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Research Article

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On homological classification of pomonoids by GP-po-flatness of S-posets

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Abstract: In this paper, we introduce GP-po-flatness property of S-posets over a pomonoid S, which lies strictly between principal weak po-flatness and po-torsion freeness. Furthermore, we investigate the homological classification problems of pomonoids by using this new property. Finally, we consider direct products of GP-po-flat S-posets. As an application, characterizations of pomonoids over which direct products of nonempty families of principally weakly po-flat S-posets are principally weakly po-flat are obtained, and some results of Khosravi, R. in a certain extent are generalized.

Keywords: S-poset, Principal weak po-flatness, GP-po-flatness, Direct product

MSC: 06F05, 20M50

1 Introduction

Let S be a monoid. It is well-known that flatness properties of S-acts play an important role in studying the homological classification problems of monoids. Different so-called flatness properties (freeness, projectivity, strong flatness, Conditions (P), (WP), (PWP), flatness, weak flatness, principal weak flatness, torsion freeness) of S-acts have been widely used in the homological classification of monoids. A recent and complete treatment of these flatness properties of S-acts appears in the monograph [1].

The study of flatness properties of partially ordered acts over a pomonoid S, or S-posets, was initiated by Fakhruddin, S. M. in the 1980s, see [2, 3]. During recent years, the ordered versions of various flatness properties of acts are defined (in a natural way) and studied [4–7], and also some new properties such as Conditions (P_w), $(WP)_w$ and $(PWP)_w$ are discovered in the studying process, see [4]. More particularly, some classes of pomonoids, such as (po-)cancellable, left PP, left PSF, (order) regular, regularly almost regular and poperfect pomonoids etc., are characterized by using flatness properties of S-posets.

In [8], Qiao and Wei introduced GP-flatness of acts and showed that the class of acts having this property lies strictly between the classes of principally weakly flat acts and torsion free acts. Moreover, using GP-flatness, some important monoids are generalized, such as regular monoids, left almost regular monoids and so on, and also a new class of monoids, called generally regular monoids, are characterized. Our aim in this paper is to carry over some of these results to the setting of S-posets over a pomonoid S. Firstly, in Section 2, we define GP-po-flat S-posets, and describe GP-po-flatness by certain subpullback diagrams. We then give an equivalent condition under which the amalgamated coproduct $A(I)$ of two copies of S over a proper ideal I is GP-po-flat in Section 3. In Section 4, we characterize pomonoids S over which all (cyclic, Rees factor) S-posets are GP-po-flat, and pomonoids S over

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which all po-torsion free S-posets are GP-po-flat. Moreover, we present examples which distinguish between GP-po-flatness and principal weak po-flatness (respectively, po-torsion freeness).

Flatness properties of product acts over a monoid have been extensively studied in recent decades, see [9–11]. However, the research on flatness properties of product S-posets over a pomonoid S is so far less advanced. To our knowledge, the work on this aspect first appeared in [12]. In that paper, the author gave conditions on a pomonoid S under which the S-poset S^I is principally weakly po-flat for each nonempty set I. Moreover, the author proved that direct products of S-posets satisfying Condition (P) (Conditions (E) and (P_w)) again satisfy that condition, if and only if the S-poset S^I is so for each nonempty set I. However, the situation for GP-po-flatness and principal weak po-flatness is markedly different. Thereby, in Section 5, we determine a condition under which principal weak po-flat and GP-po-flat S-posets are preserved under direct products, and extend some results from [12].

2 Definitions and general properties

Throughout this paper, S always stands for a pomonoid and $\mathbb{N}$ for the set of natural numbers. A nonempty poset $(A, \leq)$ is called a *right S-poset*, usually denoted A_S, if there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, which satisfies the conditions: (1) the action is monotonic in each variable, (2) $a(ss') = (as)s'$ and $a1 = a$ for all $a \in A$ and $s, s' \in S$. Left S-posets ${}_SB$ are defined analogously, and by $\Theta_S = \{\theta\}$ we denote the one-element right S-poset. A nonempty subset I of S is called a *left ideal of* S if I satisfies $SI \subseteq I$, whereas an *ordered left ideal I of* S is a left ideal I of S for which $a \leq b \in I$ implies $a \in I$ for all $a, b \in S$. Similarly, (ordered) right ideals of S are defined.

Various flatness properties are defined in terms of tensor products. To define the tensor product $A \otimes_S B$ of a right S-poset A_S and a left S-poset ${}_SB$ [7], we first equip the Cartesian product $A \times B$ with component-wise order. Let $A \otimes_S B = (A \times B)/\rho$, where ρ is the order-congruence on the right S-poset $A \times B$ (on which S acts trivially) generated by the relation

$$H = \{((as, b), (a, sb)) \mid a \in A, b \in B, s \in S\}.$$

The equivalence class of (a, b) in $A \otimes_S B$ is denoted by $a \otimes b$. The order relation on $A \otimes_S B$ will be described in Lemma 2.3. In this way, a functor $A_S \otimes -$ from the category of left S-posets into the category of posets is obtained. It is easily established, as for S-acts, that $A \otimes_S S$ can be equipped with a natural right S-action, and $A \otimes_S S \cong A_S$ for all S-posets A_S.

