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Singularities of lightcone pedals of spacelike curves in Lorentz-Minkowski 3-space

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Abstract: In this paper, geometric properties of spacelike curves on a timelike surface in Lorentz-Minkowski 3-space are investigated by applying the singularity theory of smooth functions from the contact viewpoint.

Keywords: Singularities, Spacelike curve, Pedal curve, Height function, Contact

MSC: 53A35, 58k05

1 Introduction

This paper is written as a part of our research project on the study of Lorentz pairs in semi-Euclidean space with index 2 from the viewpoint of Lagrangian/Legendrian singularity theory. Our aim is to investigate the geometric properties of different Lorentzian pairs by constructing a unified way. A Lorentzian pair consists of a Lorentzian hypersurface W in semi-Euclidean space with index two and a timelike hypersurface M in W . AdS^4/AdS^5 , for example, is a Lorentzian pair which is one of the space-time models in physics. As the first step of this research, we consider the simplest Lorentzian pair, i.e., a timelike curve on a Lorentzian surface in semi-Euclidean 3-space with index 2. However, a Lorentz-Minkowski 3-space is diffeomorphic to a semi-Euclidean 3-space with index 2, although the causalities of these two spaces are different. For the geometric properties, we can investigate a spacelike curve on a timelike surface in Lorentz-Minkowski 3-space instead of a timelike curve on a Lorentzian surface in semi-Euclidean 3-space with index 2.

On the other hand, singularity theory tools are useful in the investigation of geometric properties of submanifolds immersed in different ambient spaces, from both the local and global viewpoint [1–16]. The natural connection between geometry and singularities relies on the basic fact that the contacts of a submanifold with the models (invariant under the action of a suitable transformation group) of the ambient space can be described by means of the analysis of the singularities of appropriate families of contact functions, or equivalently, of their associated Lagrangian and/or Legendrian maps. This is our main motivation for the investigation of spacelike curves on a timelike surface in Lorentz-Minkowski 3-space from the viewpoint of singularity theory.

The organization of the paper is as follows. We construct the framework of local differential geometry of spacelike curves on a timelike surface in Section 2. We give the Frenet-Serret type formula corresponding to the spacelike curves. Moreover, we define the lightcone Gauss image and the normalized Gauss map. We also define new invariants K_L^\pm and \tilde{K}_L^\pm and call them lightcone Gauss-Kronecker curvature and normalized lightcone Gauss-Kronecker curvature, respectively. We investigate their relations. We can prove that these two Gauss-Kronecker curvature functions have the same zero sets. In Section 3 we introduce the notion of height functions on spacelike curves on a timelike surface which are useful to show that the normalized lightcone Gauss map has a singular point if and only if the lightcone Gauss-Kronecker curvature vanishes at such point. According to the general results on

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the singularity theory for families of function germs (cf. [17]), we study the relationship of these height functions (cf. Theorem 3.5). In the last section, we define a curve in the lightcone, named the lightcone pedal, as a tool to study the geometric properties of singularities of the normalized lightcone Gauss map from the contact viewpoint.

2 The local differential geometry of spacelike curves on a timelike surface

In this section, we investigate the basic ideas on semi-Euclidean $(n+1)$ -space with index two and the local differential geometry of Lorentzian pairs in semi-Euclidean $(n+1)$ -space. For details about semi-Euclidean geometry, see [18].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_i \in \mathbb{R}, i = 1, 2, 3\}$ be a 3-dimensional vector space. For any vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in \mathbb{R}^3 , the *pseudo scalar product* of \mathbf{x} and \mathbf{y} is defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$. We call $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ the *Minkowski 3-space* and write \mathbb{R}_1^3 instead of $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$.

We say that a non-zero vector \mathbf{x} in \mathbb{R}_1^3 is *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. The *norm* of the vector $\mathbf{x} \in \mathbb{R}_1^3$ is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$.

For any $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}_1^3$, we define a vector $\mathbf{x} \wedge \mathbf{y}$ by

$$\mathbf{x} \wedge \mathbf{y} = \begin{vmatrix} -\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the canonical basis of \mathbb{R}_1^3 . For any $\mathbf{w} \in \mathbb{R}_1^3$, we can easily check that

$$\langle \mathbf{w}, \mathbf{x} \wedge \mathbf{y} \rangle = \det(\mathbf{w}, \mathbf{x}, \mathbf{y}),$$

so that $\mathbf{x} \wedge \mathbf{y}$ is pseudo-orthogonal to both \mathbf{x} and \mathbf{y} . Moreover, by a straightforward calculation, we have the following simple lemma.

Lemma 2.1. *For any non-zero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}_1^3$, we assume that $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and $\mathbf{x} \wedge \mathbf{y} = \mathbf{z}$. Then we have the following assertions:*

- (1) *If \mathbf{x} is a timelike vector, \mathbf{y} is a spacelike vector, then $\mathbf{z} \wedge \mathbf{x} = \mathbf{y}$, $\mathbf{y} \wedge \mathbf{z} = -\mathbf{x}$.*
- (2) *If \mathbf{x} is a spacelike vector and \mathbf{y} is a timelike vector, then $\mathbf{z} \wedge \mathbf{x} = -\mathbf{y}$, $\mathbf{y} \wedge \mathbf{z} = \mathbf{x}$.*
- (3) *If both \mathbf{x} and \mathbf{y} are spacelike vectors, then $\mathbf{z} \wedge \mathbf{x} = -\mathbf{y}$, $\mathbf{y} \wedge \mathbf{z} = -\mathbf{x}$.*

For a vector $\mathbf{v} \in \mathbb{R}_1^3$ and a real number c , we define the *plane* with the pseudo-normal \mathbf{v} by

$$P(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

We call $P(\mathbf{v}, c)$ a *timelike plane*, *spacelike plane* or *lightlike plane* if \mathbf{v} is *spacelike*, *timelike* or *lightlike*, respectively.

