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On CSQ-normal subgroups of finite groups

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Abstract: We introduce a new subgroup embedding property of finite groups called CSQ-normality of subgroups. Using this subgroup property, we determine the structure of finite groups with some CSQ-normal subgroups of Sylow subgroups. As an application of our results, some recent results are generalized.

Keywords: CSQ-normal subgroup, Nilpotent subgroup, Supersolvable subgroup

MSC: 20D10, 20D15

1 Introduction

All groups in this paper are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group G. The other notations and terminologies in this paper are standard (see [1]).

Let $H \leq G$ and $g \in G$, then $H \leq \langle H, H^g \rangle \leq \langle H, g \rangle$. It is clear that $H = \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \trianglelefteq G$. In [2], H is called abnormal in G if $\langle H, H^g \rangle = \langle H, g \rangle$ for all $g \in G$. In [3], the famous Wielandt Theorem shows that $H \lhd\lhd \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \lhd\lhd G$. In [4], H is called pronormal in G if H is conjugate to H^g in $\langle H, H^g \rangle$ for all $g \in G$. These show that the normalities of a subgroup H in G may be determined by the normalities of a subgroup H in $\langle H, H^g \rangle$. This leads us to investigate the properties of G from the relationship between the subgroup H of G and the union of $\langle H, H^g \rangle$ for all $g \in G$. On the other hand, Kegel in [5] introduced the concept of S-quasinormal subgroups. A subgroup H of a group G is said to be s-permutable, S-quasinormal, or π-quasinormal in G if $PH = HP$ for all Sylow subgroups P of G. In this paper, we introduce a new generalized normality of subgroups, CSQ-normality, and obtain a criterion for nilpotency and supersolvablity of a group by using the CSQ-normality of subgroups. Now we recall the following definitions. Let G be a finite group. For every $n \mid |G|$, if G has a subgroup of order n, then G is called a CLT-group. Furthermore, G is called a $QCLT$-group if the image of G under every homomorphism is a CLT-group. As an application of our results, some recent results are generalized. For example, Humphreys [6] proved that a $QCLT$-group of odd order is supersolvable, and we will prove that a $QCLT$-group of even order is also supersolvable if the maximal subgroups of its Sylow 2-subgroup are all CSQ-normal subgroups.

Definition 1.1. *Let H be a subgroup of a group G. We say that H is CSQ-normal in G if H is S-quasinormal in $\langle H, H^g \rangle$ for all $g \in G$.*

By [5, Lemma 3], we know that all S-quasinormal subgroups are CSQ-normal subgroups. The following example shows that a CSQ-normal subgroup is not necessarily a S-quasinormal subgroup.

Example 1.2. *Let $G = A_4$, $H = \langle (12)(34) \rangle$. Obviously, H is not S-quasinormal in G but CSQ-normal in G.*

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2 Basic definitions and preliminary results

The lemma presented below is crucial in the sequel. The proof is a routine check, and we omit its details.

Lemma 2.1. *Let H be a CSQ-normal subgroup of a group G and $N \trianglelefteq G$. Then*
(a) *If $H \leq K$, then H is CSQ-normal in K.*
(b) *HN/N is CSQ-normal in G/N.*

Lemma 2.2. *Suppose that every proper subgroup of a group G is nilpotent but G itself is not nilpotent. Then*
(1) *There exist some primes p and q such that $|G| = p^{\alpha}q^{\beta}$.*
(2) *G has a normal Sylow q-subgroup Q; if $q > 2$, then $exp(Q) = q$ and if $q = 2$, then $exp(Q) \leq 4$; G also has a cyclic Sylow p-subgroup $P = \langle a\rangle$.*
(3) *Let $c \in Q$. Then c is a generator if and only if $[c, a] \neq 1$.*
(4) *If c is a generator of Q, then $[c, a] = c^{-1}c^{a}$ is also a generator of Q.*
(5) *If c is a generator of Q, then $Q = \langle c, c^{a}, \cdots, c^{a^{p-2}}, c^{a^{p-1}}\rangle$, namely, $Q = \langle [c,a], [c,a]^{a}, \cdots, [c,a]^{a^{p-1}}\rangle$.*

Proof. By [7, Theorem 1.1], the result is true. □

As in [8], a *minimal nonsupersolvable group* is a nonsupersolvable group whose proper subgroups and quotients are supersolvable.

