Open Math. 2016; 14: 946–954 DE GRUYTER OPEN

Open Mathematics Open Access

Research Article

Mustafa Yeneroğlu*

On new characterization of inextensible flows of space-like curves in de Sitter space

DOI 10.1515/math-2016-0071
Received June 22, 2016; accepted August 24, 2016.

Abstract: Elastica and inextensible flows of curves play an important role in practical applications. In this paper, we construct a new characterization of inextensible flows by using elastica in space. The inextensible flow is completely determined for any space-like curve in de Sitter space $\mathbb{S}_1^3$. Finally, we give some characterizations for curvatures of a space-like curve in de Sitter space $\mathbb{S}_1^3$.

Keywords: Fluid flow, Minkowski space-time, Partial differential equation, de Sitter space $\mathbb{S}_1^3$

MSC: 53A04, 53A05

1 Introduction

In mathematics and physics, a de Sitter space is the analogue in Minkowski space, or spacetime, of a sphere in ordinary, Euclidean space. The n-dimensional de Sitter space, denoted dS_n, is the Lorentzian manifold analogue of an n-sphere (with its canonical Riemannian metric). It is maximally symmetric, has constant positive curvature, and is simply connected for n at least 3. More recently, it has been considered as the setting for special relativity rather than using Minkowski space, since a group contraction reduces the isometry group of de Sitter space to the Poincaré group, allowing a unification of the spacetime translation subgroup and Lorentz transformation subgroup of the Poincaré group into a simple group rather than a semi-simple group. This alternate formulation of special relativity is called de Sitter relativity, [1–6].

The elastica caught the attention of many of the brightest minds in the history of mathematics, including Galileo, the Bernoullis, Euler, and others. It was present at the birth of many important fields, most notably the theory of elasticity, the calculus of variations, and the theory of elliptic integrals. The path traced by this curve illuminates a wide range of mathematical style, from the mechanics-based intuition of the early work, through a period of technical virtuosity in mathematical technique, to the present day where computational techniques dominate [16–20].

The flow of a curve or surface is said to be inextensible if, in the former case, the arc-length is preserved, and in the latter case, if the intrinsic curvature is preserved [7]. Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from the motion. In [15], Kwon investigated inextensible flows of curves and developable surfaces in R^3. Necessary and sufficient conditions for an inextensible curve flow were first expressed as a partial differential equation involving the curvature and torsion. Then, they derived the corresponding equations for the inextensible flow of a developable surface, and showed that it suffices to describe its evolution in terms of two inextensible curve flows, [15]. Flows of curves of a given curve are also widely studied, [8–14].

***Corresponding Author: Mustafa Yeneroğlu:** Fırat University, Department of Mathematics, 23119, Elazığ, Turkey,
E-mail: mustafayeneroglu@gmail.com

This study is organised as follows: Firstly, we construct a new method for inextensible flows of space-like curves in de Sitter space $\mathbb{S}_1^3$. Secondly, using the Frenet frame of the given curve, we present partial differential equations. Finally, we give some characterizations for curvatures of a curve in de Sitter space $\mathbb{S}_1^3$.

2 New geometry of space-like curves in $\mathbb{S}_1^3$-space

It is well-known that the Lorentzian space form with a positive curvature, more precisely [1], a positive sectional curvature is called de Sitter space $\mathbb{S}_1^3$. We define de Sitter 3-space by

$$\mathbb{S}_1^3 = \left\{ x \in \mathbb{R}_1^4 \mid \langle x, x \rangle = 1 \right\}.$$

It is well-known that to each unit speed space-like curve $\gamma : I \longrightarrow \mathbb{S}_1^3$ one can associate a pseudo-orthonormal frame $\{\gamma, \mathbf{T}, \mathbf{N}, \mathbf{B}\}$. Denote by $\mathbf{T}, \mathbf{N}, \mathbf{B}$ the space-like tangent vector, the space-like principal normal vector, and the time-like binomial vector, respectively. In this situation, the Frenet-Serret equations satisfied by the Frenet vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ formally given by

$$\begin{aligned} \gamma' &= \mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{T} &= -\gamma + \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\delta(\gamma)\mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau \mathbf{N}, \end{aligned} \tag{1}$$

where $\delta(\gamma) = -sign(\mathbf{N})$, and κ, τ are the curvature and the torsion of a curve γ respectively and given by

$$\begin{aligned} \kappa &= \left\| \mathbf{T}' + \gamma \right\|, \\ \tau(s) &= \frac{\delta(\gamma)}{R^2} \det(\gamma, \gamma', \gamma'', \gamma'''), \end{aligned}$$

with $R(s) \neq 0$.