In S-acts, principal weak flatness and GP-flatness are formulated as follows.

- An S-act A_S is called *principally weakly flat* if the functor $A_S \otimes -$ (from the category of left S-acts to the category of sets) preserves all embeddings of principal left ideals of a monoid S into S. In the language of elements this means that, for any $s \in S$ and $a, a' \in A$, $a \otimes s = a' \otimes s$ in $A \otimes_S S$ implies $a \otimes s = a' \otimes s$ in $A \otimes_S Ss$ (see [1, III, Lemma 10.1]).
- An S-act A_S is called *GP-flat* [8] if for any $a, a' \in A$ and $s \in S$, $a \otimes s = a' \otimes s$ in $A \otimes_S S$ implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S Ss^n$.

In [6], Shi introduced an ordered version of principal weak flatness as follows.

- An S-poset A_S is called *principally weakly po-flat* if the functor $A_S \otimes -$ preserves order embeddings of principal left ideals I of a monoid S into S. This means, for any $s \in S$ and $a, a' \in A$, $a \otimes s \leq a' \otimes s$ in $A \otimes_S S$ implies $a \otimes s \leq a' \otimes s$ in $A \otimes_S Ss$.

Inspired by the work of [6] and generalizing [8], we define here GP-po-flatness property in S-posets.

Definition 2.1. *A right S-poset A_S is called* GP-po-flat *if for any $a, a' \in A$ and $s \in S$, $a \otimes s \leq a' \otimes s$ in $A \otimes_S S$ implies that there exists $n \in \mathbb{N}$ such that $a \otimes s^n \leq a' \otimes s^n$ in $A \otimes_S Ss^n$.*

Indeed, the example from [7] shows that GP-po-flat S-posets do exist. For any pomonoid S, let $A = \{a, a'\}$ be a two-elements chain with $a < a'$ and $as = a$, $a's = a'$ for every $s \in S$. Then A is a right S-poset. We can verify that A is GP-po-flat by Definition 2.1.

Remark 2.2. *In Definition 2.1, if $n = 1$, then every GP-po-flat S-poset is in fact principally weakly po-flat. So principal weak po-flatness implies GP-po-flatness, but in Section 4 we will show that this implication is strict.*

Similar to principal weak flatness of S-posets, GP-flatness for S-posets can be defined by replacing "$\leq$" by "=" in Definition 2.1. It is obvious that every GP-po-flat S-poset is GP-flat, but the converse is not true by [13, Example 8].

In what follows, we will provide some basic properties about GP-po-flat S-posets. We start with a description of GP-po-flatness, and the following lemma is needed.

Lemma 2.3 ([7]). *Let A_S be a right S-poset, and ${}_SB$ a left S-poset. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes_S B$ for $a, a' \in A$, $b, b' \in B$ if and only if there exist $a_1, a_2, \cdots, a_n \in A$, $b_2, \cdots, b_n \in B$ and $s_1, t_1, \cdots, s_n, t_n \in S$ such that*

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 b \leq t_1 b_2 \\ \cdots & \cdots \\ a_n t_n \leq a' & s_n b_n \leq t_n b'. \end{array}$$

Applying Definition 2.1 and Lemma 2.3, the following result holds.

Lemma 2.4. *A right S-poset A_S is GP-po-flat if and only if for any $a, a' \in A$ and $s \in S$, $as \leq a's$ in A_S implies that there exist $m, n \in \mathbb{N}$, $a_1, a_2, \cdots, a_m \in A$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{ll} a \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 s^n \leq t_1 s^n \\ \cdots & \cdots \\ a_m t_m \leq a' & s_m s^n \leq t_m s^n. \end{array}$$

In the above lemma, the natural numbers m and n are called the *length and degree* of the scheme connecting (a, s^n) to (a', s^n), respectively. In particular, the minimum length and degree of the existing schemes will be denoted by $l_s(a, a')$ and $d_s(a, a')$, respectively.

Recall that an element c of a pomonoid S is called *right po-cancellable* if, for any $s, t \in S$, $sc \leq tc$ implies $s \leq t$. A right S-poset A_S is called *po-torsion free* if, for any $a, b \in A$ and any right po-cancellable element c of S, $ac \leq bc$ implies $a \leq b$.

The following result, which counterpart is true for S-acts, establishes a connection between GP-po-flatness and po-torsion freeness for S-posets.

Proposition 2.5. *For any pomonoid S, every GP-po-flat S-poset is po-torsion free.*

Proof. Using Lemma 2.4, the proof is routine. □

Note that Example 4.18 below illustrates in particular that the necessary condition in the above proposition is not sufficient. But, for a right po-cancellable pomonoid, GP-po-flatness coincides with po-torsion freeness.

