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Research Article

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On the saturated numerical semigroups

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Abstract: In this study, we characterize all families of saturated numerical semigroups with multiplicity four. We also present some results about invariants of these semigroups.

Keywords: Saturated numerical semigroup, Frobenius number, Gaps

MSC: 20M14

1 Introduction

Let $\mathbb{N} = \{1, 2, ..., n, ..\}$ and $\mathbb{Z}$ be the set of integers. A subset S of the set $\mathbb{N}$ of nonnegative integers is called a numerical semigroup if it satisfies the following conditions:

(i) $0 \in S$,
(ii) $a, b \in S \Rightarrow a + b \in S$,
(iii) $\mathbb{N} \setminus S$ has a finite number of elements.

Condition (iii) is equivalent to $gcd(S) = 1$ (Here, $gcd(S)$ is the greatest common divisor of the element of S).

All numerical semigroups are finitely generated, i.e.

$$S = \langle a_1, a_2, \ldots, a_r \rangle = \{\sum_{k=1}^{r} c_i a_i : c_1, \ldots, c_r \in \mathbb{N}\}$$

where $a_1, a_2, \ldots, a_r \in S$ and $r \geq 1$. In this case, $\{a_1, a_2, \ldots, a_r\}$ is a minimal system of generators if no proper subset of $\{a_1, a_2, \ldots, a_r\}$ generates S. The numbers $e(S) = r$ and $m(S) = \min\{a \in S : a > 0\}$ are called the embedding dimension and multiplicity of S respectively. In general, it holds that $e(S) \leq m(S)$. We say that S has maximal embedding dimension if $e(S) = m(S)$ (see [6]).

We define the following invariants of numerical semigroups:

$$F(S) = \max\{x : x \in \mathbb{Z} \setminus S\}$$

and

$$n(S) = |\{0, 1, 2, ..., F(S)\} \cap S|.$$

$F(S)$ and $n(S)$ are called the Frobenius number of S and the number determiner of S, respectively.

We can write

$$S = \langle a_1, a_2, \ldots, a_r \rangle = \langle s_0 = 0, s_1, s_2, \ldots, s_{n-1}, s_n = F(S) + 1, \rightarrow \ldots \rangle$$

where $s_i < s_{i+1}$ and $n = n(S)$. The arrow means that every integer greater than $F(S) + 1$ belongs to S, for $i = 1, 2, \ldots, n = n(S)$ (see [2]).

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The set $\mathbb{N} \setminus S$ is the gap of S, and the set of gaps of S is denoted by $H(S)$. $g(S) = |H(S)|$ is called the genus of S. It is clear that $g(S) = F(S)+1-n(S)$. An element $x \in H(S)$ is called a fundamental gap of S if $2x, 3x \in S$. The set of all the fundamental gaps of S is denoted by $FH(S)$, i.e.

$$FH(S) = \{x \in H(S) : 2x, 3x \in S\}.$$

An element $x \in \mathbb{Z}$ is called a Pseudo-Frobenius number of S if $x \notin S$ and $x + s \in S$, for $s \in S \setminus \{0\}$. We denote by $PF(S)$ the set of all Pseudo-Frobenius numbers of S, i.e.

$$PF(S) = \{x \in \mathbb{Z} \setminus S : x + s \in S, \text{ for all } s \in S \setminus \{0\}\}$$

(see [7]). Given a numerical semigroup S and $x \in S \setminus \{0\}$, we define the Apery set of x in S as $Ap(S, x) = \{s \in S : s - x \notin S\}$ (for details see [9]).

If a numerical semigroup S satisfies the condition $x + y - z \in S$, for every $x, y, z \in S$ such that $x \geq y \geq z$, then S is called Arf. If S is an Arf numerical semigroup, then S has maximal embedding dimension.