Let $\gamma\,(u, w)$ be a one parameter family of smooth space-like curves in $\mathbb{S}_1^3$.

$$\frac{\partial \gamma}{\partial w} = \pi_1 \mathbf{T} + \pi_2 \mathbf{N} + \pi_3 \mathbf{B}$$

Putting

$$\begin{aligned} \mathbf{W} = \mathbf{W}(w, t) &= \frac{\partial \gamma}{\partial w}, \\ \mathbf{V}(u, w) &= \frac{\partial \gamma}{\partial u} = v(u, w)\mathbf{T}(u, w), \end{aligned}$$

which gives

$$[\mathbf{W}, \mathbf{T}] = -\frac{\mathbf{W}(v)}{v} T = g\mathbf{T}.$$

Finally [18], we obtain that

$$\mathbf{W}(v) = -gv, g = - < \nabla_{\mathbf{T}}\mathbf{W}, \mathbf{T} > .$$

Definition 2.1. *The flow* $\frac{\partial \gamma}{\partial w}$ *in de Sitter space* $\mathbb{S}_1^3$ *is said to be inextensible if*

$$\frac{\partial}{\partial w} \left\| \frac{\partial \alpha}{\partial u} \right\| = 0. \tag{2}$$

Theorem 2.2. *Let* $\frac{\partial \gamma}{\partial w}$ *be a smooth flow of* γ*. The flow is inextensible if and only if*

$$\pi_2 v \kappa \delta(\gamma) + \frac{\partial \pi_1}{\partial u} = 0. \tag{3}$$

Now, assume that γ is arc-length parametrized curve. Then, we have

Lemma 2.3.

$$\nabla_w \mathbf{T} = -\pi_1\gamma + [\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\mathbf{N} + [\pi_2\tau + \frac{\partial \pi_3}{\partial s}]\mathbf{B}, \tag{4}$$

where π_1, π_2, π_3 are smooth functions of time and arc-length.

Proof. From the definition of inextensible flow, we have

$$\nabla_w \mathbf{T} = -\pi_1\gamma + [\pi_2\kappa\delta(\gamma) + \frac{\partial \pi_1}{\partial s}]\mathbf{T} + [\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\mathbf{N} + [\pi_2\tau + \frac{\partial \pi_3}{\partial s}]\mathbf{B}.$$

Using Eq. (3), we obtain Eq. (4). This completes the proof. □

Now we give the characterization of evolution of first curvature as below:

Theorem 2.4. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}^3_1$. If $\frac{\partial \gamma}{\partial w}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}^3_1$, then the evolution of κ is given by*

$$\frac{\partial \kappa}{\partial w} = \frac{\partial}{\partial s}[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}] + \tau[\pi_2\tau + \frac{\partial \pi_3}{\partial s}] + \frac{1}{\kappa}\frac{\partial \pi_1}{\partial s} + \pi_2,$$

where π_1, π_2, π_3 are smooth functions of time and arc-length.

Proof. A differentiation in Eq. (4) and the Frenet formulas give us that

$$\begin{aligned}\nabla_s\nabla_w \mathbf{T} = & -\frac{\partial \pi_1}{\partial s}\gamma - [\pi_1 - [\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)]\mathbf{T} \\ & + [\frac{\partial}{\partial s}[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}] + \tau[\pi_2\tau + \frac{\partial \pi_3}{\partial s}]]\mathbf{N} \\ & + [\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial \pi_3}{\partial s}] + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]]\mathbf{B}.\end{aligned}$$

Using the formula of the curvature, we write a relation

$$\nabla_w\nabla_s \mathbf{T} - \nabla_s\nabla_w \mathbf{T} = R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial w})\mathbf{T}.$$

We immediately arrive at

$$R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial w})\mathbf{T} = \pi_2 R(\mathbf{T}, \mathbf{N})\mathbf{T} + \pi_3 R(\mathbf{T}, \mathbf{B})\mathbf{T}.$$