Corollary 2.6. *Let S be a right po-cancellable pomonoid and A_S a right S-poset. Then the following statements are equivalent.*

(1) *A_S satisfies Condition $(PWP)_w$.*
(2) *A_S is principally weakly po-flat.*
(3) *A_S is GP-po-flat.*
(4) *A_S is po-torsion free.*

Proof. This follows from Proposition 2.5 and [14, Corollary 2.3]. □

At the end of this section, we give a characterization of GP-po-flatness by using subpullback diagrams. For information on subpullback diagrams in the category of S-posets, we refer the reader to [4, 15].

Proposition 2.7. *Let A_S be a right S-poset. The following statements are equivalent.*
(1) *A_S is GP-po-flat.*
(2) *Every subpullback diagram $P(Ss, Ss, \iota, \iota, S)$, where $s \in S$ and $\iota : {}_S(Ss) \to {}_S S$ is an order-embedding of left S-posets, satisfies the following condition:*
$(\forall a, a' \in A)(\forall u, v, s \in S)$

$$[a \otimes \iota(us) \leq a' \otimes \iota(vs)] \Longrightarrow [(\exists n \in \mathbb{N})(\exists a'' \in A)(\exists u', v' \in S)(\iota(u's^n) \leq \iota(v's^n) \wedge a \otimes us^n \leq a'' \otimes u's^n \wedge a'' \otimes v's^n \leq a' \otimes vs^n].$$

Proof. (1) $\Rightarrow$ (2). Let $a \otimes \iota(us) \leq a' \otimes \iota(vs)$ in $A \otimes_S S$, for any $a, a' \in A$, $u, v, s \in S$, and an order-embedding $\iota : {}_S(Ss) \to {}_S S$. Denoting $\iota(s) = t$ we have $a \otimes ut \leq a' \otimes vt$ in $A \otimes_S S$, and so $aut \leq a'vt$ in A_S. Since A_S is GP-po-flat, by Lemma 2.4, there exists a scheme

$$\begin{array}{ll} au \leq a_1 s_1 & \\ a_1 t_1 \leq a_2 s_2 & s_1 t^n \leq t_1 t^n \\ \cdots & \cdots \\ a_m t_m \leq a' v & s_m t^n \leq t_m t^n, \end{array}$$

where $m, n \in \mathbb{N}$, $a_i \in A$, $s_i, t_i \in S$, $i = 1, \cdots, m$. Since ι is an order-embedding, from $\iota(s) = t$, we can see that the above scheme implies $au \otimes s^n \leq a'v \otimes s^n$ in $A \otimes_S (Ss^n)$. Then we have $a \otimes us^n \leq a \otimes us^n$ and $a \otimes us^n = au \otimes s^n \leq a'v \otimes s^n = a' \otimes vs^n$ in $A \otimes_S (Ss^n)$, exactly as needed.

(2) $\Rightarrow$ (1). Assume $a, a' \in A$ and $s \in S$ are such that $a \otimes s \leq a' \otimes s$ in $A \otimes_S S$. Now consider the order-embedding $\iota : {}_S(Ss) \to {}_S S$. Then we have $a \otimes \iota(s) \leq a' \otimes \iota(s)$ in $A \otimes_S S$. By (2), there exist $n \in \mathbb{N}$, $a'' \in A$ and $u, v \in S$ such that $a \otimes s^n \leq a'' \otimes us^n$ and $a'' \otimes vs^n \leq a' \otimes s^n$ in $A \otimes_S (Ss^n)$, and $\iota(us^n) \leq \iota(vs^n)$. Since ι is an order-embedding, the last inequality implies $us^n \leq vs^n$. Thus we may compute that $a \otimes s^n \leq a'' \otimes us^n \leq a'' \otimes vs^n \leq a' \otimes s^n$ in $A \otimes_S (Ss^n)$. This means that A_S is GP-po-flat. □

3 GP-po-flatness of the S-poset $A(I)$

Let I be a proper right ideal of a pomonoid S. As it is known, the amalgamated coproduct $A(I)$ of two copies of S over I is an important tool to study the homological classification of pomonoids. In this section, we will investigate GP-po-flatness of the S-poset $A(I)$.

Suppose that I is a proper right ideal of a pomonoid S. For any $x, y, z \notin S$, let $A(I) = (\{x, y\} \times (S - I)) \cup (\{z\} \times I)$. Define a right S-action on $A(I)$ by

$$(x, u)s = \begin{cases} (x, us), & \text{if } us \notin I, \\ (z, us), & \text{if } us \in I, \end{cases}$$

$$(y, u)s = \begin{cases} (y, us), & \text{if } us \notin I, \\ (z, us), & \text{if } us \in I, \end{cases}$$

$$(z, u)s = (z, us).$$

The order on $A(I)$ is defined by

$$(w_1, s) \leq (w_2, t) \Longleftrightarrow (w_1 = w_2, s \leq t) \text{ or } (w_1 \neq w_2,\ s \leq i \leq t \text{ for some } i \in I).$$

In [16] it is proved that $A(I)$ is a right S-poset.