Lemma 2.3. *Suppose that a group G is minimal nonsupersolvable. Then G is isomorphic to a group of the form G_t for $1 \leq t \leq 6$, where the groups G_t are defined in the following way.*
(I) *G_1 is a minimal nonabelian group and $|G_1| = pq^{\beta}$, where $p \nmid q-1$, $\beta \geq 2$.*
(II) *$G_2 = \langle a, c_1\rangle$ and $|G_2| = p^{\alpha}r^{p}$ and $p^{\alpha-1} \| r-1$, where $\alpha \geq 2$. $a^{p^{\alpha}} = c_1^{r} = c_2^{r} = \cdots = c_p^{r} = 1$; $c_ic_j = c_jc_i$; $c_i^{a} = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^{a} = c_1^{t}$, where the exponent of $t \pmod r$ is $p^{\alpha-1}$.*
(III) *$G_3 = \langle a, b, c_1\rangle$ and $|G_3| = 8r^2$ and $4 \mid r-1$, $a^4 = c_1^r = c_2^r = 1$, $a^2 = b^2$, $ba = a^{-1}b$, $c_1^a = c_2$, $c_2^a = c_1^{-1}$, $c_1^b = c_1^s$, $c_2^b = c_2^s$, where the exponent of $s \pmod r$ is 4.*
(IV) *$G_4 = \langle a, b, c_1\rangle$ and $|G_4| = p^{\alpha+\beta}r^{p}$ and $p^{max\{\alpha,\beta\}} \mid r-1$, where $\beta \geq 2$. $a^{p^{\alpha}} = b^{p^{\beta}} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_ic_j = c_jc_i$, $ab = b^{1+p^{\beta-1}}a$; $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$, $c_i^b = c_i^{u^{1+ip^{\beta-1}}}$, $i = 1, 2, \ldots, p$; where the exponents of t and $u \pmod r$ are $p^{\alpha-1}$ and p^{β}, respectively.*
(V) *$G_5 = \langle a, b, c, c_1\rangle$ and $|G_5| = p^{\alpha+\beta+1}r^{p}$ and $p^{max\{\alpha,\beta\}} \mid r-1$. $a^{p^{\alpha}} = b^{p^{\beta}} = c^{p} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_ic_j = c_jc_i$, $ba = abc$, $ca = ac$, $cb = bc$, $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$, $c_i^c = c_i^u$, $c_i^b = c_i^{vu^{p-i+1}}$, where the exponents of t, v and $u \pmod r$ are $p^{\alpha-1}$, p^{β} and p, respectively.*
(VI) *$G_6 = \langle a, b, c_1\rangle$, $|G_6| = p^{\alpha}qr^{p}$ and $p^{\alpha}q \mid r-1$, $p \mid q-1$, $\alpha \geq 1$. $a^{p^{\alpha}} = b^{q} = c_1^r = c_2^r = \cdots = c_p^r = 1$; $c_ic_j = c_jc_i$, $i, j = 1, 2, \ldots, p$; $c_i^a = c_{i+1}$, $i = 1, 2, \ldots, p-1$; $c_p^a = c_1^t$; $b^a = b^u$, $c_i^b = c_i^{vu^{i-1}}$, $i = 1, 2, \cdots, p$; where the exponents of t, $v \pmod r$ are $p^{\alpha-1}$ and q, respectively, and the exponent of $u \pmod q$ is p.*

Proof. See [8, Corollary 2.2]. □

Lemma 2.4. *Let H be a CSQ-normal subgroup of G. Then*
(a) *H^x is also a CSQ-normal subgroup of G for any $x \in G$.*
(b) *H is subnormal in G.*