Another important fact is that the curvature operator R on de Sitter space $\mathbb{S}^3_1$ has a simple expression, i.e.,

$$R(X_1, X_2)X_3 = g(X_1, X_3)X_2 - g(X_2, X_3)X_1.$$

Then,

$$\begin{aligned} R(\mathbf{T}, \mathbf{N})\mathbf{T} &= g(\mathbf{T}, \mathbf{T})\mathbf{N} - g(\mathbf{N}, \mathbf{T})\mathbf{T} = \mathbf{N}, \\ R(\mathbf{T}, \mathbf{B})\mathbf{T} &= g(\mathbf{T}, \mathbf{T})\mathbf{B} - g(\mathbf{B}, \mathbf{T})\mathbf{T} = \mathbf{B}. \end{aligned}$$

From above equations, we get

$$R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial w})\mathbf{T} = \pi_2\mathbf{N} + \pi_3\mathbf{B}.$$

Then, we can write

$$\nabla_w\nabla_s \mathbf{T} = \nabla_s\nabla_w \mathbf{T} + R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial w})\mathbf{T}.$$

Thus it is easy to obtain that

$$\nabla_w\nabla_s \mathbf{T} = -\frac{\partial \pi_1}{\partial s}\gamma - [\pi_1 - [\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)]\mathbf{T}$$

$$+[\frac{\partial}{\partial s}[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+\tau[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\pi_2]\mathbf{N}$$
$$+[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+\pi_3]\mathbf{B}.$$

On the other hand, we have

$$\frac{\partial\kappa}{\partial w}=\frac{\partial}{\partial w}g\left(\nabla_s\mathbf{T},\mathbf{N}\right).$$

Since, we express

$$\frac{\partial\kappa}{\partial w}=g\left(\nabla_w\nabla_s\mathbf{T},\mathbf{N}\right)+g\left(\nabla_s\mathbf{T},\nabla_w\mathbf{N}\right).$$

Moreover, by the definition of metric tensor, we have

$$g\left(\mathbf{N},\nabla_w\mathbf{N}\right)=0.$$

Then

$$g\left(\nabla_s\mathbf{T},\nabla_w\mathbf{N}\right)=-g\left(\gamma,\nabla_w\mathbf{N}\right)=\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}.$$

Combining these we have

$$\frac{\partial\kappa}{\partial w}=\frac{\partial}{\partial s}[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+\tau[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}+\pi_2.$$

Thus, we obtain the theorem. This completes the proof. □

From the above theorem, we have

Theorem 2.5.

$$\nabla_w\mathbf{N}=-\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}\gamma+\frac{1}{\kappa}[[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]\mathbf{T}+\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+2\pi_3]\mathbf{B},$$

where π_1,π_2,π_3 are smooth functions of time and arc-length.

Proof. Using Frenet equations, we have

$$\nabla_w\nabla_s\mathbf{T}=-\frac{\partial\gamma}{\partial w}+\frac{\partial\kappa}{\partial w}\mathbf{N}+\kappa\nabla_w\mathbf{N}.$$

Then,

$$\kappa\nabla_w\mathbf{N}=-\frac{\partial\pi_1}{\partial s}\gamma+[[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]\mathbf{T}$$
$$+[-\frac{\partial\kappa}{\partial t}+\frac{\partial}{\partial s}[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+\tau[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+2\pi_2]\mathbf{N}$$
$$+[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+2\pi_3]\mathbf{B}.$$

Therefore

$$g\left(\mathbf{N},\nabla_w\mathbf{N}\right)=0.$$

From above equation we obtain

$$\nabla_w\mathbf{N}=-\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}\gamma+\frac{1}{\kappa}[[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]\mathbf{T}$$
$$+\frac{1}{\kappa}[-\frac{\partial\kappa}{\partial t}+\frac{\partial}{\partial s}[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+\tau[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+2\pi_2]\mathbf{N}$$
$$+\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+2\pi_3]\mathbf{B},$$

which completes the proof. □

Theorem 2.6. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}_1^3$. If $\frac{\partial\gamma}{\partial w}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}_1^3$, then*

$$\begin{aligned}\nabla_w \mathbf{B} = {} & \frac{1}{\tau}[\pi_1\kappa\delta(\gamma) + \frac{\partial}{\partial s}[-\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}] - [\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]]]\gamma \\ & + \frac{1}{\tau}[-\pi_2 + \frac{\partial}{\partial s}[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] - \frac{1}{\kappa}\frac{\partial\pi_1}{\partial s} - \frac{\partial}{\partial t}\kappa\delta(\gamma)]\mathbf{T} \\ & + \frac{1}{\tau}[\kappa[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] + \tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial\pi_3}{\partial s}] \\ & + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}] + 2\pi_3]] - \kappa\delta(\gamma)[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]]\mathbf{N},\end{aligned}$$

where π_1, π_2, π_3 are smooth functions of time and arc-length.