The investigation of combinatorial properties of semigroups is very important, because they often occur in applications ([1, 3, 5]) and are related to automata theory (see [4]). A numerical semigroup S is saturated if $s + c_1s_1 + c_2s_2 + \ldots + c_ks_k \in S$, where $s, s_i \in S$ and $c_i \in \mathbb{Z}$ such that $c_1s_1 + c_2s_2 + \ldots + c_ks_k \geq 0$ and $s_i \leq s$ for $i = 1, 2, \ldots, k$. Also, all saturated numerical semigroup are Arf. However an Arf numerical semigroup need not be to be saturated. The numerical semigroup

$$S = \langle 7, 12, 15, 16, 17, 18, 20\rangle$$

is Arf, but it is not saturated since $12 + (-5).7 + 3.12 = 13 \notin S$.

In this study, we show that all families of numerical semigroups with multiplicity four are saturated numerical semigroups; these are numerical semigroups of the form $S = \langle 4, k, k+2, k+3\rangle$, for $k \equiv 3(mod 4)$ and $k \geq 7$ and $S = \langle 4, k, k+t, k+t+2\rangle$, for $k \equiv 2(mod 4)$ and $k \geq 6$ and t an odd integer. We also give the formulae for $F(S)$, $n(S)$, $PF(S)$, $g(S)$, $H(S)$ and $FH(S)$ of these numerical semigroups.

2 Main results

In this section we provide some results for numerical semigroups with multiplicity four; i.e. numerical semigroups of the form $S = \langle 4, k, k+2, k+3\rangle$ (for $k \equiv 3(mod 4)$ and $k \geq 7$) and $S = \langle 4, k, k+t, k+t+2\rangle$, (for $k \equiv 2(mod 4)$ and $k \geq 6$ and t an odd integer).

Proposition 2.1 ([8])**.** *Let S be a numerical semigroup, then the following conditions are equivalent:*
(i) *S is a saturated numerical semigroup.*
(ii) *$a + d_S(a) \in S$ for all $a \in S$, $a > 0$ where $d_S(a) = gcd\{x \in S : x \leq a\}$.*
(iii) *$a + kd_S(a) \in S$ for all $a \in S$, $a > 0$ and $k \in \mathbb{N}$.*

Theorem 2.2 ([10])**.** *If $S = \langle 4, k, k+1, k+2\rangle$, then S is a saturated numerical semigroup, for $k \equiv 1(mod 4)$ and $k \geq 5$.*

Theorem 2.3. *Let $S = \langle 4, k, k+2, k+3\rangle$ be numerical semigroup, where $k \equiv 3(mod 4)$ and $k \geq 7$. Then S is saturated.*

Proof. Let $S = \langle 4, k, k+2, k+3\rangle$ be numerical semigroup, where $k \equiv 3(mod 4)$ and $k \geq 7$. We note that $k = 4r + 3$, $r \geq 1$ and $r \in \mathbb{Z}$. Thus, we have

$$S = \langle 4, k, k+2, k+3\rangle = \{0, 4, 8, \ldots, k-7, k-3, k, \rightarrow \ldots,\}$$

$$= \{0, 4, 8, \ldots, 4r-4, 4r, 4r+3, \rightarrow \ldots,\}.$$

In this case,

(a) If $a < 4r + 3$, then $d_S(a) = 1$. So, we find that $a + d_S(a) \in S$ since $a + d_S(a) = a + 1 \geq 4r + 4 \in S$, for all $a \in S, a > 0$.
(b) If $a \geq 4r + 3$, then $d_S(a) = 4$. So, we have $a + d_S(a) = a + 4 \in S$, for all $a \in S, a > 0$.