Proof. (a) By the hypothesis, H is S-quasinormal in $\langle H, H^g\rangle$ for all $g \in G$. Then for any $x \in G$, we have that H^x is S-quasinormal in $\langle H^x, H^{gx}\rangle$=$\langle H^x, (H^x)^{g^x}\rangle$ for all $g \in G$. Then one checks easily that $\tau : G \to G$, defined by

$$\tau(g) = g^x, \quad \text{where } x \in G,$$

is a bijective map. Since g^x runs over G as g does for fixed x, we get that H^x is S-quasinormal in $\langle H^x, (H^x)^{g^x}\rangle$ for all $g^x \in G$. Thus H^x is a CSQ-normal subgroup of G.

(b) By the hypothesis, H is S-quasinormal in $\langle H, H^g\rangle$ for all $g \in G$. By [5, Theorem 1], we know that H is subnormal in $\langle H, H^g\rangle$ for all $g \in G$, so H is subnormal in G by Wielandt's theorem. □

3 Main results

Let $\mathcal{Z}$ be a complete set of Sylow subgroups of a group G, that is, for each prime p dividing the order of G, $\mathcal{Z}$ contains exactly one Sylow p-subgroup of G. Let $\mathcal{Z} \cap E = \{P \cap E \mid P \in \mathcal{Z}\}$.

Theorem 3.1. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. Suppose that $E \trianglelefteq G$ such that G/E is nilpotent and G is G_1-free. If every cyclic subgroup of a Sylow subgroup of E contained in $\mathcal{Z} \cap E$ is a CSQ-normal subgroup of G, then G is nilpotent.*

Proof. Assume that the result is false, and let G be a counterexample with least $(|G| + |E|)$.

Let $H < G$. Of course, H is G_1-free. Obviously, $H/H \cap E \cong HE/E$ is nilpotent. Suppose that $K = H \cap E$ and K_p is a Sylow p-subgroup of K, so $\overline{\mathcal{Z}} = \{K_p | p \in \pi(H \cap E)\}$ is a complete set of Sylow subgroups of $H \cap E$. Assume that T is a cyclic subgroup of K_p. Since $K \leq E$, there exists $x \in E$ such that $K_p^x \leq P \cap E$, where $P \in \mathcal{Z}$. By the hypothesis and Lemma 2.4 (a), we get that T is CSQ-normal in G. Then T is CSQ-normal in H by Lemma 2.1 (a). Hence all cyclic subgroups of K_p contained in $\overline{\mathcal{Z}}$ are CSQ-normal in H, and thus H and its normal subgroup K satisfy the hypothesis. By the minimal choice of $|G| + |E|$, H is nilpotent. By Lemma 2.2, we may assume that $G = P^*Q$, where Q is a normal Sylow q-subgroup of G and P^* is a cyclic Sylow p-subgroup of G.

Suppose that $N \lhd G$. We shall prove that $(G/N, EN/N)$ satisfies the hypothesis. Clearly, $(G/N)/(EN/N) \cong G/EN$ is nilpotent and G/N is G_1-free. Let H/N be a cyclic subgroup of a Sylow subgroup of $EN/N \cap \mathcal{Z}N/N$. Then we may assume $H = \langle xN\rangle$ and $\langle x\rangle$ is a cyclic subgroup of a Sylow subgroup in $E \cap \mathcal{Z}$. By the hypothesis, $\langle x\rangle$ is CSQ-normal in G and by Lemma 2.1 (b), H/N is CSQ-normal in G/N. Then $(G/\Phi(G), E/\Phi(G))$ satisfies the hypothesis of the theorem. The minimality of $|G|+|E|$ implies that $G/\Phi(G)$ is nilpotent and so is G, a contradiction. Thus $\Phi(G) = 1$ and so $G \cong G_1$, again a contradiction. This shows that there exists no counterexample, so the result is true. □