We define the *hyperbolic 2-space* by

$$H^2(-1) = \{\mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{x} \rangle = -1\},$$

the *de Sitter 2-space* by

$$S_1^2 = \{\mathbf{x} \in \mathbb{R}_1^3 | \langle \mathbf{x}, \mathbf{x} \rangle = 1\},$$

the *(open) lightcone* at the origin by

$$LC^* = \{\mathbf{x} \in \mathbb{R}_1^3 \setminus \{\mathbf{0}\} | \langle \mathbf{x}, \mathbf{x} \rangle = 0\}.$$

We call $LC_+^* = \{\mathbf{x} \in LC^* | x_1 > 0\}$ the *future lightcone*. We also define the *spacelike lightcone circle* by

$$S_+^1 = \{\mathbf{x} = (x_1, x_2, x_3) \in LC^* | x_1 = 1\}.$$

For any lightlike vector $\mathbf{x} = (x_1, x_2, x_3) \in LC^*$, we have

$$\tilde{\mathbf{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) = \frac{1}{x_1} \mathbf{x} \in S_+^1.$$

We study the local differential geometry of spacelike curves on a timelike surface as follows. Firstly, let $Y : V \rightarrow \mathbb{R}_1^3$ be a regular surface (i.e., an embedding), where $V \subset \mathbb{R}^2$ is an open subset. We denote $W = Y(V)$ and identify W with V via the embedding Y . The embedding Y is said to be timelike if the induced metric \mathbf{I} of W is Lorentzian. Throughout the remainder of this paper we assume that W is a timelike surface in \mathbb{R}_1^3 . We define a vector $N(v)$ by

$$N(v) = \frac{Y_{v_1}(v) \wedge Y_{v_2}(v)}{\|Y_{v_1}(v) \wedge Y_{v_2}(v)\|}.$$

By the definition of wedge product, we have $\langle N(v), Y_{v_i}(u) \rangle \equiv 0$ (for $i = 1, 2$). This means that $N(v) \in N_q W$, where $v = (v_1, v_2) \in V$, $q = Y(v) \in W$ and $N_q W$ denotes the normal space of W in \mathbb{R}_1^3 at q . Since the embedding Y is timelike, N is unit spacelike (i.e., $N(v) \in S_1^2$). Moreover, we define a regular curve on W by $\gamma : I \rightarrow W$, where $I \subset \mathbb{R}$ is an open interval. We call this curve the *spacelike curve*, if $\mathbf{t}(t) = (d\gamma/dt)(t)$ is spacelike at any point $t \in I$, and denote $\gamma(I) = M$. Since γ is a regular spacelike curve on W , it may admit the arc length parametrization $s = s(t)$. Therefore, we can assume that γ is a unit speed spacelike curve, namely, $\mathbf{t}(s) = \gamma'(s) = (d\gamma/ds)(s) \in S_1^2$. Throughout the remainder in this paper we assume that M is a spacelike curve on the timelike surface W .

We define a smooth mapping $\bar{\gamma} : I \rightarrow V$ by $\bar{\gamma}(s) = v$. For any $s \in I$, we have $\gamma(s) = Y(\bar{\gamma}(s))$. It follows that the unit spacelike normal vector field of W along γ can be defined by $\mathbf{n}(s) = N(\bar{\gamma}(s))$, for any $s \in I$. We can also define another unit normal vector field \mathbf{e} by $\mathbf{e}(s) = \mathbf{n}(s) \wedge \mathbf{t}(s)$. Since \mathbf{t} and \mathbf{n} are spacelike, \mathbf{e} is timelike. Then we have a pseudo-orthonormal frame $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ of \mathbb{R}_1^3 along $\gamma(s)$. By a straightforward calculation, we arrive at the following *Frenet-Serret type formula*:

$$\begin{cases} \mathbf{t}'(s) = \kappa_n(s)\mathbf{n}(s) - \kappa_g(s)\mathbf{e}(s), \\ \mathbf{n}'(s) = -\kappa_n(s)\mathbf{t}(s) + \tau_g(s)\mathbf{e}(s), \\ \mathbf{e}'(s) = -\kappa_g(s)\mathbf{t}(s) + \tau_g(s)\mathbf{n}(s), \end{cases}$$

where $\kappa_n(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle$, $\kappa_g(s) = \langle \mathbf{t}'(s), \mathbf{e}(s) \rangle$, $\tau_g(s) = \langle \mathbf{e}'(s), \mathbf{n}(s) \rangle$. We call them the *normal curvature*, *geodesic curvature* and *geodesic torsion* of γ at point $p = \gamma(s)$, respectively. We remark that γ is a spacelike geodesic curve on W if and only if $\kappa_g \equiv 0$; γ is a spacelike asymptotic curve on W if and only if $\kappa_n \equiv 0$; γ is a spacelike principal curve on W if and only if $\tau_g \equiv 0$.

