $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, then $I(\pi_d|\pi_C) = \sum_{X_i \in \pi_C^{inc}} p(X_i)\sum_{j=1}^{n} p(Y_j|X_i)(1 - p(Y_j|X_i))$.*

Corollary 3.15. *Let $DS = (U, C \cup \{d\}, V, f)$, $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, DS is consistent decision system iff $I(\pi_d|\pi_C) = 0$.*

Definition 3.16. *In decision system $DS = (U, C \cup \{d\}, V, f)$, $a \in C$ is called d- dispensable if $I(\pi_d|\pi_{(C-\{a\})}) = I(\pi_d|\pi_C)$.*

Corollary 3.17. *Let $DS = (U, C \cup \{d\}, V, f)$ be a consistent decision system, $a \in C$ is d-dispensable iff $\forall X \in \pi_{C-\{a\}}$, $I(\pi_d|X) = 0$.*

Corollary 3.18. *Let $DS = (U, C \cup \{d\}, V, f)$ be a consistent decision system, condition attribute C is indispensable with respect to d iff $\forall a \in C$,$\exists X \in \pi_{C-\{a\}}$, $I(\pi_d|X)) > 0$.*

Definition 3.19. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $B \subseteq C$, $\forall a \in B$, the significance measure (inner measure) of a in B is defined as $Sig^{inner}(a, B, D) = I(\pi_d|\pi_{(B-\{a\})}) - I(\pi_d|\pi_B)$.*

Definition 3.20. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $B \subseteq C$, $\forall a \in C - B$, the significance measure (outer measure) of a in B is defined as $Sig^{outer}(a, B, D) = I(\pi_d|\pi_B) - I(\pi_d|\pi_{(B\cup\{a\})})$.*

Theorem 3.21. *In decision system $DS = (U, C \cup \{d\}, V, f)$, $a \in C$ is d- dispensable, then $POS_{C-\{a\}}(d) = POS_C(d)$.*

Proof. On the one hand, according to Theorem 3.7, if we combine two condition classes of decision table, the condition information quantity will increase monotonically, and only if the two condition classes X_i and X_j satisfy $\frac{|Y_k \bigcap X_i|}{|X_i|} = \frac{|Y_k \bigcap X_j|}{|X_j|}$, for each $Y_k \in \pi_d$, then the condition information quantity doesn't change, that is, if $\pi_C = \{X_1, X_2, \cdots, X_i, \cdots, X_j, \cdots, X_n\}$, $\pi_B = \{X_1, X_2, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_n, X_i \cup$

$X_j\}$, and $I(\pi_d|\pi_C) = I(\pi_d|\pi_B)$, then for each $Y_k \in \pi_d$, we have $\frac{|Y_k \bigcap X_i|}{|X_i|} = \frac{|Y_k \bigcap X_j|}{|X_j|}$,that is $POS_B(d) = POS_C(d)$.

On the other hand, since $\pi_C \preceq \pi_{C-\{a\}}$, then $\pi_{C-\{a\}}$ can be obtained through combining the classes of π_C. According to the analysis of the above, if $I(\pi_d|\pi_C) = I(\pi_d|\pi_{C-\{a\}})$, we must have $POS_C(d) = POS_{(C-\{a\})}(d)$. □

Definition 3.22. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $B \subseteq C$ is a relative reduct of C relative to decision attributed d, if*
(1) $I(\pi_d|\pi_C) = I(\pi_d|\pi_B)$
(2) $\forall B' \subset B$, $I(\pi_d|\pi_{B'}) \neq I(\pi_d|\pi_B)$

Theorem 3.23. *Let $DS = (U, C \cup \{d\}, V, f)$ be a consistent decision system, then $B \subseteq C$ is a relative reduct of C relative to decision attributed d if and only if (1) $POS_C(d) = POS_B(d)$; (2) $\forall B' \subset B$, $POS_{B'}(d) \neq POS_B(d)$.*

4 Updating mechanism of information quantity for decision systems

Given a dynamic decision table, based on the information quantity, this section presents the updating mechanisms of the information quantity for dynamically varying data values.