Proof. Assume that $\frac{\partial\gamma}{\partial w}$ be inextensible flow of γ.

$$\begin{aligned}\nabla_s\nabla_w \mathbf{N} = {} & [\frac{\partial}{\partial s}[-\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}] - [\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]]]\gamma \\ & + [\frac{\partial}{\partial s}[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] - \frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}]\mathbf{T} \\ & + [\kappa[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] + \tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau \\ & + \frac{\partial\pi_3}{\partial s}] + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}] + 2\pi_3]]]\mathbf{N} \\ & + \frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial\pi_3}{\partial s}] + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}] + 2\pi_3]]\mathbf{B}.\end{aligned}$$

Under the assumption of space-like curve, we have

$$\nabla_w\nabla_s \mathbf{N} = -\pi_1\kappa\delta(\gamma)\gamma + \frac{\partial}{\partial t}\kappa\delta(\gamma)\mathbf{T} + \kappa\delta(\gamma)[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\mathbf{N} + [\kappa\delta(\gamma)[\pi_2\tau + \frac{\partial\pi_3}{\partial s}] + \frac{\partial\tau}{\partial t}]\mathbf{B} + \tau\nabla_w\mathbf{B}.$$

Using the formula of the curvature, we write a relation

$$\nabla_w\nabla_s\mathbf{N} - \nabla_s\nabla_w\mathbf{N} = R(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial w})\mathbf{N}.$$

Thus, it is seen that

$$R(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial w})\mathbf{N} = \pi_2 R(\mathbf{T},\mathbf{N})\mathbf{N} + \pi_3 R(\mathbf{T},\mathbf{B})\mathbf{N}.$$

By using formula of curvature, we have

$$\begin{aligned}R(\mathbf{T},\mathbf{N})\mathbf{N} &= g\,(\mathbf{T},\mathbf{N})\,\mathbf{N} - g\,(\mathbf{N},\mathbf{N})\,\mathbf{T} = -\,\mathbf{T}, \\ R(\mathbf{T},\mathbf{B})\mathbf{N} &= g\,(\mathbf{T},\mathbf{N})\,\mathbf{B} - g\,(\mathbf{B},\mathbf{N})\,\mathbf{T} = 0.\end{aligned}$$

Arranging the last equations, we obtain

$$R(\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial w})\mathbf{N} = -\pi_2\mathbf{T}.$$

Therefore, we can easily see that

$$\begin{aligned}\tau\nabla_w \mathbf{B} = {} & [\pi_1\kappa\delta(\gamma) + \frac{\partial}{\partial s}[-\frac{1}{\kappa}\frac{\partial\pi_1}{\partial s}] - [\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]]]\gamma \\ & + [-\pi_2 + \frac{\partial}{\partial s}[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] - \frac{1}{\kappa}\frac{\partial\pi_1}{\partial s} - \frac{\partial}{\partial t}\kappa\delta(\gamma)]\mathbf{T} \\ & + [\kappa[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]\kappa\delta(\gamma)]] + \tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial\pi_3}{\partial s}] \\ & + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}] + 2\pi_3]] - \kappa\delta(\gamma)[\pi_1\kappa + \pi_3\tau + \frac{\partial\pi_2}{\partial s}]]\mathbf{N}\end{aligned}$$

$$+[\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+2\pi_3]]$$
$$-[\kappa\delta(\gamma)[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\frac{\partial\tau}{\partial t}]]\mathbf{B}.$$

By this way, we conclude

$$g\left(\mathbf{B}_1,\nabla_w\mathbf{B}_1\right)=0.$$

Thus, we obtain the theorem. The proof of theorem is completed. □

Now we give the characterization of evolution of second curvature as below:

Theorem 2.7. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}_1^3$. If $\frac{\partial\gamma}{\partial t}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}_1^3$, then the evolution of τ is given by*

$$\frac{\partial\tau}{\partial w}=\frac{\partial}{\partial s}[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau+\frac{\partial\pi_3}{\partial s}]+\tau[\pi_1\kappa+\pi_3\tau+\frac{\partial\pi_2}{\partial s}]+2\pi_3]]-\kappa\delta(\gamma)[\pi_2\tau+\frac{\partial\pi_3}{\partial s}],$$

where π_1,π_2,π_3 are smooth functions of time and arc-length.