Let $\mathbf{n}(s) = (n_1(s), n_2(s), n_3(s))$ and $\mathbf{e}(s) = (e_1(s), e_2(s), e_3(s))$. Since $\mathbf{n}(s) \pm \mathbf{e}(s) \in LC^*$, we can define a mapping $\mathbb{G}_L^\pm : I \rightarrow LC^*$ by $\mathbb{G}_L^\pm(s) = \mathbf{n}(s) \pm \mathbf{e}(s)$. We call it the *lightcone Gauss image*. Moreover, we define another mapping $\tilde{\mathbb{G}}_L^\pm : I \rightarrow S_+^1$ by $\tilde{\mathbb{G}}_L^\pm(s) = \frac{1}{\xi^\pm(s)} \mathbb{G}_L^\pm(s)$, where $\xi^\pm(s) = n_1(s) \pm e_1(s)$. We call it the *normalized lightcone Gauss map*. By a straightforward calculation, we have

$$\mathbb{G}_L^{\pm'}(s) = \mathbf{n}'(s) \pm \mathbf{e}'(s) = -(\kappa_n(s) \pm \kappa_g(s))\mathbf{t}(s) \pm \tau_g(s)\mathbb{G}_L^\pm(s).$$

Let $\pi^T : T_p M \oplus N_p M \rightarrow T_p M$. It follows that $-\pi^T \circ \mathbb{G}_L^{\pm'}(s) = (\kappa_n(s) \pm \kappa_g(s))\mathbf{t}(s)$. Moreover, we also have $-\pi^T \circ \tilde{\mathbb{G}}_L^{\pm'}(s) = \frac{1}{\xi^\pm(s)}(\kappa_n(s) \pm \kappa_g(s))\mathbf{t}(s)$. According to the above calculations, we can define new invariants K_L^\pm and \tilde{K}_L^\pm by $K_L^\pm(s) = \kappa_n(s) \pm \kappa_g(s)$ and $\tilde{K}_L^\pm(s) = \frac{1}{\xi^\pm(s)}(\kappa_n(s) \pm \kappa_g(s))$, respectively. We call them the *lightcone Gauss-Kronecker curvature* (or, *lightcone G-K curvature*) of M and *normalized lightcone Gauss-Kronecker curvature* (or, *normalized lightcone G-K curvature*) of M , respectively. By the definitions, we have $K_L^\pm(s) = 0$ if and only if $\tilde{K}_L^\pm(s) = 0$, for $s \in I$.

For $\mathbf{v} \in S_+^1$, $c \in \mathbb{R}$, we denote $S_L(\mathbf{v}, c) = P(\mathbf{v}, c) \cap W$ and call it the *lightlike slice*. Then we have the following proposition.

Proposition 2.2. *Using the above notations, the following conditions are equivalent:*

- (1) $K_L^\pm \equiv 0$.
- (2) $\tilde{\mathbb{G}}_L^\pm$ is constant.

(3) There exists a constant lightlike vector $\mathbf{v} \in S_+^1$ such that $\text{Im}\boldsymbol{\gamma} \subset S_L(\mathbf{v}, c)$.

Proof. We first assume that $K_L^\pm \equiv 0$. This condition is equivalent to $\widetilde{K}_L^\pm \equiv 0$. It follows that

$$\widetilde{\mathbb{G}}_L^{\pm'}(s) = -\frac{\xi^{\pm'}(s)}{\xi^{\pm 2}(s)}\mathbb{G}_L^\pm(s) \pm \frac{\tau_g(s)}{\xi^\pm(s)}\mathbb{G}_L^\pm(s) = \left(-\frac{\xi^{\pm'}(s)}{\xi^\pm(s)} \pm \tau_g(s)\right)\widetilde{\mathbb{G}}_L^\pm(s).$$

Since the first component of $\widetilde{\mathbb{G}}_L^\pm(s)$ is 1, the first component of $\widetilde{\mathbb{G}}_L^{\pm'}(s)$ is 0. Therefore $\widetilde{\mathbb{G}}_L^{\pm'}(s) = 0$ for any $s \in I$. It follows that $\widetilde{\mathbb{G}}_L^\pm$ is constant. Moreover, if $\widetilde{\mathbb{G}}_L^\pm$ is constant, then $K_L^\pm \equiv 0$. Thus the conditions (1) and (2) are equivalent.

On the other hand, suppose that $\widetilde{\mathbb{G}}_L^\pm$ is constant. We assume that $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm$. Then $\langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle' = 0$. This means that $\langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle = c$, where $c \in \mathbb{R}$ is a constant real number. Therefore, $\boldsymbol{\gamma}$ is a part of $P(\mathbf{v}, c) \cap W$. Moreover, we assume that there exists a constant lightlike vector $\mathbf{v} \in S_+^1$ such that $\text{Im}\boldsymbol{\gamma} \subset P(\mathbf{v}, c) \cap W$. It follows that $\langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle = c$. Therefore, $\langle \mathbf{t}(s), \mathbf{v} \rangle = 0$. This means that $\mathbf{v} \in N_P M \cap S_+^1$. Thus, $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s)$ for any $s \in I$. It follows that the conditions (2) and (3) are equivalent. This completes the proof. \square

As an application of the above proposition, we have the following corollary.