Theorem 4.1. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $\pi_d = \{Y_1, Y_2, \cdots, Y_q, \cdots, Y_n\}$, if $x \in X(|X| > 1)$, $x \in Y_q$ $(q \in \{1, 2, \cdots, n\})$, when x exits out of this system, the new information quantity of block X*

$$I(\pi_d'|X') = \frac{|U||X|}{(|U|-1)(|X|-1)} I(\pi_d|X) - \frac{2}{(|X_p|-1)(|U|-1)}(|X_p| - |Y_q \cap X_p|).$$

where $X' = X - \{x\}$, $\pi_d' = \{Y_1, Y_2, \cdots, Y_q' = Y_q - \{x\}, \cdots, Y_n\}$.

Proof.

$$\begin{aligned}
I(\pi_d'|X') =& P(X') \sum_{j=1}^{n} P(Y_j|X')(1 - P(Y_j|X')) \\
=& P(X') \sum_{j=1, j \neq q'}^{n} P(Y_j|X')(1 - P(Y_j|X')) + P(X')P(Y_q'|X')(1 - P(Y_q'|X')) \\
=& \frac{|X|-1}{|U|-1} \sum_{j=1, j\neq q'} \frac{|Y_j \cap X|}{|X|-1}\left(1 - \frac{|Y_j \cap X|}{|X|-1}\right) + \frac{|X|-1}{|U|-1}\frac{|Y_q \cap X|-1}{|X|-1}\left(1 - \frac{|Y_q \cap X|-1}{|X|-1}\right) \\
=& \frac{|X||U|}{(|X|-1)(|U|-1)} \cdot \frac{|X|}{|U|} \sum_{j=1, j\neq q'} \frac{|Y_j \cap X|}{|X|}\left(1 - \frac{|Y_j \cap X|}{|X|} - \frac{1}{|X|}\right) \\
&+ \frac{|X||U|}{(|X|-1)(|U|-1)} \cdot \frac{|X|}{|U|}\left[\frac{|Y_q \cap X|}{|X|} \cdot \left(1 - \frac{|Y_q \cap X|}{|X|}\right) - \frac{|X| - |Y_q \cap X|}{|X|^2}\right] \\
=& \frac{|X||U|}{(|X|-1)(|U|-1)} \cdot \frac{|X|}{|U|} \sum_{j=1, j\neq q'} \frac{|Y_j \cap X|}{|X|}\left(1 - \frac{|Y_j \cap X|}{|X|}\right) \\
&- \frac{|X||U|}{(|X|-1)(|U|-1)} \cdot \frac{|X|}{|U|} \sum_{j=1, j\neq q'} \frac{|Y_j \cap X|}{|X|^2} \\
&+ \frac{|X||U|}{(|X|-1)(|U|-1)} \cdot \frac{|X|}{|U|} \frac{|Y_q \cap X|}{|X|} \cdot \left(1 - \frac{|Y_q \cap X|}{|X|}\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{(|X|-1)(|U|-1)}(|X|-|Y_q\cap X|)\\
=&\frac{|U|}{|U|-1}\Big[\frac{|X|}{(|X|-1)}\frac{|X|}{|U|}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}\right)\\
&-\frac{1}{(|X|-1)|U|}(\sum_{j=1}^{n}|Y_j\cap X|-|Y_q\cap X|)\\
&+\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{1}{|X|-1}\cdot\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&-\frac{1}{(|X|-1)|U|}(|X|-|Y_q\cap X|)\Big]\\
=&\frac{|U|}{|U|-1}\Big[\frac{|X|}{|X|-1}\frac{|X|}{|U|}\sum_{j=1,j\neq q}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}\right)\\
&+\frac{|X|}{|X|-1}\cdot\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&-\frac{2}{(|X|-1)|U|}(|X|-|Y_q\cap X|)\Big]\\
=&\frac{|U|}{|U|-1}\cdot\frac{|X|}{|X|-1}I(\pi_d|X)-\frac{2}{(|X|-1)|U|}(|X|-|Y_q\cap X|)
\end{aligned}
$$

□

Theorem 4.2. *Let $DS=(U,C\cup\{d\},V,f)$ be a decision system, $\pi_d=\{Y_1,Y_2,\cdots,Y_q,\cdots,Y_n\}$, if $x\notin X$, $x\in Y_q$ $(q\in\{1,2,\cdots,n\})$, when x exits out of this system, the new information quantity of block X*

$$I(\pi'_d|X)=\frac{|U|}{|U|-1}I(\pi_d|X).$$

where $\pi'_d=\{Y_1,Y_2,\cdots,Y'_q=Y_q-\{x\},\cdots,Y_n\}$.