Proof. It is obvious from Theorem 2.6. This completes the proof. □

Since $\gamma(t)$ is an immersed curve, it has velocity vector $\mathbf{V}=v\mathbf{T}$ and squared geodesic curvature

$$\kappa^2+1=\left\|\nabla_{\mathbf{T}}\mathbf{T}\right\|^2.$$

Theorem 2.8 (Main Theorem)**.**

$$\mathbf{W}(\kappa^2+1)=2<\nabla_s\nabla_s\mathbf{W},\nabla_s\mathbf{T}>+4g(\kappa^2+1)+2<R(\mathbf{W},\mathbf{T})\mathbf{T},\nabla_s\mathbf{T}>,$$

where $g=-<\nabla_{\mathbf{T}}\mathbf{W},\mathbf{T}>$.

Proof. From Euler equations, we easily have

$$\mathbf{W}(\kappa^2+1)=2<\nabla_s\nabla_s\mathbf{W},\nabla_s\mathbf{T}>+2<R(\mathbf{W},\mathbf{T})\mathbf{T},\nabla_s\mathbf{T}>+4g<\nabla_s\mathbf{T},\nabla_s\mathbf{T}>.$$ □

Corollary 2.9.

$$\mathbf{W}(\kappa^2+1)=\mathbf{W}(\kappa^2)$$

In what follows, $\gamma:[0,1]\to M$ is a curve of length L. Now for fixed constant λ let

$$\mathfrak{F}^\lambda(\gamma)=\frac{1}{2}\int_0^L\kappa^2+1+\lambda ds=\frac{1}{2}\int_0^1(\|\nabla_s\mathbf{T}\|^2+\lambda)v(t)dt.$$

For a variation γ_w with variation field $\mathbf{W}$, we compute

$$\frac{d}{dw}\mathfrak{F}^\lambda(\gamma_w)=\frac{1}{2}\int_0^1 W(\kappa^2+1)v+(\kappa^2+1+\lambda)\mathbf{W}(v)dt=\frac{1}{2}\int_0^1 W(\kappa^2+1)-(\kappa^2+1+\lambda)gds$$
$$=\int_0^1<\nabla_s\nabla_s\mathbf{W},\nabla_s\mathbf{T}>+2g(\kappa^2+1)+<R(\mathbf{W},\mathbf{T})\mathbf{T},\nabla_s\mathbf{T}>-\frac{1}{2}(\kappa^2+1+\lambda)gds.$$

This condition implies that

$$\frac{d}{dw}\mathfrak{F}^\lambda(\gamma_w)=\int_0^1<\nabla_s\nabla_s\mathbf{W},\nabla_s\mathbf{T}>-<\nabla_s\mathbf{W},2(\kappa^2+1)\mathbf{T}>+$$

$$+ < R(\nabla_s \mathbf{T}, \mathbf{T})\mathbf{T}, \mathbf{W} > + \frac{1}{2} < \nabla_s \mathbf{W}, (\kappa^2 + 1 + \lambda)\mathbf{T} > ds$$

$$= \int_0^L < \mathbf{E}, \mathbf{W} > ds + [< \nabla_s \mathbf{W}, \nabla_s \mathbf{T} > + < \mathbf{W}, -(\nabla_s)^2 \mathbf{T} + \Lambda \mathbf{T} >]_0^L,$$

where

$$\mathbf{E} = (\nabla_s)^3 \mathbf{T} - \nabla_s(\Lambda \mathbf{T}) + R(\nabla_s \mathbf{T}, \mathbf{T})\mathbf{T}, g = - < \nabla_{\mathbf{T}} \mathbf{W}, \mathbf{T} >$$

and

$$\Lambda = \frac{\lambda - 3\kappa^2 - 4}{2}.$$

Thus, we can state the following.

Lemma 2.10. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}_1^3$. If $\frac{\partial \gamma}{\partial w}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}_1^3$, then*

$$g = - < \nabla_s \mathbf{W}, \mathbf{T} >= \pi_2 \kappa \delta(\gamma) + \frac{\partial \pi_1}{\partial s},$$

where π_1, π_2 are smooth functions of time and arc-length.

Theorem 2.11. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}_1^3$. If $\frac{\partial \gamma}{\partial w}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}_1^3$, then*

$$W(\kappa^2) = 2\frac{\partial \pi_1}{\partial s} + \kappa[\frac{\partial}{\partial s}[\pi_1 \kappa + \pi_3 \tau + \frac{\partial \pi_2}{\partial s}] + \tau[\pi_2 \tau + \frac{\partial \pi_3}{\partial s}]] + 4g(\kappa^2 + 1) + \pi_2 \kappa,$$

where π_1, π_2, π_3 are smooth functions of time and arc-length.