We now present an equivalent condition under which $A(I)$ is GP-po-flat. This condition will be useful to characterize pomonoids over which all S-posets are GP-po-flat

Proposition 3.1. *Let I be a proper right ideal of a pomonoid S. Then the right S-poset $A(I)$ is GP-po-flat if and only if, for every $u, v, s \in S$ and $i \in I$,*

$$us \le i \le vs \Longrightarrow (\exists n \in \mathbb{N})(\exists j \in I)(us^n \le js^n \le vs^n).$$

Proof. **Necessity**. Suppose that the right S-poset $A(I)$ is GP-po-flat. If $us \le i \le vs$ for $u, v, s \in S$ and $i \in I$, then there are two cases to be considered:

Case 1. $u \in I$ or $v \in I$. Then we have $us \in I$ or $vs \in I$, and so it suffices to take $j = u$ or $j = v$.

Case 2. $u \notin I$ and $v \notin I$. Then we have two possibilities.

Subcase 1. $us \in I$ or $vs \in I$. If $us \in I$, then $(x, u)s \le (y, u)s$ in $A(I)$. Since $A(I)$ is GP-po-flat, by Lemma 2.4, there exist $m, n \in \mathbb{N}$, and $(w_1, u_1), \cdots, (w_m, u_m) \in A(I)$, $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{array}{ll} (x,u) \le (w_1,u_1)s_1 & \\ (w_1,u_1)t_1 \le (w_2,u_2)s_2 & s_1s^n \le t_1s^n \\ (w_2,u_2)t_2 \le (w_3,u_3)s_3 & s_2s^n \le t_2s^n \\ \cdots & \cdots \\ (w_m,u_m)t_m \le (y,u) & s_ms^n \le t_ms^n. \end{array}$$

Denote x by w_0 and y by w_{n+1}, then there exists $k \in \{0, 1, \cdots, m\}$ such that $w_k \ne w_{k+1}$, and so, according to the order relation on $A(I)$, there exists $j \in I$ such that $u_kt_k \le j \le u_{k+1}s_{k+1}$. Thus we can compute that

$$us^n \le u_1s_1s^n \le \cdots \le u_kt_ks^n \le js^n \le u_{k+1}s_{k+1}s^n \le \cdots \le u_mt_ms^n \le us^n.$$

But the order is antisymmetric, we have $us^n = js^n$, the result follows. If $vs \in I$, a similar argument can be used.

Subcase 2. $us \notin I$ and $vs \notin I$. By definition of the order on $A(I)$, we have $(x, u)s \le (y, v)s$. The remainder of the proof is similar to the Subcase 1.

From what has been discussed above, we obtain the desired conclusion.

Sufficiency. Assume $(w_1, u), (w_2, v) \in A(I)$ where $w_1, w_2 \in \{x, y, z\}$, and $u, v, s \in S$ are such that $(w_1, u) \otimes s \le (w_2, v) \otimes s$ in $A(I) \otimes_S S$. Then we have four cases as follows:

Case 1. $(w_1, u), (w_2, v) \in (x, 1)S$. Since $(x, 1)S \cong S$ is free, it follows that $(x, 1)S$ is GP-po-flat, and so $(w_1, u) \otimes s^n \le (w_2, v) \otimes s^n$ holds in $(x, 1)S \otimes_S Ss^n$ for some $n \in \mathbb{N}$, and hence also in $A(I) \otimes_S Ss^n$, exactly as needed.

Case 2. $(w_1, u), (w_2, v) \in (y, 1)S$. This case is analogous to the previous one.

Case 3. $w_1 = x$ and $w_2 = y$. In this case, it necessarily implies $u, v \in S - I$. Then we have $(x, u)s \le (y, v)s$ in $A(I)$, and so there exists $i \in I$ such that $us \le i \le vs$. By the assumed condition, there exist $n \in \mathbb{N}$ and $j \in I$ such that $us^n \le js^n \le vs^n$, then we can calculate that, in $A(I) \otimes_S Ss^n$,

$$\begin{aligned} (x,u) \otimes s^n &= (x,1) \otimes us^n \le (x,1) \otimes js^n = (z,j) \otimes s^n \\ &= (y,1)j \otimes s^n = (y,1) \otimes js^n \le (y,1) \otimes vs^n = (y,v) \otimes s^n. \end{aligned}$$

Case 4. $w_1 = y$ and $w_2 = x$. This is similar to the Case 3.

In conclusion, $A(I)$ is GP-po-flat, and the proof is complete. □

4 Homological classification of pomonoids

Based on the preparation of the previous section, in this section, we are going to consider the homological classification of pomonoids by GP-po-flatness of (cyclic, Rees factor) S-posets.

Recall that a pomonoid S is called *regular*, if for every $s \in S$, there exists $x \in S$ such that $s = sxs$.

Definition 4.1. *A pomonoid S is called* generally regular, *if for every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$.*

It is obvious that every regular pomonoid is generally regular. But, [8, Example 3.3] shows that the converse is not true in general.