Proof.

$$
\begin{aligned}
I(\pi'_d|X)=&\frac{|X|}{|U|-1}\sum_{j=1}^{n}P(Y_j|X)(1-P(Y_j|X))=\frac{|U|}{|U|-1}\cdot\frac{|X|}{|U|}\sum_{j=1}^{n}P(Y_j|X)(1-P(Y_j|X))\\
=&\frac{|U|}{|U|-1}I(\pi_d|X).
\end{aligned}
$$

□

Theorem 4.3. *Let $DS=(U,C\cup\{d\},V,f)$ be a decision system, $\pi_C=\{X_1,X_2,\cdots,X_m\}$, $\pi_d=\{Y_1,Y_2,\cdots,Y_n\}$, if $x\in X_p(|X_p|>1)$, $x\in Y_q$ $(p\in\{1,2,\cdots,m\};q\in\{1,2,\cdots,n\})$, when x exits out of this system, the new information quantity*

$$
\begin{aligned}
I(\pi'_d|\pi'_C)=\frac{|U|}{|U|-1}\Big[&I(\pi_d|\pi_C)+\frac{1}{|X_p|-1}I(\pi_d|X_p)\\
&-\frac{2}{(|X_p|-1)|U|}(|X_p|-|Y_q\cap X_p|)\Big].
\end{aligned}
$$

where $\pi'_C=\{X_1,X_2,\cdots,X'_p=X_p-\{x\},\cdots,Y_n\}$,$\pi'_d=\{Y_1,Y_2,\cdots,Y'_q=Y_q-\{x\},\cdots,Y_n\}$.

Proof. According to Theorem 4.1 and Theorem 4.2, we have

$$
\begin{aligned}
I(\pi'_d|\pi'_C)=&\sum_{i=1}^{m}I(\pi'_d|X_i)=\sum_{i=1,i\neq p'}^{m}I(\pi'_d|X_i)+I(\pi'_d|X'_p))\\
=&\frac{|U|}{|U|-1}\sum_{i=1,i\neq p}^{m}I(\pi_d|X_i)+\frac{|U||X_p|}{(|U|-1)(|X_p|-1)}I(\pi_d|X_p)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2}{(|X_p|-1)(|U|-1)}(|X_p|-|Y_q\cap X_p|).\\
=&\frac{|U|}{|U|-1}\sum_{i=1}^{m} I(\pi_d|X_i)-\frac{|U|}{|U|-1}I(\pi_d|X_p)\\
&+\frac{|U||X_p|}{(|U|-1)(|X_p|-1)}I(\pi_d|X_p)\\
&-\frac{2}{(|X_p|-1)(|U|-1)}(|X_p|-|Y_q\cap X_p|)\\
=&\frac{|U|}{|U|-1}I(\pi_d|\pi_C)+\frac{|U|}{(|U|-1)(|X_p|-1)}I(\pi_d|X_p)\\
&-\frac{2}{(|X_p|-1)(|U|-1)}(|X_p|-|Y_q\cap X_p|)\\
=&\frac{|U|}{|U|-1}\big[I(\pi_d|\pi_C)+\frac{1}{|X_p|-1}I(\pi_d|X_p)\\
&-\frac{2}{(|X_p|-1)|U|}(|X_p|-|Y_q\cap X_p|)\big].
\end{aligned}
$$
□

Corollary 4.4. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, if $x \in X_p$ ($p \in \{1, 2, \cdots, m\}$ and $X_p \in \pi_C^{con}$, when x exits out of this system, then $I(\pi'_d|\pi'_C) = \frac{|U|}{|U|-1} I(\pi_d|\pi_C)$.*

Theorem 4.5. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, when x goes into this system, if x adds to X and Y_q ($q \in \{1, 2, \cdots, n\}$), then*

$$
\begin{aligned}
I(\pi'_d|X') =& \frac{|U||X|}{(|U|+1)(|X|+1)}I(\pi_d|X)\\
&+\frac{2}{(|X|+1)(|U|+1)}(|X|-|Y_q\cap X|).
\end{aligned}
$$

where $X' = X \cup \{x\}$,$\pi'_d = \{Y_1, Y_2, \cdots, Y'_q = Y_q \cup \{x\}, \cdots, Y_n\}$.