Corollary 2.3. Using the same notations, with the above proposition, s_0 is a singular point of the normalized lightcone Gauss map $\widetilde{\mathbb{G}}_L^\pm$ if and only if $K_L^\pm(s_0) = 0$.

3 Height functions on spacelike curves

In this section we define three families of functions on the spacelike curve $\boldsymbol{\gamma}$ on W which are helpful for investigating the geometric properties of the spacelike curve.

Let $\boldsymbol{\gamma} : I \rightarrow W$ be a spacelike curve on W . We first define a function $H : I \times S_+^1 \rightarrow \mathbb{R}$ by $H(s, \mathbf{v}) = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle$. We call it the *lightcone circle height function* on $\boldsymbol{\gamma}$. For any fixed $\mathbf{v} \in S_+^1$, we denote $h_v(s) = H(s, \mathbf{v})$. Then we have the following proposition.

Proposition 3.1. Suppose that $\boldsymbol{\gamma} : I \rightarrow W$ is a unit-speed spacelike curve on W . Then we have the following assertions:

- (1) $h_v'(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$.
- (2) $h_v'(s_0) = h_v''(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$ and $K_L^\pm(s_0) = 0$.
- (3) $h_v'(s_0) = h_v''(s_0) = h_v'''(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $K_L^\pm(s_0) = 0$ and $K_L^{\pm'}(s_0) = 0$.

Proof. Since $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{e}(s)\}$ is a pseudo orthonormal frame of \mathbb{R}_1^3 along $\boldsymbol{\gamma}(s)$ and $\mathbf{v} \in S_+^1$, there exist $\eta, \alpha, \lambda \in \mathbb{R}$ with $\eta^2 + \alpha^2 - \lambda^2 = 0$ such that $\mathbf{v} = \eta\mathbf{t}(s) + \alpha\mathbf{n}(s) + \lambda\mathbf{e}(s)$. By the definition of $h_v(s)$, we can show that $h_v'(s_0) = 0$ if and only if $\eta = 0$. Therefore, we have $\mathbf{v} = \alpha(\mathbf{n}(s_0) \pm \mathbf{e}(s_0))$. Since $\mathbf{v} \in S_+^1$, we have $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$.

(2) Since $h_v'(s_0) = 0$, we have $\langle \mathbf{t}'(s_0), \mathbf{v} \rangle = 0$. By the assertion (1) and the Frenet-Serret type formula, we have $h_v'(s_0) = h_v''(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $\kappa_n(s_0) \pm \kappa_g(s_0) = 0$. Therefore, $K_L^\pm(s_0) = 0$.

(3) Since $h_v'''(s_0) = 0$, we have $\langle \mathbf{t}''(s_0), \mathbf{v} \rangle = 0$. By the assertions (1), (2) and the Frenet-Serret type formula, we have $h_v'(s_0) = h_v''(s_0) = h_v'''(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $K_L^\pm(s_0) = 0$, $K_L^{\pm'}(s_0) = 0$. \square

Moreover, we define another function $\overline{H} : I \times S_+^1 \times \mathbb{R}^* \rightarrow \mathbb{R}$ by $\overline{H}(s, \mathbf{v}, r) = \langle \boldsymbol{\gamma}(s), \mathbf{v} \rangle - r$. We call it the *extended lightcone circle height function*. For any fixed $\mathbf{v} \in S_+^1$, $r \in \mathbb{R}^*$, we have $\overline{h}_{(v,r)}(s) = \overline{H}(s, \mathbf{v}, r)$. By almost the same arguments as in above proposition, we have the following proposition.

Proposition 3.2. Suppose that $\boldsymbol{\gamma} : I \rightarrow W$ is a unit speed spacelike curve on W . Then we have the following assertions:

- (1) $\overline{h}_{(v,r)}(s_0) = \overline{h}'_{(v,r)}(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $r = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle$.
- (2) $\overline{h}_{(v,r)}(s_0) = \overline{h}'_{(v,r)}(s_0) = \overline{h}''_{(v,r)}(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $r = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle$ and $K_L^\pm(s_0) = 0$.

- (3) $\bar{h}_{(v,r)}(s_0) = \cdots = \bar{h}_{(v,r)}'''(s_0) = 0$ if and only if $\mathbf{v} = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $r = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle$, $K_L^\pm(s_0) = 0$ and $K_L^{\pm'}(s_0) = 0$.

Furthermore, we define the third function $\widetilde{H} : I \times LC^* \rightarrow \mathbb{R}$ by $\widetilde{H}(s, \mathbf{v}) = \langle \boldsymbol{\gamma}(s), \widetilde{\mathbf{v}} \rangle - v_1$, where $\mathbf{v} = (v_1, v_2, v_3) \in LC^*$. We call it the *lightcone height function* on $\boldsymbol{\gamma}$. For any fixed $\mathbf{v} \in LC^*$, we have $\widetilde{h}_v(s) = \widetilde{H}(s, \mathbf{v})$. By a straightforward calculation, we have the following proposition.