Remark 3.2. *We cannot replace the condition "cyclic subgroup of Sylow subgroup" by "minimal subgroup of a Sylow subgroup" in Theorem 3.1. For example, let $G = E = (Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) \rtimes Z_9$. Obviously, the pair (G, E) satisfy the hypothesis. Nevertheless, it is not nilpotent.*

Remark 3.3. *The condition of "G is G_1-free" cannot be removed. For example, let $G = S_3$ and choose $E = A_3$. Then the pair (S_3, A_3) satisfy the hypothesis of Theorem 3.1. Nevertheless, S_3 is not nilpotent.*

Corollary 3.4. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. If every cyclic subgroup of a Sylow subgroup of G contained in $\mathcal{Z}$ is a CSQ-normal subgroup of G, then G is nilpotent.*

Proof. By the proof of Theorem 3.1, we just need to check that $G \cong G_1$. By the hypothesis, we have that a p-Sylow subgroup G_p is a CSQ-normal subgroup of G. Then $G_p \lhd\lhd G$ by Lemma 2.4 (b), thus $G_p \lhd G$, so G is nilpotent. The proof is completed. □

To prove Theorem 3.6, we need the following Lemma 3.5.

Lemma 3.5. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. Suppose that P is a Sylow p-subgroup of G contained in $\mathcal{Z}$, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is CSQ-normal in G, then $G/O_p(G)$ is p-nilpotent and hence G is solvable.*

Proof. Assume that the result is false and let G be a counterexample of smallest order.

First of all, we show that $O_p(G) = 1$. Assume that $O_p(G) = P$. Then $G/O_p(G)$ is a p'-group and of course it is p-nilpotent, a contradiction. Assume that $1 < O_p(G) < P$. Obviously, $O_p(G)\mathcal{Z}/O_p(G)$ is a complete set of Sylow subgroups of $G/O_p(G)$ and $G/O_p(G)$ satisfies the hypothesis by Lemma 2.1 (b). The minimal choice implies that $G/O_p(G) \cong (G/O_p(G))/O_p(G/O_p(G))$ is p-nilpotent, a contradiction. Thus we have $O_p(G) = 1$.

Let P_1 be a maximal subgroup of P. By the hypothesis, P_1 is CSQ-normal subgroup of G. Then P_1 is subnormal in G by Lemma 2.4, and thus $P_1 \leq O_p(G) = 1$. Hence P is a cyclic subgroup of order p. Since $N_G(P)/C_G(P) \lesssim Aut(P)$, we get that the order of $N_G(P)/C_G(P)$ must divide $(|G|, p-1) = 1$. Then $N_G(P) = C_G(P)$. Thus G is p-nilpotent by [1, Burnside's theorem] , a contradiction. We conclude that there is no counterexample and Lemma 3.5 is proved. □

Theorem 3.6. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. Suppose that G is G_t-free with $t \in \{1, 2, 6\}$ and every maximal subgroup of any non-cyclic Sylow subgroup of G contained in $\mathcal{Z}$ is CSQ-normal in G. Then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample of smallest order. We proceed in a number of steps.

If every Sylow subgroup of G contained in $\mathcal{Z}$ is cyclic, then every Sylow subgroup of G is cyclic, thus G is supersolvable. Next we assume that there is a non-cyclic Sylow p-subgroup contained in $\mathcal{Z}$.

Step 1. G is solvable.

Let $p = \min \pi(G)$ and P be a Sylow p-subgroup of G contained in $\mathcal{Z}$. If P is cyclic, then G is p-nilpotent, so G is solvable. If P is not cyclic, then $G/O_p(G)$ is p-nilpotent by Lemma 3.5, thus G is solvable. Hence we have Step 1.