Proof.

$$
\begin{aligned}
I(\pi'_d|X') =&P(X')\sum_{j=1}^{n} P(Y_j|X')(1-P(Y_j|X))\\
=&P(X')\sum_{j=1,j\neq q'}^{n} P(Y_j|X')(1-P(Y_j|X'))+P(X')P(Y'_q|X')(1-P(Y'_q|X'))\\
=&\frac{|X|+1}{|U|+1}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|+1}\left(1-\frac{|Y_j\cap X|}{|X|+1}\right)\\
&+\frac{|X|+1}{|U|+1}\frac{|Y_q\cap X|+1}{|X|+1}\left(1-\frac{|Y_q\cap X|+1}{|X|+1}\right)\\
=&\frac{|X||U|}{(|X|+1)(|U|+1)}\cdot\frac{|X|}{|U|}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}+\frac{1}{|X|}\right)\\
&+\frac{|X||U|}{(|X|+1)(|U|+1)}\cdot\frac{|X|}{|U|}\Big[\frac{|Y_q\cap X|}{|X|}\cdot\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{|X|-|Y_q\cap X|}{|X|^2}\Big]\\
=&\frac{|X||U|}{(|X|+1)(|U|+1)}\cdot\frac{|X|}{|U|}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|X||U|}{(|X|+1)(|U|+1)}\cdot\frac{|X|}{|U|}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|^2}\\
&+\frac{|X||U|}{(|X|+1)(|U|+1)}\cdot\frac{|X|}{|U|}\frac{|Y_q\cap X|}{|X|}\cdot\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{1}{(|X|-1)(|U|-1)}(|X|-|Y_q\cap X|)\\
=&\frac{|U|}{|U|+1}\Big[\frac{|X|}{(|X|+1)}\frac{|X|}{|U|}\sum_{j=1,j\neq q'}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}\right)\\
&+\frac{1}{(|X|+1)|U|}(\sum_{j=1}^{n}|Y_j\cap X|-|Y_q\cap X|)\\
&+\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{1}{|X|-1}\cdot\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{1}{(|X|+1)|U|}(|X|-|Y_q\cap X|)\Big]\\
=&\frac{|U|}{|U|+1}\Big[\frac{|X|}{|X|+1}\frac{|X|}{|U|}\sum_{j=1,j\neq q}\frac{|Y_j\cap X|}{|X|}\left(1-\frac{|Y_j\cap X|}{|X|}\right)\\
&+\frac{|X|}{|X|+1}\cdot\frac{|X|}{|U|}\cdot\frac{|Y_q\cap X|}{|X|}\left(1-\frac{|Y_q\cap X|}{|X|}\right)\\
&+\frac{2}{(|X|+1)|U|}(|X|-|Y_q\cap X|)\Big]\\
=&\frac{|U|}{|U|+1}\cdot\frac{|X|}{|X|+1}I(\pi_d|X)+\frac{2}{(|X|+1)|U|}(|X|-|Y_q\cap X|)
\end{aligned}
$$ □

Theorem 4.6. *Let $DS=(U,C\cup\{d\},V,f)$ be a decision system, $\pi_C=\{X_1,X_2,\cdots,X_m\}$, $\pi_d=\{Y_1,Y_2,\cdots,Y_n\}$, when x adds to this system, then*

$$
\begin{aligned}
I(\pi'_d|\pi'_C)=&\frac{|U|}{|U|+1}\big[I(\pi_d|\pi_C)-\frac{1}{|X_p|+1}I(\pi_d|X_p)\\
&+\frac{2}{(|X_p|+1)|U|}(|X_p|-|Y_q\cap X_p|)\big].
\end{aligned}
$$

where $\pi'_C=\{X_1,X_2,\cdots,X'_p=X_p\cup\{x\},\cdots,X_m\}$,$\pi'_d=\{Y_1,Y_2,\cdots,Y'_q=Y_q\cup\{x\},\cdots,Y_n\}$.