Using the amalgamated coproduct $A(I)$, we first give a characterization of pomonoids S over which all S-posets are GP-po-flat. Its corresponding result for S-acts is true (see [8, Theorem 3.4]).

Theorem 4.2. *For any pomonoid S, the following statements are equivalent.*
(1) *All right S-posets are GP-po-flat.*
(2) *All right S-posets satisfying Condition (E) are GP-po-flat.*
(3) *S is a generally regular pomonoid.*

Proof. The implication (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). For any $s \in S$, if $sS = S$, then there exists $x \in S$ such that $s = sxs$. Otherwise, $I = sS$ is a proper right ideal of S, by [16, Lemma 2.4], $A(I)$ satisfies Condition (E), and so $A(I)$ is GP-po-flat. In view of Proposition 3.1, from the inequalities $1 \cdot s \leq s \leq 1 \cdot s$ we obtain $n \in \mathbb{N}$ and $j \in I$ with $s^n \leq js^n \leq s^n$. This means that there exists $x \in S$ such that $j = sx$, and so $s^n = sxs^n$, as required.

(3) $\Rightarrow$ (1). It is straightforward to verify. □

From Theorem 4.2 we can deduce the following.

Corollary 4.3. *For a commutative pomonoid S, the following statements are equivalent.*
(1) *All right S-posets are GP-po-flat.*
(2) *For every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = s^n x s^n$.*

We stated that in view of Proposition 2.5, GP-po-flatness implies po-torsion freeness, but the converse is not true. So we naturally consider the question of when all po-torsion free S-posets are GP-po-flat.

The following definition is a generalization of the duality for regularly right almost regular pomonoids which is introduced by Zhang and Laan in [17].

Definition 4.4. *An element s of S is called* generally regularly left almost regular*, if there exist natural numbers $m, n \in \mathbb{N}$, elements $r, r_1, \cdots, r_m$, $s_1, s_2, \cdots, s_m$, $s'_1, s'_2, \cdots, s'_m \in S$, and right po-cancellable elements $c_1, c_2, \cdots, c_m \in S$ such that*

$$\begin{gathered} s_1 c_1 \leq s r_1 \leq s'_1 c_1 \\ s_2 c_2 \leq s_1 r_2 \leq s'_1 r_2 \leq s'_2 c_2 \\ \cdots \\ s_m c_m \leq s_{m-1} r_m \leq s'_{m-1} r_m \leq s'_m c_m \\ s^n = s_m s^n = s'_m s^n. \end{gathered}$$

In particular, when $n = 1$, we say the element s of S is regularly left almost regular.

A pomonoid S is *(generally) regularly left almost regular* (denoted by *(G)RLAR* for short) if all its elements are (generally) regularly left almost regular.

It is easy to see that every (generally) regular pomonoid is *(G)RLAR*, and every *RLAR* pomonoid is *GRLAR*. But Example 4.5 below and [8, Example 3.3] illustrate that these two implications are both strict, respectively.

Example 4.5. *((G)RLAR $\not\Rightarrow$ (generally) regular). Let $S = \langle e, s, c \mid e^2 = e, es = se = ec = ce = s, sc = cs = s^2 \rangle$. Equip S with the order induced by the relations $e < s$ and $s < s^2$, thereby, obtaining a commutative pomonoid S. Actually, $S = \{1, e, s^k, c^k (k \in \mathbb{N})\}$, and the elements of the form c^k $(k \in \mathbb{N})$ and 1 are the only po-cancellable elements. It is not difficult to see that e and 1 are the only regular elements. But since $ec^k = s^k$ and $s^k = es^k$, the elements of the form s^k $(k \in \mathbb{N})$ are also* RLAR*, although they are not generally regular. This shows that there exists a pomonoid, which is not (generally) regular, is (*G*)*RLAR.

Proposition 4.6. *If S is a GRLAR pomonoid, then all po-torsion free right S-posets are GP-po-flat.*

Proof. Let S be a *GRLAR* pomonoid. Assume A_S is a po-torsion free S-poset. Let $as \leq a's$ for $a, a' \in A$ and $s \in S$. Since s is *GRLAR*, there exist $m, n \in \mathbb{N}$, $r, r_1, \cdots, r_m, s_1, s_2, \cdots, s_m, s'_1, s'_2, \cdots, s'_m \in S$, and right po-cancellable elements $c_1, c_2, \cdots, c_m \in S$ such that

$$s_1c_1 \leq sr_1 \leq s'_1c_1$$
$$s_2c_2 \leq s_1r_2 \leq s'_1r_2 \leq s'_2c_2$$
$$\cdots$$
$$s_mc_m \leq s_{m-1}r_m \leq s'_{m-1}r_m \leq s'_mc_m$$
$$s^n = s_ms^n = s'_ms^n.$$