Step 2. G has a unique minimal normal subgroup N and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G, then $\mathcal{Z}N/N$ be a complete set of Sylow subgroups of G/N. Let $PN/N \in Syl_p(G/N)$, where $P \in \mathcal{Z}$ and PN/N is non-cyclic. (Of course, P is non-cyclic.) Assume that T/N be a maximal subgroup of PN/N. Then $T = T \cap PN = (T \cap P)N$. Suppose that $T \cap P = P_1$. Then $P_1 \cap N = T \cap P \cap N = P \cap N$. Hence

$$|P : P_1| = |PN/N : P_1N/N| = |PN/N : T/N| = p.$$

By the hypothesis, P_1 is CSQ-normal in G, so $P_1N/N = T/N$ is CSQ-normal in G/N by Lemma 2.1 (b). Thus G/N satisfies the hypothesis. By the choice of G, we obtain that G/N is supersolvable. Similarly, if N_1 is another minimal normal subgroup of G. Then G/N_1 is also supersolvable. Now it follows that $G \cong G/N \cap N_1$ is supersolvable, a contradiction. Hence, N is the unique minimal normal subgroup of G. If $N \leq \Phi(G)$, then the supersolvability of G/N implies the supersolvability of G. Hence, $\Phi(G) = 1$. Therefore, we have Step 2.

Step 3. $N = O_p(G) = P$, $C_G(N) = N$ and $|G| = p^n r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_s^{\alpha_s}$, the Sylow r_i-subgroup of G is cyclic, where $1 \leq i \leq s, \alpha_i \geq 1$.

By Step 1 and Step 2, we know that N is an elementary abelian p-subgroup and $N = F(G) = O_p(G) \leq P$, so $C_G(N) = N$. Assume that $N < P$. Given a maximal subgroup P_1 of P, by the hypothesis, P_1 is a CSQ-normal subgroup of G, then P_1 is subnormal in G by Lemma 2.4, so $P_1 \leq O_p(G) = N < P$. If $N = P_1 \lhd G$, we get that P has a unique maximal subgroup, so P is cyclic and hence so is N. By Step 2, we obtain that G/N is supersolvable, hence so is G, a contradiction. Therefore, we have $N = P$. Suppose that R_i is a non-cyclic Sylow r_i-subgroup of G contained in $\mathcal{Z}$ for some natural number i, $1 \leq i \leq s$, and $|R_i| = r_i^{\alpha_i}$. Then $\alpha_i \geq 2$, so we can choose $1 \neq R_{i1}$ to be a maximal subgroup of $R_i \in Syl_{r_i}(G)$. By the hypothesis, R_{i1} is CSQ-normal in G, so R_{i1} is subnormal in G by Lemma 2.4, so $1 \neq R_{i1} \leq O_{r_i}(G)$. By the uniqueness of N, this is impossible. Hence R_i is cyclic, and thus all Sylow subgroups B of G are cyclic except $B = P$. Hence we have the assertion in Step 3.

Step 4. Let E be a maximal subgroup of G. We show that $|G : E| = |P| = p^n$ or $r_i^{\beta_i}$, where $\beta_i \leq \alpha_i$. Then E satisfies the hypothesis, so E is supersolvable.

Since G is solvable, $|G : E| = p^j$ or $r_i^{\beta_i}$, where $j \leq n$, $\beta_i \leq \alpha_i$. Suppose that $|G : E| = p^j$. By Step 2 and Step 3, it is easy to show $G = NE$ and $N \cap E = 1$, so $E = R_1 R_2 \cdots R_s$ and $j = n$, where $R_i \in Syl_{r_i}(G)$ $(1 \leq i \leq s)$. It is clear that E satisfies the hypothesis by Lemma 2.1 (a), so E is supersolvable.

Step 5. Final contradiction.

By Step 2 and Step 4, we know that G is minimal nonsupersolvable. On the other hand, by Step 4 and the hypothesis, G is not isomorphic to any group G_i in Lemma 2.3. We conclude that there is no minimal counterexample and Theorem 3.6 is proved. □

If we remove "non-cyclic" in the hypothesis of Theorem 3.6, we can get the following Theorem.