Proposition 3.3. *Suppose that $\boldsymbol{\gamma} : I \rightarrow W$ is a unit speed spacelike curve on W . Then we have the following assertions:*

- (1) $\widetilde{h}_v(s_0) = \widetilde{h}_v'(s_0) = 0$ if and only if $\mathbf{v} = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle \widetilde{\mathbb{G}}_L^\pm(s_0)$.
- (2) $\widetilde{h}_v(s_0) = \widetilde{h}_v'(s_0) = \widetilde{h}_v''(s_0) = 0$ if and only if $\mathbf{v} = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle \widetilde{\mathbb{G}}_L^\pm(s_0)$ and $K_L^\pm(s_0) = 0$.
- (3) $\widetilde{h}_v(s_0) = \cdots = \widetilde{h}_v'''(s_0) = 0$ if and only if $\mathbf{v} = \langle \boldsymbol{\gamma}(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle \widetilde{\mathbb{G}}_L^\pm(s_0)$, $K_L^\pm(s_0) = 0$ and $K_L^{\pm'}(s_0) = 0$.

On the other hand, we will introduce some general results on the singularity theory for families of function germs as follows. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f , where $f(s) = F(s, \mathbf{x}_0)$. We say that f has an A_k -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has an $A_{\geq k}$ -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ have an A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k-1)$ -jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{k-1}(\frac{\partial F}{\partial x_i}(s, \mathbf{x}_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji} s^j$ for $i = 1, \dots, r$. Then F is called an \mathcal{R}^+ -versal unfolding if and only if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k-1 \leq r$). Moreover, we call F an \mathcal{R} -versal unfolding if and only if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k ($k \leq r$), where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, \mathbf{x}_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *discriminant set* of F is the set $\mathcal{D}_F = \{\mathbf{x} \in \mathbb{R}^r \mid \text{there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, \mathbf{x})\}$.

The *catastrophe set* of F is the set

$$\mathcal{C}_F = \{(s, \mathbf{x}) \mid \frac{\partial F}{\partial s}(s, \mathbf{x}) = 0\}.$$

The *bifurcation set* of F is the set $\mathcal{B}_F = \{\mathbf{x} \in \mathbb{R}^r \mid \text{there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, \mathbf{x})\}$. Then we have the following well-known result (cf. [17]).

Theorem 3.4. *Let $F : (\mathbb{R} \times \mathbb{R}^2, (s_0, \mathbf{x}_0)) \rightarrow \mathbb{R}$ be a 2-parameter unfolding of $f(s)$ which has an A_2 -singularity at s_0 . Suppose that F is a \mathcal{R} -versal unfolding. Then \mathcal{D}_F is locally diffeomorphic to C . Here, $C = \{(x_1, x_2) \mid x_1^2 - x_2^3 = 0\}$ is the ordinary cusp.*

Now we can apply the above arguments to our case. Let $\boldsymbol{\gamma} : I \rightarrow W$ be a unit speed spacelike curve on W , H be the lightcone circle height function on $\boldsymbol{\gamma}$, \bar{H} be the extended lightcone circle height function on $\boldsymbol{\gamma}$ and \widetilde{H} be the lightcone height function on $\boldsymbol{\gamma}$. According to Proposition 3.1, we have $\mathcal{C}_H = \{(s, \widetilde{\mathbb{G}}_L^\pm(s)) \mid s \in I\}$. By Propositions 3.2 and 3.3, we have $\mathcal{D}_{\bar{H}} = \{(\widetilde{\mathbb{G}}_L^\pm(s), \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{G}}_L^\pm(s) \rangle) \mid s \in I\}$ and $\mathcal{D}_{\widetilde{H}} = \{(\boldsymbol{\gamma}(s), \widetilde{\mathbb{G}}_L^\pm(s)) \mid s \in I\}$. Then we have the following proposition.

Proposition 3.5. *Let $\boldsymbol{\gamma} : I \rightarrow W$ be a regular spacelike curve on W , H be the lightcone circle height function on $\boldsymbol{\gamma}$ and \bar{H} be the extended lightcone circle height function on $\boldsymbol{\gamma}$. Suppose that $\mathbf{v}_0 = \widetilde{\mathbb{G}}_L^\pm(s_0)$ and $h_{v_0}(s) = H(s, \mathbf{v}_0)$. Then the following assertions are equivalent.*

- (1) h_{v_0} has A_2 -singularity at s_0 .
- (2) H is a \mathcal{R}^+ -versal unfolding of h_{v_0} .
- (3) \bar{H} is a \mathcal{R} -versal unfolding of h_{v_0} .

Proof. Let $\mathbf{v} = (1, \cos \theta, \sin \theta) \in S_+^1$ and $\boldsymbol{\gamma}(s) = (r_1(s), r_2(s), r_3(s)) \in W$, we have

$$H(s, \mathbf{v}) = -r_1(s) + r_2(s) \cos \theta + r_3(s) \sin \theta = H(s, \theta).$$

By a straightforward calculation, we have

$$\frac{\partial H}{\partial s}(s, \theta) = -r'_1(s) + r'_2(s) \cos \theta + r'_3(s) \sin \theta, \quad \frac{\partial H}{\partial \theta}(s, \theta) = -r_2(s) \sin \theta + r_3(s) \cos \theta.$$

It follows that

$$\frac{\partial^2 H}{\partial s \partial \theta}(s, \theta) = -r'_2(s) \sin \theta + r'_3(s) \cos \theta.$$