Using the first inequality we get $as_1c_1 \leq asr_1 \leq a'sr_1 \leq a's'_1c_1$. Since A_S is po-torsion free, we see that $as_1 \leq a's'_1$. Further, for the second inequality, we have $as_2c_2 \leq as_1r_2 \leq a's'_1r_2 \leq a's'_2c_2$. So po-torsion freeness of A_S implies $as_2 \leq a's'_2$. In this way we finally arrive at $as_m \leq a's'_m$, and so we can now compute that

$$a \otimes s^n = a \otimes s_ms^n = as_m \otimes s \leq a's'_m \otimes s = a' \otimes s'_ms^n = a' \otimes s^n$$

in $A \otimes_S Ss^n$. This means that A_S is GP-po-flat. □

In particular, when $n = 1$ in the proof of the above proposition, we can deduce

Corollary 4.7. *If S is a RLAR pomonoid, then all po-torsion free right S-posets are principally weakly flat.*

In addition, from [4] we remark that Condition $(PWP)_w$ implies GP-po-flatness, but [4, Example 6.3] shows that this implication is strict. So it is natural to ask for pomonoids over which GP-po-flatness of S-posets implies Condition $(PWP)_w$. To reach the target, we need some more preliminary material.

Recall that a pomonoid S is called *left PSF* if all principal left ideal of S is strongly flat (as a left S-poset). It is shown in [6] that a pomonoid S is left PSF if and only if for $s, t, u \in S$, $su \leq tu$ implies that there exists $r \in S$ such that $ru = u$ and $sr \leq tr$.

Lemma 4.8. *The following statements on a pomonoid S are equivalent.*
(1) *For every proper right ideal I of S there exists $j \in I - Ij$.*
(2) *For every infinite sequence $(x_0, x_1, x_2, \cdots)$ with $x_i = x_{i+1}x_i$, $x_i \in S$, $i = 0, 1, \cdots$, there exists a positive integer n such that $x_n = x_{n+1} = \cdots = 1$.*

Proof. A similar argument as [18, Proposition 2.1] can be used. □

The following proposition is the ordered analogue of [10, Proposition 2.5]. The technique for the proof is taken from the unordered case.

Lemma 4.9. *Let S be a left PSF pomonoid. Then the following statements are equivalent.*
(1) *A_S is GP-po-flat.*
(2) *For any $a, a' \in A$, $s \in S$, $as \leq a's$ implies that there exist $n \in \mathbb{N}$ and $u \in S$ such that $us^n = s^n$ and $au \leq a'u$.*

Now we can address the above matter.

Theorem 4.10. *Let S be a left PSF pomonoid and 1 the identity of S, in which 1 is incomparable with every other element of S. If for every proper right ideal I of S there exists $i \in I - Ii$, then all GP-po-flat right S-posets satisfy Condition $(PWP)_w$.*

Proof. Suppose that A_S is a GP-po-flat right S-poset. Let $as \leq a's$ for $a, a' \in A$ and $s \in S$. Then by Lemma 4.9, there exist $n \in \mathbb{N}$, $u \in S$ such that $au \leq a'u$ and $us^n = s^n$. Since S is left PSF, from $us^n \leq s^n$ we get $x_1 \in S$ with $x_1s^n = s^n$ and $ux_1 \leq x_1$. Further, from the inequality $ux_1 \leq x_1$ we obtain $x_2 \in S$ with $x_2x_1 = x_1$ and $ux_2 \leq x_2$. By continuing this process, letting $x_0 = s^n$ we can find an infinite sequence $(x_0, x_1, \cdots)$, such that

$$x_{i+1}x_i = x_i, \quad ux_i \leq x_i, \quad i = 0, 1, \cdots .$$

By Lemma 4.8, there exists a positive integer m such that $x_m = x_{m+1} = \cdots = 1$. Thus, we get $u \leq 1$. But 1 is isolated, we obtain $u = 1$ and so $a \leq a'$. This shows that A_S satisfies Condition $(PWP)_w$. □

Notice that the proof of the above theorem also allows us to deduce the following.

Theorem 4.11. *Let S be a left PSF pomonoid and 1 the identity of S, in which 1 is either the minimal or the maximal element of S. If for every proper right ideal I of S there exists $i \in I - Ii$, then all GP-po-flat right S-posets satisfy Condition $(PWP)_w$.*

Next, we turn our attention to GP-po-flatness of cyclic S-posets. We need some more preliminary material.

Recall that a relation σ on an S-poset A_S is called a *pseudo-order* on A_S if it is transitive, compatible with the S-action, and contains the relation $\leq$ on A_S. For information pertaining to pseudo-orders on S-posets, we refer the reader to [19], and for further information about order congruence on S-posets to [13, 20].

Suppose ρ is a right order congruence on a pomonoid S. Define a relation $\widehat{\rho}$ by

$$s \,\widehat{\rho}\, t \iff [s]_\rho \leq [t]_\rho \quad \text{in } S/\rho.$$

It is clear that $\widehat{\rho}$ is a pseudo-order on S_S.

The following lemma is useful in dealing with GP-po-flat cyclic S-posets.