In view of Proposition 2.1, we find that S is saturated a numerical semigroup. □

Theorem 2.4. *Let $S = \langle 4, k, k + t, k + t + 2\rangle$ be numerical semigroup, where $k \equiv 2(mod 4)$, $k \geq 6$, and t is an odd integer. Then S is saturated.*

Proof. It is trivial that $gcd\{4, k, k + t, k + t + 2\} = 1$ since k is even and t is an odd integer. If we put $k = 4r + 2$, $r \geq 1$ and $r \in \mathbb{Z}$, then we have

$$S = \langle 4, k, k + t, k + t + 2\rangle = \{0, 4, 8, \ldots, k - 6, k - 2, k, k + 2, \ldots, k + t - 3, k + t - 1, \rightarrow \ldots, \}$$
$$= \{0, 4, 8, \ldots, 4r - 4, 4r, 4r + 2, 4r + 4, \ldots, 4r + t - 1, 4r + t + 1, \rightarrow \ldots, \}.$$

In this case,

(i) If $a > 4r + t + 1$, then $d_S(a) = 1$. So, we obtain $a + d_S(a) \in S$ from the inequality $a + d_S(a) = a + 1 \geq 4r + t + 2 \in S$, for all $a \in S, a > 0$.
(ii) If $4r \leq a \leq 4r + t + 1$, then $d_S(a) = 2$. So, we obtain $a + d_S(a) = a + 2 \in S$ from the inequality $4r \leq a \leq 4r + t + 1$, for all $a \in S, a > 0$.
(iii) If $a < 4r$, then $d_S(a) = 4$. So, we obtain $a + d_S(a) = a + 4 \in S$ since $a + 4r < 4r + 4$, for all $a \in S$, $a > 0$.

In view of Proposition 2.1, we have that S is a saturated numerical semigroup. □

Proposition 2.5 ([6]). *Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \ldots < n_r\}$. Then S has maximal embedding dimension if and only if $Ap(S, n_1) = \{0, n_2, n_3, \ldots, n_r\}$.*

Corollary 2.6 ([6]). *Let S be a numerical semigroup minimally generated by $\{n_1 < n_2 < \ldots < n_r\}$. Then the following conditions are true:*
(1) *If S has maximal embedding dimension, then $F(S) = n_r - n_1$.*
(2) *S has maximal embedding dimension if and only if*

$$g(S) = \frac{n_2 + n_3 + \ldots + n_r}{n_1} - \frac{n_1 - 1}{2}.$$

Theorem 2.7. *If $S = \langle 4, k, k + 2, k + 3\rangle$ is a numerical semigroup, where $k \equiv 3(mod 4)$ and $k \geq 7$. Then we obtain following equalities:*
(a) $F(S) = k - 1$,
(b) $g(S) = \frac{3k-1}{4}$,
(c) $PF(S) = \{k - 4, k - 2, k - 1\}$,
(d) $n(S) = \frac{k+1}{4}$,
(e) $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, k - 6, k - 5, k - 4, k - 2, k - 1\}$.

Proof. We have $Ap(S, 4) = \{0, k, k + 2, k + 3\}$ since S has maximal embedding dimension. Thus,
(a) We have $F(S) = (k + 3) - 4 = k - 1$ from Corollary 2.6 (1).
(b) We obtain $g(S) = \frac{k+k+2+k+3}{4} - \frac{4-1}{2} = \frac{3k-1}{4}$ from Corollary 2.6 (2).
(c) It is obvious that $PF(S) = \{k - 4, k + 2 - 4, k + 3 - 4\}$. So we find $PF(S) = \{k - 4, k - 2, k - 1\}$.
(d) We have $n(S) = (k - 1) + 1 - \frac{3k-1}{4} = \frac{k+1}{4}$ from $g(S) = F(S) + 1 - n(S)$.
(e) We find that $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, 4r - 3, 4r - 2, 4r - 1, 4r + 1, 4r + 2\} = \{1, 2, 3, 5, 6, 7, \ldots, k - 6, k - 5, k - 4, k - 2, k - 1\}$ from the equality $S =< 4, k, k + 2, k + 3 >= \{0, 4, 8, \ldots, k - 7, k - 3, k, \rightarrow \ldots, \} = \{0, 4, 8, \ldots, 4r - 4, 4r, 4r + 3, \rightarrow \ldots, \}$. □

Theorem 2.8. *Let $S = \langle 4, k, k + t, k + t + 2\rangle$ be a numerical semigroup, where $k \equiv 2(mod 4)$, $k \geq 6$, and t is an odd integer. Then, we have following equalities:*

(a) $F(S) = k + t - 2$,
(b) $g(S) = \frac{3k+2t-4}{4}$,
(c) $PF(S) = \{k-4, k+t-4, k+t-2\}$,
(d) $n(S) = \frac{k+2t}{4}$,
(e) $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, k-5, k-4, k-3, k-1, k+1, k+3, \ldots, k+t-2\}$.