For $s = s_0$, we suppose that $-r'_2(s_0) \sin \theta_0 + r'_3(s_0) \cos \theta_0 = 0$ and $\mathbf{t}(s_0) = (r'_1(s_0), r'_2(s_0), r'_3(s_0))$. Since $\langle \mathbf{t}(s_0), \mathbf{v}_0 \rangle = -r'_1(s_0) + r'_2(s_0) \cos \theta_0 + r'_3(s_0) \sin \theta_0 = 0$, we have $-r'_1(s_0) \sin \theta_0 + r'_3(s_0) = 0$. Moreover, we also have $-r'_1(s_0) \cos \theta_0 + r'_2(s_0) = 0$. Therefore, $\mathbf{t}(s_0) = r'_1(s_0)(1, \cos \theta_0, \sin \theta_0)$. It follows that $\mathbf{t}(s)$ is a unit spacelike vector for any $s \in I$, so we have a contradiction. Thus, $(\partial^2 H / \partial s \partial \theta)(s_0, \theta_0) \neq 0$. Therefore, the rank of the matrix $(-r'_2(s_0) \sin \theta_0 + r'_3(s_0) \cos \theta_0)$ is one. We have thus shown that the assertions (1) and (2) are equivalent.

On the other hand, for extended lightcone circle height function \overline{H} , we have

$$\overline{H}(s, v, r) = -r_1(s) + r_2(s) \cos \theta + r_3(s) \sin \theta - r = \overline{H}(s, \theta, r).$$

Thus,

$$\frac{\partial \overline{H}}{\partial r}(s, \theta, r) = -1, \quad \frac{\partial \overline{H}}{\partial \theta}(s, \theta, r) = -r_2(s) \sin \theta + r_3(s) \cos \theta.$$

It follows that

$$\frac{\partial^2 \overline{H}}{\partial s \partial r}(s, \theta, r) = 0, \quad \frac{\partial^2 \overline{H}}{\partial s \partial \theta}(s, \theta, r) = -r'_2(s) \sin \theta + r'_3(s) \cos \theta = \frac{\partial^2 H}{\partial s \partial \theta}(s, \theta).$$

Suppose that

$$\begin{aligned} A &= \begin{pmatrix} (\partial \overline{H} / \partial \theta)(s_0, \theta_0, r_0) & (\partial^2 \overline{H} / \partial s \partial \theta)(s_0, \theta_0, r_0) \\ (\partial \overline{H} / \partial r)(s_0, \theta_0, r_0) & (\partial^2 \overline{H} / \partial s \partial r)(s_0, \theta_0, r_0) \end{pmatrix} \\ &= \begin{pmatrix} -r_2(s_0) \sin \theta_0 + r_3(s_0) \cos \theta_0 & -r'_2(s_0) \sin \theta_0 + r'_3(s_0) \cos \theta_0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Since $(\partial^2 H / \partial s \partial \theta)(s_0, \theta_0) \neq 0$, we have $(\partial^2 \overline{H} / \partial s \partial \theta)(s_0, \theta_0, r_0) \neq 0$. This means that $\text{rank } A = 2$. Therefore, the assertions (1) and (3) are equivalent. This completes the proof. \square

4 Lightcone pedal curves

We now define a mapping $P_L^\pm : I \rightarrow LC^*$ by $P_L^\pm(s) = \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{G}}_L^\pm(s) \rangle \widetilde{\mathbb{G}}_L^\pm(s)$. We call it the *lightcone pedal curve*. We also define another mapping $CP_L^\pm : I \rightarrow S_+^1 \times \mathbb{R}^*$ by $CP_L^\pm(s) = (\widetilde{\mathbb{G}}_L^\pm(s), \langle \boldsymbol{\gamma}(s), \widetilde{\mathbb{G}}_L^\pm(s) \rangle)$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. We call CP_L^\pm the *cylindrical lightcone pedal curve*. By definitions, we have $\{P_L^\pm(s) | s \in I\} = \mathcal{D}_{\widetilde{H}}$ and $\{CP_L^\pm(s) | s \in I\} = \mathcal{D}_{\overline{H}}$. In order to investigate the relationship between the lightcone pedal curve and the cylindrical lightcone pedal, we define a mapping $\phi : S_+^1 \times \mathbb{R}^* \rightarrow LC^*$ by $\phi(v, r) = rv$. It is easy to check that ϕ is a diffeomorphism and $\phi(CP_L^\pm(s)) = P_L^\pm(s)$, this means that CP_L^\pm and P_L^\pm are diffeomorphic. Therefore, the singular sets of $\mathcal{D}_{\widetilde{H}}$ and $\mathcal{D}_{\overline{H}}$ are diffeomorphic.

On the other hand, let $F : W \rightarrow \mathbb{R}$ be a submersion and $\boldsymbol{\gamma} : I \rightarrow W$ be a spacelike curve on W . We say that $\boldsymbol{\gamma}$ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$ provided the function g defined by $g(t) = F(\boldsymbol{\gamma}(t))$ satisfies

$$g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0, \quad g^{(k)}(t_0) \neq 0.$$

We also say that the *order of contact* is k . Dropping the condition $g^{(k)}(t_0) \neq 0$ we say that there is *at least k-point contact* (cf. [17]). Then we have the following result.