Lemma 4.12. *Let ρ be a right order congruence on S and $s \in S$. Then $[u]_\rho \otimes s^n \leq [v]_\rho \otimes s^n$ in $S/\rho \otimes_S Ss^n$ for $u, v \in S$ and $n \in \mathbb{N}$, if and only if $(u, v) \in \widehat{\rho} \sqcup \overrightarrow{ker}\rho_{s^n}$.*

Proof. It is similar to that of [20, Lemma 3.18]. □

Proposition 4.13. *Let ρ be a right order congruence on S. Then S/ρ is GP-po-flat if and only if for $u, v, s \in S$, $[us]_\rho \leq [vs]_\rho$ implies $(u, v) \in \widehat{\rho} \sqcup \overrightarrow{ker}\rho_{s^n}$ for some $n \in \mathbb{N}$.*

Proof. **Necessity.** Let $[us]_\rho \leq [vs]_\rho$ in S/ρ for $u, v, s \in S$. Then we have $[u]_\rho s \leq [v]_\rho s$, and so $[u]_\rho \otimes s \leq [v]_\rho \otimes s$ in $S/\rho \otimes_S S$. Since S/ρ is GP-po-flat, we have $[u]_\rho \otimes s^n \leq [v]_\rho \otimes s^n$ in $S/\rho \otimes_S Ss^n$ for some $n \in \mathbb{N}$. This implies that $(u, v) \in \widehat{\rho} \sqcup \overrightarrow{ker}\rho_{s^n}$ by Lemma 4.12.

Sufficiency. If $[u]_\rho s \leq [v]_\rho s$ in S/ρ, then $[us]_\rho \leq [vs]_\rho$, and so by assumption, $(u, v) \in \widehat{\rho} \sqcup \overrightarrow{ker}\rho_{s^n}$ for some $n \in \mathbb{N}$. Lemma 4.12 implies that $[u]_\rho \otimes s^n \leq [v]_\rho \otimes s^n$ in $S/\rho \otimes_S Ss^n$. Therefore, S/ρ is GP-po-flat. □

Proposition 4.13 immediately implies the following fact about one-element S-posets.

Corollary 4.14. *For any pomonoid S, the one-element S-poset Θ_S is GP-po-flat.*

Recall that a subpomonoid P of a pomonoid S is called *convex*, if $P = [P]$ where

$$[P] = \{x \in S \mid \exists p, q \in P,\ p \leq x \leq q\}.$$

For Rees factor S-posets, we have the following description of GP-po-flatness.

Proposition 4.15. *Let K be a convex, proper right ideal of a pomonoid S. Then the right Rees factor S-poset S/K is GP-po-flat if and only if, for every $k \in K$ and $u, s \in S$,*

$$k \leq us \Longrightarrow (\exists n \in \mathbb{N})(\exists k' \in K)(k's^n \leq us^n) \;\; and$$
$$us \leq k \Longrightarrow (\exists n \in \mathbb{N})(\exists k'' \in K)(us^n \leq k''s^n).$$

Proof. **Necessity.** Assume first that the right Rees factor S-poset S/K is GP-po-flat. Let $k \in K$ and $u, s \in S$ with $k \leq us$. Then we see $[ks]_{\rho_K} \leq [us]_{\rho_K}$ in S/K and so, GP-po-flatness of S/K implies that $[1]_{\rho_K} \otimes ks^n \leq [1]_{\rho_K} \otimes us^n$ in $S/K \otimes_S Ss^n$ for some $n \in \mathbb{N}$. In view of [13, Lemma 4], if $ks^n \leq us^n$, there is nothing to prove. Otherwise, there exists an array

$$ks^n \leq k_1 s^n$$

$$k'_1 s^n \leq k_2 s^n$$
$$\cdots$$
$$k'_m s^n \leq us^n$$

for some $k_i, k'_i \in K$, $i = 1, \cdots, m$. The last line of this array gives what we want. In case $us \leq k$, a similar argument can be used.

Sufficiency. Suppose that $u, v, s \in S$ are such that $[u]_{\rho_K} \otimes s \leq [v]_{\rho_K} \otimes s$ in $S/K \otimes_S S$. Then we see $[us]_{\rho_K} \leq [vs]_{\rho_K}$ in S/K. In light of [13, Lemma 3.1], if $us \leq vs$, then immediately $[1]_{\rho_K} \otimes us \leq [1]_{\rho_K} \otimes vs$, the result follows. Otherwise, $us \leq k$ and $l \leq vs$ for some $k, l \in K$. By the assumed condition, there exist $n_1, n_2 \in \mathbb{N}$ and $k', l' \in K$ such that $us^{n_1} \leq k's^{n_1}$ and $l's^{n_2} \leq vs^{n_2}$. Set $n = max\{n_1, n_2\}$. Then $us^n \leq k's^n$ and $l's^n \leq vs^n$, and so we may now compute that

$$[u]_{\rho_K} \otimes s^n = [1]_{\rho_K} \otimes us^n \leq [1]_{\rho_K} \otimes k's^n = [k']_{\rho_K} \otimes s^n = [l']_{\rho_K} \otimes s^n \leq [1] \otimes l's^n = [v]_{\rho_K} \otimes s^n$$

in $S/K \otimes_S Ss^n$, and the proof is complete. □

In [8], Qiao and Wei proved that generally regular monoids are precisely the monoids over which all Rees factor S-acts are GP-flat. We shall prove an analogue of this result for S-posets.