Proof. Firstly, we obtain

$$\begin{aligned}\nabla_s \nabla_w \mathbf{T} = & -\frac{\partial \pi_1}{\partial s}\gamma - [\pi_1 - [\pi_1 \kappa + \pi_3 \tau + \frac{\partial \pi_2}{\partial s}]\kappa \delta(\gamma)]\mathbf{T} \\ & + [\frac{\partial}{\partial s}[\pi_1 \kappa + \pi_3 \tau + \frac{\partial \pi_2}{\partial s}] + \tau[\pi_2 \tau + \frac{\partial \pi_3}{\partial s}]]\mathbf{N} \\ & + [\frac{\partial}{\partial s}[\pi_2 \tau + \frac{\partial \pi_3}{\partial s}] + \tau[\pi_1 \kappa + \pi_3 \tau + \frac{\partial \pi_2}{\partial s}]]\mathbf{B}.\end{aligned}$$

Since, we immediately arrive at

$$R(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t})\mathbf{T} = \pi_2 \mathbf{N} + \pi_3 \mathbf{B}.$$

Therefore,

$$\begin{aligned}W(\kappa^2) & = 2 < \nabla_T \nabla_T W, \nabla_T T > + 4g(\kappa^2 + 1) + 2 < R(W, T)T, \nabla_T T > \\ & = 2\frac{\partial \pi_1}{\partial s} + \kappa[\frac{\partial}{\partial s}[\pi_1 \kappa + \pi_3 \tau + \frac{\partial \pi_2}{\partial s}] + \tau[\pi_2 \tau + \frac{\partial \pi_3}{\partial s}]] + 4g(\kappa^2 + 1) + \pi_2 \kappa. \quad \square\end{aligned}$$

Now, we can obtain following equation in terms of flows.

Lemma 2.12.

$$\begin{aligned}\mathbf{E} = & (1 - \kappa^2 \delta(\gamma) + \Lambda)\gamma + [3\frac{\partial \kappa}{\partial s}\kappa \delta(\gamma) - \frac{\partial \Lambda}{\partial s}]\mathbf{T} \\ & + (\frac{\partial^2 \kappa}{\partial s^2} - \kappa + \kappa^3 \delta(\gamma) + \kappa \tau^2 - \kappa \Lambda - \kappa)\mathbf{N} + (2\frac{\partial \kappa}{\partial s}\tau + \kappa \frac{\partial \tau}{\partial s})\mathbf{B}.\end{aligned}$$

Theorem 2.13. *Let γ be one parameter family curves in de Sitter space $\mathbb{S}_1^3$. If $\dfrac{\partial \gamma}{\partial w}$ is inextensible flow of space-like γ in de Sitter space $\mathbb{S}_1^3$, then*