Proof. According to Theorem 4.5, we have

$$
\begin{aligned}
I(\pi'_d|\pi'_C)=&\sum_{i=1}^{m}I(\pi'_d|X_i)=\sum_{i=1,i\neq p'}^{m}I(\pi'_d|X_i)+I(\pi'_d|X'_p))\\
=&\frac{|U|}{|U|+1}\sum_{i=1,i\neq p}^{m}I(\pi_d|X_i)+\frac{|U||X_p|}{(|U|+1)(|X_p|+1)}I(\pi_d|X_p)\\
&+\frac{2}{(|X_p|+1)(|U|+1)}(|X_p|-|Y_q\cap X_p|).\\
=&\frac{|U|}{|U|+1}\sum_{i=1}^{m}I(\pi_d|X_i)-\frac{|U|}{|U|+1}I(\pi_d|X_p)\\
&+\frac{|U||X_p|}{(|U|+1)(|X_p|+1)}I(\pi_d|X_p)\\
&+\frac{2}{(|X_p|+1)(|U|+1)}(|X_p|-|Y_q\cap X_p|)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{|U|}{|U|+1} I(\pi_d|\pi_C) - \frac{|U|}{(|U|+1)(|X_p|+1)} I(\pi_d|X_p) \\
&\quad + \frac{2}{(|X_p|+1)(|U|+1)}(|X_p| - |Y_q \cap X_p|) \\
&= \frac{|U|}{|U|+1}\big[I(\pi_d|\pi_C) - \frac{1}{|X_p|+1} I(\pi_d|X_p) \\
&\quad + \frac{2}{(|X_p|+1)|U|}(|X_p| - |Y_q \cap X_p|)\big].
\end{aligned}
$$

□

Corollary 4.7. *Let* $DS = (U, C \cup \{d\}, V, f)$ *be a decision system,* $\pi_C = \{X_1, X_2, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_n\}$, *when* x *goes into this system, if* $x \in X_p$ ($p \in \{1, 2, \cdots, m\}$ *and* $X_p \in \pi_C^{con}$ *, then* $I(\pi'_d|\pi'_C) = \frac{|U|}{|U|+1} I(\pi_d|\pi_C)$.

In the following, we discuss how the information quantity is changed if attribute values of one object x are varied.

Theorem 4.8. *Let* $DS = (U, C \cup \{d\}, V, f)$ *be a decision system,* $\pi_C = \{X_1, X_2, \cdots, X_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_{q1}, \cdots, Y_{q2}, \cdots, Y_n\}$. $x \in X_{p1}$, $x \in Y_{q1}$($p1 \in \{1, 2, \cdots, m\}; q1 \in \{1, 2, \cdots, n\}$). *If the object* x *is changed to* x', *and in the new decision system,* $\pi'_C = \{X_1, X_2, \cdots, X'_{p1} = X_{p1} - \{x\}, \cdots, X'_{p2} = X_{p2} \cup \{x'\}, \cdots, X_m\}$, $\pi'_d = \{Y_1, Y_2, \cdots, Y'_{q1} = Y_{q1} - \{x\}, \cdots, Y'_{q2} = Y_{q2} \cup \{x'\}, \cdots, Y_n\}$, *then*

$$
\begin{aligned}
I(\pi'_d|\pi'_C) = I(\pi_d|\pi_C) &+ \frac{1}{(|X_{p1}|-1)|U|} I(\pi_d|X_{p1}) \\
&- \frac{2}{(|X_{p1}|-1)(|U|)}(|X_{p1}| - |Y_{q1} \cap X_{p1}|) \\
&- \frac{1}{(|X_{p2}|+1)|U|} I(\pi_d|X_{p2}) \\
&+ \frac{2}{(|X_{p2}|+1)(|U|)}(|X_{p2}| - |Y_{q2} \cap X_{p2}|).
\end{aligned}
$$

Proof. When x changes to x', and $\pi_C = \{X_1, X_2, \cdots, X_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_{q1}, \cdots, Y_{q2}, \cdots, Y_n\}$ turn into $\pi'_C = \{X_1, X_2, \cdots, X'_{p1} = X_{p1} - \{x\}, \cdots, X'_{p2} = X_{p2} \cup \{x'\}, \cdots, X_m\}$, $\pi'_d = \{Y_1, Y_2, \cdots, Y'_{q1} = Y_{q1} - \{x\}, \cdots, Y'_{q2} = Y_{q2} \cup \{x'\}, \cdots, Y_n\}$.