Theorem 3.7. *Let G be a group and $\mathcal{Z}$ be a complete set of Sylow subgroups of G. Suppose that G is G_1-free and $G_{6'}$-free, where $G_{6'} \lesssim G_6$ and $|G_{6'}| = pqr^p$, that is, the case $\alpha = 1$. If every maximal subgroup of every Sylow subgroup of G contained in $\mathcal{Z}$ is a CSQ-normal subgroup of G, then G is supersolvable.*

Proof. By the proof of Theorem 3.6, we only need to check $G \cong G_2$ and $G \cong G_6$, where $|G_6| = p^\alpha qr^p$ and $p^\alpha q \mid r-1$, $p \mid q-1$, $\alpha \geq 2$. Assume that $G \cong G_2$. Using the same description as in Lemma 2.3, let $V_1 = \langle a^p \rangle$. Then it is a maximal subgroup of P. By the hypothesis V_1 is a CSQ-normal subgroup of G, so V_1 is S-quasinormal in $\langle V_1, V_1^g \rangle$ for all $g \in G$. Choosing $g = c_i$. Then $((a^p)^{-1})^{c_i} = c_i^{-1}(a^p)^{-1}c_i \in \langle V_1, V_1^{c_i} \rangle$, so

$$c_i^{-1}(a^p)^{-1}c_i a^p = c_i^{-1}c_i^{a^p} = c_i^{-1}(c_i^a)^{a^{p-1}} = c_i^{-1}c_{i+1}^{a^{p-1}} = \cdots = c_i^{-1}c_i^t \in \langle V_1, V_1^{c_i} \rangle$$

where the exponent of $t \pmod r$ is $p^{\alpha-1}$. Thus r divides

$$t^{p^{\alpha-1}} - 1 = (t-1)(t^{p^{\alpha-1}-1} + t^{p^{\alpha-1}-2} + \cdots + 1).$$

If $r \mid t-1$, then c_i commutes with V_1, of course, c_i normalizes V_1. If $r \nmid t-1$, then $(t-1, r) = 1$, we get

$$c_i = c_i^{m(t-1)+nr} = (c_i^{t-1})^m \in \langle V_1, V_1^{c_i} \rangle.$$

It follows that $\langle V_1, V_1^{c_i} \rangle = \langle a^p, c_i \rangle$. Since V_1 is S-quasinormal in $\langle V_1, V_1^{c_i} \rangle$, we have that $V_1 R_i = \langle a^p \rangle R_i$ is a subgroup of G, where $R_i \in Syl_r(\langle a^p, c_i \rangle)$. By [5, Theorem 1], V_1 is subnormal in $V_1 R_i$, hence $V_1 \lhd V_1 R_i$. Therefore, R_i normalizes V_1 and, of course, c_i normalizes V_1. Since i was arbitrary, we conclude that V_1 is normalized by P and R, where $P \in Syl_p(G)$, $R \in Syl_r(G)$. If $\alpha \geq 2$, then $1 \neq V_1 \lhd G$, which is impossible. If $\alpha = 1$, then $G \cong G_1$, a contradiction. Hence G is not isomorphic to G_1. As in a similar argument above, we also get that G is not isomorphic to G_6, where $|G_6| = p^\alpha qr^p$ and $p^\alpha q \mid r-1$, $p \mid q-1$, $\alpha \geq 2$. The proof is completed. □

Corollary 3.8. *[9, Theorem 2] Let G be a group with the property that maximal subgroups of Sylow subgroups are π-quasinormal in G for $\pi = \pi(G)$. Then G is supersolvable.*