Theorem 4.1. Let $\gamma : I \rightarrow W$ be a spacelike curve on W . For $v_0^\pm = \widetilde{\mathbb{G}}_L^\pm(s_0)$, $s_0 \in I$, we assume that $h_{v_0^\pm} : I \rightarrow \mathbb{R}$ is the lightlike circle height function and $P_L^\pm : I \rightarrow LC^*$ is the lightcone pedal curve. Then we have the following:

(A) The following conditions are equivalent.

- (1) $K_L^\pm(s_0) \neq 0$.
- (2) $\widetilde{\mathbb{G}}_L^\pm$ is non-singular at s_0 .
- (3) $h_{v_0^\pm}$ has A_1 singular point.
- (4) γ and $S_L^\pm(\widetilde{\mathbb{G}}_L^\pm(s_0), \langle \gamma(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle)$ have 2-point contact at s_0 .
- (5) P_L^\pm is an immersion at s_0 .

(B) The following conditions are equivalent.

- (1) $K_L^\pm(s_0) = 0$ and $K_L^{\pm'}(s_0) \neq 0$.
- (2) $\widetilde{\mathbb{G}}_L^\pm$ has A_1 singular point at s_0 .
- (3) $h_{v_0^\pm}$ has A_2 singular point.
- (4) γ and $S_L^\pm(\widetilde{\mathbb{G}}_L^\pm(s_0), \langle \gamma(s_0), \widetilde{\mathbb{G}}_L^\pm(s_0) \rangle)$ have 3-point contact at s_0 .
- (5) P_L^\pm is an ordinary cusp at s_0 .
- (6) \overline{H} is a \mathcal{R} -versal unfolding of $h_{v_0^\pm}$.
- (7) H is a \mathcal{R}^+ -versal unfolding of $h_{v_0^\pm}$.

Proof. We first consider (A). According to Corollary 2.3, the assertions (1) and (2) are equivalent. By Proposition 3.1 (1), the $K_L^\pm(s_0) \neq 0$ if and only if $(\partial h_{v_0}/\partial s)(s_0) = 0$ and $(\partial^2 h_{v_0}/\partial s^2)(s_0) \neq 0$. This means that the assertions (1) and (3) are equivalent. Moreover, we define a mapping $\overline{H} : W \rightarrow \mathbb{R}$ by $\overline{H}(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v}_0 \rangle - r_0$, where $\mathbf{v}_0 \in S_+^1$, $r_0 \in \mathbb{R}$. It follows that $\overline{H}(\gamma(s)) = h_{v_0}(s) - r_0$. Since $\overline{H}^{-1}(0) = S_L(\mathbf{v}_0, r_0)$ and 0 is a regular value of \overline{H} , h_{v_0} has an A_k -singularity at s_0 if and only if γ and $S_L(\mathbf{v}_0, r_0)$ have $(k+1)$ -point contact for s_0 . This means that the assertions (3) and (4) are equivalent. By the definition, we have $CP_L^{\pm'}(s) = (\widetilde{\mathbb{G}}_L^\pm(s), \langle \gamma(s), \widetilde{\mathbb{G}}_L^\pm(s) \rangle)$. It follows that $CP_L^{\pm'}(s) \neq 0$ if and only if $\widetilde{\mathbb{G}}_L^{\pm'}(s) \neq 0$. Since the singular sets of CP_L^\pm and P_L^\pm are diffeomorphic, we have $CP_L^{\pm'}(s) \neq 0$ if and only if $P_L^{\pm'}(s) \neq 0$. Therefore, the assertions (2) and (5) are equivalent.

On the other hand, we consider (B). By Proposition 3.1 (2), the assertions (1) and (3) are equivalent. Moreover, by Proposition 3.1 (1), we have $(\partial H/\partial s)(s, \widetilde{\mathbb{G}}_L^\pm(s)) = 0$. If we take the derivative of the equation, then we have

$$\frac{d}{ds} \left(\frac{\partial H}{\partial s} \right) (s, \widetilde{\mathbb{G}}_L^\pm(s)) = \frac{\partial^2 H}{\partial s^2} (s, \widetilde{\mathbb{G}}_L^\pm(s)) + \frac{\partial^2 H}{\partial s \partial v} (s, \widetilde{\mathbb{G}}_L^\pm(s)) \widetilde{\mathbb{G}}_L^{\pm'}(s) = 0.$$

For $s = s_0$, we have $h_{v_0}''(s_0) + (\partial^2 H/\partial s \partial v)(s_0, \widetilde{\mathbb{G}}_L^\pm(s_0)) \widetilde{\mathbb{G}}_L^{\pm'}(s_0) = 0$, where $\mathbf{v}_0 = \widetilde{\mathbb{G}}_L^\pm(s_0)$. By the proof of Proposition 3.5, we have $(\partial^2 H/\partial s \partial v)(s_0, \widetilde{\mathbb{G}}_L^\pm(s_0)) \neq 0$. It follows that $h_{v_0}''(s_0) = 0$ if and only if $\widetilde{\mathbb{G}}_L^{\pm'}(s_0) = 0$. By a similar calculation as above, we have $h_{v_0}'''(s_0) + (\partial^2 H/\partial s \partial v)(s_0, \widetilde{\mathbb{G}}_L^\pm(s_0)) \widetilde{\mathbb{G}}_L^{\pm''}(s_0) = 0$. This means that $h_{v_0}'''(s_0) = 0$ if and only if $\widetilde{\mathbb{G}}_L^{\pm''}(s_0) = 0$. Therefore, the assertions (2) and (3) are equivalent. Moreover, if we consider the mapping $\overline{H} : W \rightarrow \mathbb{R}$ defined in the proof of (A), then we obtain that the assertions (3) and (4) are equivalent. Furthermore, we consider the cylindrical lightcone pedal curve