This process can be divided into two steps: first, $\pi_C = \{X_1, X_2, \cdots, X_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_{q1}, \cdots, Y_{q2}, \cdots, Y_n\}$ turn into $\pi''_C = \{X_1, X_2, \cdots, X'_{p1} = X_{p1} - \{x\}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi''_d = \{Y_1, Y_2, \cdots, Y'_{q1} = Y_{q1} - \{x\}, \cdots, Y_{q2}, \cdots, Y_n\}$; then $\pi''_C = \{X_1, X_2, \cdots, X'_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi''_d = \{Y_1, Y_2, \cdots, Y'_{q1}, \cdots, Y_{q2}, \cdots, Y_n\}$ turn into $\pi'_C = \{X_1, X_2, \cdots, X'_{p1}, \cdots, X'_{p2} = X_{p2} \cup \{x'\}, \cdots, X_m\}$, $\pi'_d = \{Y_1, Y_2, \cdots, Y'_{q1}, \cdots, Y'_{q2} = Y_{q2} \cup \{x'\}, \cdots, Y_n\}$;

According to Theorem 4.6, we have

$$
\begin{aligned}
I(\pi''_d|\pi''_C) = \frac{|U|}{|U|-1}\big[I(\pi_d|\pi_C) &+ \frac{1}{|X_{p1}|-1} I(\pi_d|X_{p1}) \\
&- \frac{2}{(|X_{p1}|-1)|U|}(|X_{p1}| - |Y_{q1} \cap X_{p1}|)\big].
\end{aligned}
$$

$$
\begin{aligned}
I(\pi'_d|\pi'_C) = \frac{|U|-1}{(|U|-1)+1}\big[I(\pi''_d|\pi''_C) &- \frac{1}{|X_{p2}|+1} I(\pi''_d|X_{p2}) \\
&+ \frac{2}{(|X_{p2}|+1)(|U|-1)}(|X_{p2}| - |Y_{q2} \cap X_{p2}|)\big].
\end{aligned}
$$

Since $x \notin X_{p2}$, so $I(\pi''_d|X_{p2}) = \frac{|U|}{(|U|-1)} I(\pi_d|X_{p2})$.

Thus,

$$\begin{aligned} I(\pi'_d|\pi'_C) = I(\pi_d|\pi_C) + \frac{1}{(|X_{p1}|-1)|U|} I(\pi_d|X_{p1}) \\ -\frac{2}{(|X_{p1}|-1)(|U|)}(|X_{p1}|-|Y_{q1}\cap X_{p1}|) \\ -\frac{1}{(|X_{p2}|+1)|U|} I(\pi_d|X_{p2}) \\ +\frac{2}{(|X_{p2}|+1)(|U|)}(|X_{p2}|-|Y_{q2}\cap X_{p2}|). \end{aligned}$$

□

Corollary 4.9. *Let $DS = (U, C \cup \{d\}, V, f)$ be a decision system, $\pi_C = \{X_1, X_2, \cdots, X_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_{q1}, \cdots, Y_{q2}, \cdots, Y_n\}$. When the object x is changed to x', $x \in X_{p1}$, $x' \in X_{p2}$, if $X_{p1} \in \pi_C^{con}$, $X_{p2} \in \pi'_{Ccon}$, then $I(\pi'_d|\pi'_C) = I(\pi_d|\pi_C)$.*

5 Attribute reduction algorithm for decision systems with dynamically varying attribute values

Based on the updating mechanisms of the information quantity, this section introduces an attribute reduction algorithm based on information quantity for decision systems with dynamically varying attribute values. In rough set theory, core is the intersection of all reducts of a given table, and core attributes are considered as the indispensable attributes in a reduct. First, we give an algorithm to obtain the core of a dynamically decision system.

Input: A decision system $DS = (U, C \cup \{d\}, V, f)$ and object $x \in U$ is changed to x'.

Output: Core attribute $Core_{U_{x'}}$ on $U_{x'}$, where $U_{x'}$ expresses the updated objects with $x \in U$ changed to x', and $Core_{U_{x'}}$ is the core of decision system $DS = (U_{x'}, C \cup \{d\}, V, f)$.