Proof. By the proof of Theorem 3.6 and Theorem 3.7, we only need to check that $G \cong G_1$ and $G \cong G_{6'}$, where $|G_{6'}| = pqr^p$ and $pq \mid r-1$, $p \mid q-1$. Assume that $G \cong G_1$. By Lemma 2.3, we have $G_1 = PQ$, where $|P| = p$ and $|Q| = q^\beta (\beta \geq 2)$. By Step 2 and Step 3 of Theorem 3.6, Q is a minimal normal subgroup of G_1. Choosing Q_1 to be a maximal subgroup of Q, by the hypothesis, we obtain that Q_1 is π-quasinormal in G_1. Then $O^q(G) \leq N_G(Q_1)$, so P normalizes Q_1, and thus $1 \neq Q_1 \lhd G$, contrary to the minimality of Q. Hence $G \ncong G_1$. Using a similar argument as above, we also get that G is not isomorphic to $G_{6'}$. The proof is completed. □

Corollary 3.9 ([9, Theorem 1])**.** *Let G be a group with the property that maximal subgroups of Sylow subgroups are normal in G. Then G is supersolvable.*

Theorem 3.10. *Let G be a $QCLT$-group. If every maximal subgroup of a Sylow 2-subgroup of G is CSQ-normal in G, then G is supersolvable.*

Proof. Assume that the Theorem is false and let G be a counterexample of smallest order.

Assume first that G has odd order. Since G is a $QCLT$-group, by [6], we have that G is supersolvable. Now we assume that $2 \mid |G|$. By Lemma 3.5, we have that G is solvable. For any $1 \neq N \trianglelefteq G$, if $2 \nmid |G/N|$, then G/N is a $QCLT$-group of odd order and hence G/N is supersolvable. Suppose that $2 \mid |G/N|$. Without loss of generality, we assume that every maximal subgroup of a Sylow 2-subgroup of G/N is of the form P_1N/N, where P_1 is a maximal subgroup of a Sylow 2-subgroup of G. Then P_1 is CSQ-normal in G by hypothesis, so P_1N/N is CSQ-normal in G/N by Lemma 2.1 (b). Hence the quotient group G/N satisfies the hypothesis. By the choice of G, we have that G is a solvable outer-supersolvable group. Then, by [7, Theorem 7.1], $G = ML$, where M is a maximal subgroup of G, $M \cap L = 1$, L is an elementary abelian p-group and is also the unique minimal normal subgroup of G with order p^{α}, $\alpha > 1$, the Sylow p-subgroup of M is an abelian p-group and $\Phi(G) = 1$.

If $|G_2| \leq 4$, where $G_2 \in Syl_2(G)$, then G_2 is a cyclic subgroup or an elementary abelian 2-subgroup. It follows that G is S_4-free, then G is supersolvable by [10, Theorem 4], a contradiction. Hence we may choose $1 \neq P_1$ to be a maximal subgroup of G_2. By hypothesis, P_1 is a CSQ-normal subgroup of G. Then P_1 is subnormal in G by Lemma 2.4, thus $1 \neq P_1 \leq O_2(G)$, hence $L \leq O_2(G)$, so we get $p = 2$. By [7, §6.1, Main lemma], we also get $O_2(G) = F(G) = L$.

Let M_2 be a Sylow 2-subgroup of M. Then $G_2 = M_2L$ is a Sylow 2-subgroup of G. Assume that P_1 is a maximal subgroup of M_2N containing M_2. Then $M_2 < P_1$ since $|L| = 2^{\alpha}$, where $\alpha > 1$. Then P_1 is CSQ-normal in G by the hypothesis, so P_1 is subnormal in G by Lemma 2.4. Thus $P_1 \leq O_2(G) = L$, hence $G_2 = M_2L = P_1L = L$ is an elementary abelian Sylow 2-subgroup of G. It follows that G is S_4-free, so G is supersolvable by [10, Theorem 4], a contradiction. Hence the minimal counterexample does not exist. Therefore G is supersolvable. □

Theorem 3.11. *Let G be a $QCLT$-group. If every 2-maximal subgroup of a Sylow 2-subgroup of G is CSQ-normal in G. Then G is supersolvable.*

Proof. The proof is similar to Theorem 3.10 and omitted here. □

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