$$CP_L^\pm(s) = (\widetilde{\mathbb{G}}_L^\pm(s), \langle \gamma(s), \widetilde{\mathbb{G}}_L^\pm(s) \rangle) = (\widetilde{\mathbb{G}}_L^\pm(s), H(s, \widetilde{\mathbb{G}}_L^\pm(s)))$$

and denote $\mathbf{v}^{(k)} = \widetilde{\mathbb{G}}_L^{\pm(k)}(s)$ ($k = 0, 1, 2, 3$) and $\mathbf{v}_0 = \widetilde{\mathbb{G}}_L^\pm(s_0)$. Then we have

$$CP_L^{\pm'}(s) = (\mathbf{v}', \frac{\partial H}{\partial v}(s, \mathbf{v})\mathbf{v}'),$$

$$CP_L^{\pm''}(s) = (\mathbf{v}'', \frac{\partial^2 H}{\partial s^2}(s, \mathbf{v}) + 2\frac{\partial^2 H}{\partial v \partial s}(s, \mathbf{v})\mathbf{v}' + \frac{\partial^2 H}{\partial v^2}(s, \mathbf{v})(\mathbf{v}')^2 + \frac{\partial H}{\partial v}(s, \mathbf{v})\mathbf{v}''),$$

$$\begin{aligned} CP_L^{\pm'''}(s) = & (\mathbf{v}''', \frac{\partial^3 H}{\partial s^3}(s, \mathbf{v}) + 3\frac{\partial^3 H}{\partial v \partial s^2}(s, \mathbf{v})\mathbf{v}' + 3\frac{\partial^3 H}{\partial v^2 \partial s}(s, \mathbf{v})(\mathbf{v}')^2 + \frac{\partial^3 H}{\partial v^3}(s, \mathbf{v})(\mathbf{v}')^3 \\ & + 3\frac{\partial^2 H}{\partial v^2}(s, \mathbf{v})\mathbf{v}'\mathbf{v}'' + 3\frac{\partial^2 H}{\partial v \partial s}(s, \mathbf{v})\mathbf{v}'' + \frac{\partial H}{\partial v}(s, \mathbf{v})\mathbf{v}'''). \end{aligned}$$

The previous discussion shows that $CP_L^\pm(s_0) = 0$ if and only if $\widetilde{\mathbb{G}}_L^\pm(s_0) = 0$. Then we obtain that $CP_L^\pm(s_0)$ is an ordinary cusp if and only if $\widetilde{\mathbb{G}}_L^\pm(s_0) = 0$ and $CP_L^{\pm''}(s_0), CP_L^{\pm'''}(s_0)$ are linearly independent, where we have used the well-known fact that $CP_L^\pm(s_0)$ is an ordinary cusp if and only if $CP_L^{\pm'}(s_0) = 0$ and $CP_L^{\pm''}(s_0), CP_L^{\pm'''}(s_0)$ are linearly independent. Moreover, we denote

$$A = \begin{pmatrix} \widetilde{\mathbb{G}}_L^{\pm''}(s_0) & \frac{\partial^2 H}{\partial s^2}(s_0, v_0) + \frac{\partial H}{\partial v}(s_0, v_0) \widetilde{\mathbb{G}}_L^{\pm''}(s_0) \\ \widetilde{\mathbb{G}}_L^{\pm'''}(s_0) & \frac{\partial^3 H}{\partial s^3}(s_0, v_0) + 3 \frac{\partial^2 H}{\partial s \partial v}(s_0, v_0) \widetilde{\mathbb{G}}_L^{\pm''}(s_0) + \frac{\partial H}{\partial v}(s_0, v_0) \widetilde{\mathbb{G}}_L^{\pm'''}(s_0) \end{pmatrix},$$

$$B = \begin{pmatrix} \widetilde{\mathbb{G}}_L^{\pm''}(s_0) & \frac{\partial^2 H}{\partial s^2}(s_0, v_0) \\ \widetilde{\mathbb{G}}_L^{\pm'''}(s_0) & \frac{\partial^3 H}{\partial s^3}(s_0, v_0) + 3 \frac{\partial^2 H}{\partial s \partial v}(s_0, v_0) \widetilde{\mathbb{G}}_L^{\pm''}(s_0) \end{pmatrix}.$$

Then $\text{rank } A = \text{rank } B$. Since the assertions (2) and (3) are equivalent, condition (2) or (3) holds if and only if $\widetilde{\mathbb{G}}_L^{\pm'}(s_0) = 0$, $\widetilde{\mathbb{G}}_L^{\pm''}(s_0) \neq 0$ and $h'_{v_0}(s_0) = h''_{v_0}(s_0) = 0$, $h'''_{v_0}(s_0) \neq 0$. Therefore, the assertion (2) is equivalent to the conditions $\widetilde{\mathbb{G}}_L^{\pm'}(s_0) = 0$, $\text{rank } B = 2$. Thus, the assertions (2), (3) and (5) are equivalent. By Proposition 3.5, we have the assertions (3), (6) and (7) are equivalent. This completes the proof. \square

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