Proof. We have $Ap(S, 4) = \{0, k, k+t, k+t+2\}$ since S has maximal embedding dimension. Thus,
(a) We have $F(S) = (k+t+2) - 4 = k+t-2$ from Corollary 2.6 (1).
(b) We obtain $g(S) = \frac{k+k+t+k+t+2}{4} - \frac{4-1}{2} = \frac{3k+2t-4}{4}$ from Corollary 2.6 (2).
(c) It is obvious that $PF(S) = \{k-4, k+t-4, k+t+2-4\}$. So, we find $PF(S) = \{k-4, k+t-4, k+t-2\}$.
(d) We have $n(S) = (k+t-2) + 1 - \frac{3k+2t-4}{4} = \frac{k+2t}{4}$ from $g(S) = F(S) + 1 - n(S)$.
(e) We observe that $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, 4r-3, 4r-2, 4r-1, 4r+1, 4r+3, 4r+5, \ldots, 4r+t\} = \{1, 2, 3, 5, 6, 7, \ldots, k-5, k-4, k-3, k-1, k+1, k+3, \ldots, k+t-2\}$ since $S = \langle 4, k, k+t, k+t+2\rangle = \{0, 4, 8, \ldots, k-6, k-2, k, k+2, \ldots, k+t-3, k+t-1, \rightarrow \ldots,\} = \{0, 4, 8, \ldots, 4r-4, 4r, 4r+2, 4r+4, \ldots, 4r+t-1, 4r+t+1, \rightarrow \ldots,\}$. □

Example 2.9. *Consider the numerical semigroup* $S = \langle 4, k, k+2, k+3\rangle$. *If we put* $k = 15$ *then we have that* $S = \langle 4, k, k+2, k+3\rangle = \langle 4, 15, 17, 18\rangle = \{0, 4, 8, 12, 15, \rightarrow \ldots,\}$ *is saturated. Hence, we find that*
(a) $F(S) = k - 1 = 15 - 1 = 14$,
(b) $g(S) = \frac{3k-1}{4} = \frac{45-1}{4} = 11$,
(c) $PF(S) = \{k-4, k-2, k-1\} = \{11, 13, 14\}$,
(d) $n(S) = \frac{k+1}{4} = \frac{15+1}{4} = 4$,
(e) $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, k-6, k-5, k-4, k-2, k-1\} = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14\}$ *and* $Ap(S, 4) = \{0, k, k+2, k+3\} = \{0, 15, 17, 18\}$.

Example 2.10. *Consider the numerical semigroup* $S = \langle 4, k, k+t, k+t+2\rangle$. *If we put* $k = 14$ *and* $t = 13$ *then we find that* $S = \langle 4, k, k+t, k+t+2\rangle = \langle 4, 14, 27, 29\rangle = \{0, 4, 8, 12, 14, 16, 18, 20, 22, 24, 26, \rightarrow \ldots,\}$ *is saturated. Thus, we observe that*
(a) $F(S) = k + t - 2 = 14 + 13 - 2 = 25$,
(b) $g(S) = \frac{3k+2t-4}{4} = \frac{64}{4} = 16$,
(c) $PF(S) = \{k-4, k+t-4, k+t-2\} = \{10, 23, 25\}$,
(d) $n(S) = \frac{k+2t}{4} = \frac{40}{4} = 10$,
(e) $H(S) = \{1, 2, 3, 5, 6, 7, \ldots, k-5, k-4, k-3, k-1, k+1, k+3, \ldots, k+t-2\} = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25\}$ *and* $Ap(S, 4) = \{0, k, k+t, k+t+2\} = \{0, 14, 27, 29\}$.

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