$$
\begin{aligned}
\nabla_w \mathbf{E} = {} & [\frac{\partial}{\partial w}(1-\kappa^2\delta(\gamma)+\Lambda) - \pi_1[3\frac{\partial \kappa}{\partial s}\kappa\delta(\gamma) - \frac{\partial \Lambda}{\partial s}] - \frac{1}{\kappa}\frac{\partial \pi_1}{\partial s}(\frac{\partial^2 \kappa}{\partial s^2} - \kappa + \kappa^3\delta(\gamma) + \kappa\tau^2 \\
& - \kappa\Lambda - \kappa) + (2\frac{\partial \kappa}{\partial s}\tau + \kappa\frac{\partial \tau}{\partial s})\frac{1}{\tau}[\pi_1\kappa\delta(\gamma) + \frac{\partial}{\partial s}[-\frac{1}{\kappa}\frac{\partial \pi_1}{\partial s}] - [\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)]]]]\gamma \\
& + [\pi_1(1-\kappa^2\delta(\gamma)+\Lambda) + \frac{\partial}{\partial w}[3\frac{\partial \kappa}{\partial s}\kappa\delta(\gamma) - \frac{\partial \Lambda}{\partial s}] + \frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)](\frac{\partial^2 \kappa}{\partial s^2} - \kappa \\
& + \kappa^3\delta(\gamma) + \kappa\tau^2 - \kappa\Lambda - \kappa) + \frac{1}{\tau}[-\pi_2 + \frac{\partial}{\partial s}[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)]] - \frac{1}{\kappa}\frac{\partial \pi_1}{\partial s} \\
& - \frac{\partial}{\partial t}\kappa\delta(\gamma)](2\frac{\partial \kappa}{\partial s}\tau + \kappa\frac{\partial \tau}{\partial s})]\mathbf{T} + [\pi_2(1-\kappa^2\delta(\gamma)+\Lambda) + [\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}][3\frac{\partial \kappa}{\partial s}\kappa\delta(\gamma) \\
& - \frac{\partial \Lambda}{\partial s} + \frac{\partial}{\partial w}(\frac{\partial^2 \kappa}{\partial s^2} - \kappa + \kappa^3\delta(\gamma) + \kappa\tau^2 - \kappa\Lambda - \kappa) + (2\frac{\partial \kappa}{\partial s}\tau + \kappa\frac{\partial \tau}{\partial s})\frac{1}{\tau}[\kappa[\frac{1}{\kappa}[[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]\kappa\delta(\gamma)]] \\
& + \tau[\frac{1}{\kappa}[\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial \pi_3}{\partial s}] + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}] + 2\pi_3]] - \kappa\delta(\gamma)[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}]]]]\mathbf{N} \\
& + [\pi_3(1-\kappa^2\delta(\gamma)+\Lambda) + [3\frac{\partial \kappa}{\partial s}\kappa\delta(\gamma) - \frac{\partial \Lambda}{\partial s}][\pi_2\tau + \frac{\partial \pi_3}{\partial s}] + \frac{1}{\kappa}(\frac{\partial^2 \kappa}{\partial s^2} - \kappa + \kappa^3\delta(\gamma) + \kappa\tau^2 - \kappa\Lambda \\
& - \kappa)[\frac{\partial}{\partial s}[\pi_2\tau + \frac{\partial \pi_3}{\partial s}] + \tau[\pi_1\kappa + \pi_3\tau + \frac{\partial \pi_2}{\partial s}] + 2\pi_3] + \frac{\partial}{\partial w}(2\frac{\partial \kappa}{\partial s}\tau + \kappa\frac{\partial \tau}{\partial s})]\mathbf{B}.
\end{aligned}
$$

Example 2.14. *The time-helix is parametrized by*

$$\gamma(s, w) = (A(w)\cos(s), A(w)\sin(s), B(w)s, 0),$$

where A, B are functions only of time.

Projection of γ at xyz-plane:

Fig. 1. Time-helix is illustrated using colours Magenta, Cyan, Green at the time $t = 1.2$, $t = 1.8$, $t = 2.2$, respectively.

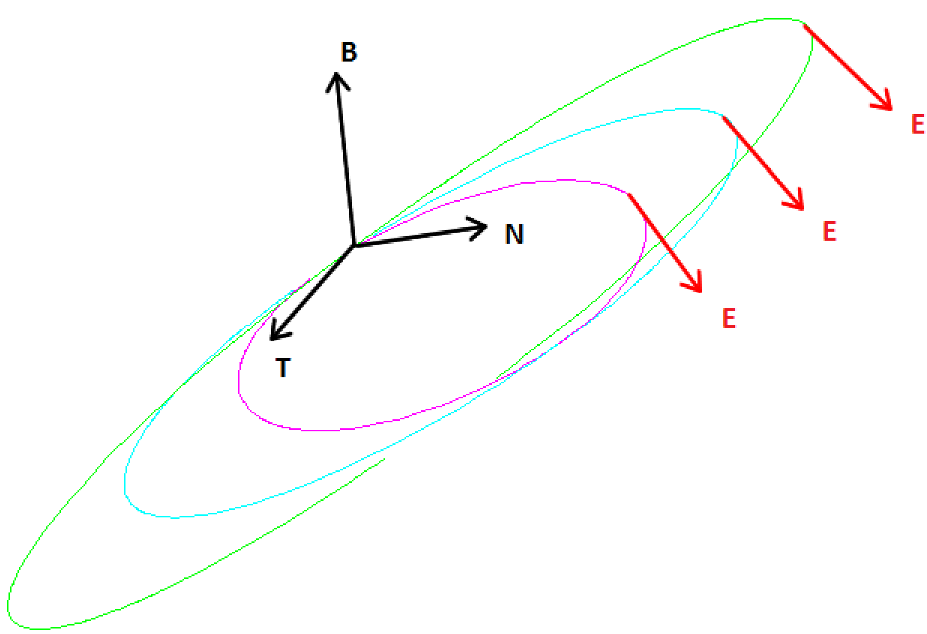
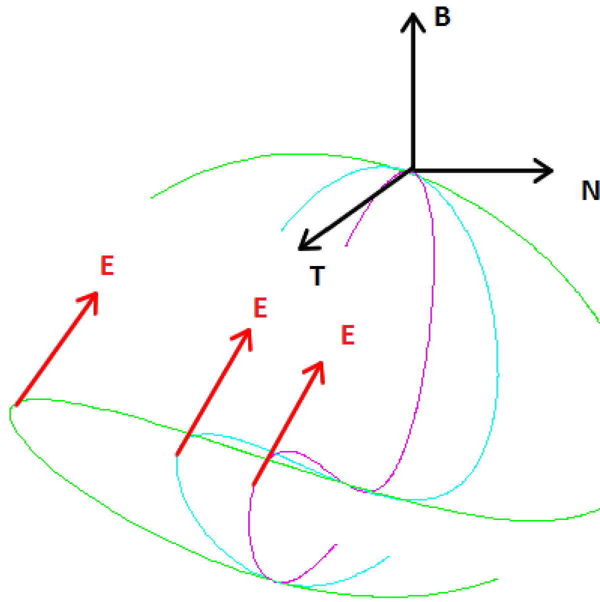