Step 1 When x is changed to x', compute $\pi_C = \{X_1, X_2, \cdots, X_{p1}, \cdots, X_{p2}, \cdots, X_m\}$, $\pi_d = \{Y_1, Y_2, \cdots, Y_{q1}, \cdots, Y_{q2}, Y_n\}$ and $\pi'_C = \{X_1, X_2, \cdots, X_{p1} - \{x\}, \cdots, X_{p2} \cup \{x'\}, \cdots, X_m\}$, $\pi'_d = \{Y_1, Y_2, \cdots, Y_{q1} - \{x\}, \cdots, Y_{q2} \cup \{x'\}, Y_n\}$ respectively.

Step 2 compute $I(\pi'_d|\pi'_C)$;

Step 3 $Core_{U'_x} = \phi$, for each $a \in C$

(1) compute $\pi_{C-\{a\}} = \{Z_1, Z_2, \cdots, Z_{t1}, \cdots, Z_{t2}, \cdots, Z_s\}$ and $\pi'_{C-\{a\}} = \{Z_1, Z_2, \cdots, Z_{t1} - \{x\}, \cdots, Z_{t2} \cup \{x'\}, \cdots, Z_s\}$;

(2) compute $I(\pi'_d|\pi'_{C-\{a\}})$;

(3) If $I(\pi'_d|\pi'_{C-\{a\}}) \neq I(\pi'_d|\pi'_C)$, then $Core_{u_{x'}} = core_{U_{x'}} \cup \{a\}$

Step 4 Return $Core_{U_{x'}}$.

Based on updating mechanisms of the information quantity, an attribute reduction algorithm for decision systems with dynamically varying attribute values is proposed in the following. In this algorithm, the existing reduction result is one of inputs, which is used to find its new reduct after data changes.

Input: A decision system $DS = (U, C \cup \{d\}, V, f)$, reduct RED_U on U, and the changed object x which is changed to x'.

Output: Attribute reduct $RED_{U_{x'}}$ on $U_{x'}$.

Step 1 $B = Core_{U_{x'}}$.

Step 2 Compute $I(\pi'_d|\pi_B)$, if $I(\pi'_d|\pi_B) = I(\pi'_d|\pi'_C)$, then $RED_{U_{x'}} = B$, turn to step 4; else turn to step 3.

Step 3 While $I(\pi'_d|\pi_B) \neq I(\pi'_d|\pi'_C)$ do

{ For each $a \in C - B$, computer $sig^{outer}_{U_{x'}}(a, B, d)$;

Select $a_0 = \max\{sig^{outer}_{U_{x'}}(a, b, d)\}, a \in C - B$;

$B \leftarrow B \cup \{a_0\}$.

}

Step 4 For each $a \in B$ do

{ compute $sig^{inner}_{U_{x'}}(a, B, d)$;

If $sig_{U_{x'}}^{inner}(a, B, d) = 0$, then $B \leftarrow B - \{a\}$. };

Step 5 $RED_{U_{x'}} = B$, return $RED_{U_{x'}}$ and end.

In this algorithm, firstly we obtain the core of dynamic information system; then, we add the attribute with the biggest significance gradually till the $RED_{U_{x'}}$ is obtained. In this way, we can get a relative optimal reduct.

6 Conclusion

The incremental technique is an effective way to maintain knowledge in the dynamic environment. An attribute selection for dynamic data sets is still a challenging issue in the field of artificial intelligence. In this paper, we put forward the information quantity for information systems and decision systems according to the information measure proposed by Professor Hu Guoding, and we also discuss the updating mechanism of information quantity for decision systems. Further, we give an attribute reduction algorithm for decision tables with dynamically varying attribute values. It should be pointed out that updating mechanisms of the information quantity introduced in this paper are only applicable when data are varied one by one, whereas many real data may vary in groups in application. This gives rise to many difficulties for the proposed feature selection algorithm to deal with. In our further work, we will focus on improving the incremental algorithm for updating knowledge by varying some objects simultaneously. Furthermore, as an decision system consists of the objects, the attributes, and the domain of attributes values, all of the elements in the decision system will change as time goes by under the dynamic environment. In the future, the variation of attributes and the domain of attributes values in decision system will also be taken into consideration in terms of incremental updating knowledge.

Acknowledgement: The authors would like to thank the Editors and the anonymous reviewers for their valuable comments and suggestions in improving this paper. This research was supported by the National Natural Science Foundation of China(Grant Nos. 11401052) and Innovation Platform and Talent Project of Hunnan Province(Grant Nos. 2015RS4049), the author(s) declare(s) that there is no conflict of interest regarding the publication of this article.

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