Fig. 2. Time-helix is illustrated using colours Magenta, Cyan, Green at the time $t = 1.2,\ t = 1.8,\ t = 2.2$, respectively.



References

[1] Abdel Aziz H.S., New Special Surfaces in de Sitter 3-Space, Applied Mathematics & Information Sciences, 2008, 2(3), 345-352.
[2] Altschuler S.J., Grayson M.A., Shortening space curves and flow through singularities, IMA preprint series, 1991, 823.
[3] Andrews B., Evolving convex curves, Calculus of Variations and Partial Differential Equations, 1998, 7, 315-371.
[4] Do Carmo M.P., Differential Geometry of Curves and Surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.
[5] Einstein A., Relativity: The Special and General Theory, New York: Henry Holt, 1920.
[6] Gage M., Hamilton R.S., The heat equation shrinking convex plane curves, J. Differential Geom. 1981, 23, 69-96.
[7] Grayson M., The heat equation shrinks embedded plane curves to round points, J. Differential Geom. 1987, 26, 285-314.
[8] Körpınar T., On the Fermi-Walker Derivative for Inextensible Flows, Zeitschrift für Naturforschung A, 2015, 70 (7), 477-482.
[9] Körpınar T., A new method for inextensible flows of timelike curves in 4-dimensional LP-Sasakian manifolds, Asian-European Journal of Mathematics, 2015, 8 (4), DOI: 10.1142/S1793557115500734.
[10] Körpınar T., B-tubular surfaces in Lorentzian Heisenberg Group H3, Acta Scientiarum. Technology, 2015, 37(1), 63-69.
[11] Körpınar T., Bianchi Type-I Cosmological Models for Inextensible Flows of Biharmonic Particles by Using Curvature Tensor Field in Spacetime, Int J Theor Phys, 2015, 54, 1762-1770.
[12] Körpınar T., New characterization of b-m2 developable surfaces, Acta Scientiarum. Technology, 2015, 37(2), 245-250.
[13] Körpınar T., Turhan E., A New Version of Inextensible Flows of Spacelike Curves with Timelike B2 in Minkowski Space-Time E41, Differ. Equ. Dyn. Syst., 2013, 21 (3), 281-290.
[14] Körpınar T., A New Method for Inextensible Flows of Timelike Curves in Minkowski Space-Time E41, International Journal of Partial Differential Equations, 2014, Article ID 517070, 7 pages.
[15] Kwon D.Y., Park F.C., Chi D.P., Inextensible flows of curves and developable surfaces, Appl. Math. Lett. 2005, 18, 1156-1162.
[16] Ma L., Chen D., Curve shortening in Riemannian manifold, (preprint).
[17] McKinley G.H., Dimensionless Groups For Understanding Free Surface Flows of Complex Fluids, (preprint).
[18] Singer D., Lectures on elastic curves and rods, AIP Conf. Proc. 2008, 1002(1), 3-32.
[19] Post F.H., van Walsum T., Fluid flow visualization. In Focus on Scientific Visualization, 1993, 4, 1-40.
[20] Wilcox D.C., Turbulence Modeling for CFD. DCW Industries, 2006.