Theorem 4.16. *For any pomonoid S, the following statements are equivalent.*
(1) *All Rees factor right S-posets are GP-po-flat.*
(2) *For every $s \in S$, there exist $n \in \mathbb{N}$, $s', s'' \in S$ such that $ss's^n \leq s^n \leq ss''s^n$.*

Proof. (1) ⇒ (2). For every $s \in S$, $[sS]$ is a convex right ideal of S. If $[sS] = S$, then there exist $w, v \in S$ such that $sw \leq 1 \leq sv$. Postmultiplying by s^n for any $n \in \mathbb{N}$ we obtain $sws^n \leq s^n \leq svs^n$, exactly as needed. If $[sS] \neq S$ then $[sS]$ is a convex, proper right ideal of S. Obviously, $s \in [sS]$ and $s = 1 \cdot s$, from Proposition 4.15 we obtain $n \in \mathbb{N}$ and $k, k' \in [sS]$ with $ks^n \leq s^n$ and $s^n \leq k's^n$. Also, since $k, k' \in [sS]$, there exist $s', s'', s_1, s_2 \in S$ such that

$$ss' \leq k \leq ss_1 \text{ and } ss_2 \leq k' \leq ss''.$$

Thus we may compute that

$$ss's^n \leq ks^n \leq s^n \leq k's^n \leq ss''s^n.$$

(2) ⇒ (1). Let K be a convex right ideal of S. If $K = S$, then by Corollary 4.14, $S/K \cong \Theta_S$ is GP-po-flat. If K is proper, we will use Proposition 4.15 to check that S/K is GP-po-flat. So for every $k \in K$ and $u, s \in S$, for s by (2) there exist $w, v \in S$ such that $sws^n \leq s^n \leq svs^n$. If $k \leq us$, then we get $(kw)s^n \leq usws^n \leq us^n$. If $us \leq k$, then we have $us^n \leq usvs^n \leq (kv)s^n$. Setting $k' = kw$ or $k'' = kv$, the desired result is obtained. □

As we saw in Section 2, principally weakly po-flat ⇒ GP-po-flat ⇒ po-torsion free. Now our crucial thing is to verify the distinctness of these properties.

Example 4.17. *(GP-po-flat ⇏ p. w. po-flat) Let $S = K \cup \{I\}$ with*

$$K = \left\{ \begin{pmatrix} 0 & m & n \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} \middle| m, n, t \in \mathbb{N} \right\}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The order on S is defined by

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \leq \begin{pmatrix} a' & b' & c' \\ 0 & d' & e' \\ 0 & 0 & f' \end{pmatrix} \Longleftrightarrow a \leq a', b \leq b', c \leq c', d \leq d', e \leq e' \text{ and } f \leq f'.$$

Then K is a convex, proper right ideal of the pomonoid S. By Proposition 4.15, we can verify that the Rees factor S-poset S/K is GP-po-flat. On the other hand, note that $k = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in K$, but there is no $k' \in K$ such that $k \leq k'k$. From [13, Proposition 10] it follows that S/K is not principally weakly po-flat.

Example 4.18 ([13, Example 7]). *(po-t. free $\not\Rightarrow$ GP-po-flat) Let S denote an infinite monogenic monoid $\{1, s, s^2, \cdots\}$, equipped with the order in which*

$$s^2 < s^3 < s^4 < \cdots$$

and s and 1 are isolated. Let $K = \{s^2, s^3, s^4, \cdots\}$. Then by [13, Example 7], S/K is po-torsion free. However, because $s^2 \le s \cdot s$, there cannot exist $n \in \mathbb{N}$ and $k \in K$ such that $k \cdot s^n \le s \cdot s^n$, and by Proposition 4.15, S/K is not GP-po-flat.

5 GP-po-flatness of product S-posets

In this section, we first show that GP-po-flatness is preserved under coproducts and directed colimits, respectively. Furthermore, we mainly consider the question of when GP-po-flat transfers from S-posets to their products. As an application, we also consider the same question to principally weakly po-flat, and extend some results from [12].

The following two propositions show that GP-po-flatness is closed under coproducts and directed colimits, respectively. For more information about coproducts and directed colimits in the category of S-posets, the reader is referred to [15, 21].

Proposition 5.1. *Let $A_S = \coprod_{i \in I} A_i$, where A_i, $i \in I$, is a strongly convex S-subposet of A_S. Then A_S is GP-po-flat if and only if every A_i is GP-po-flat, $i \in I$.*

Proof. It is a direct consequence of the definition. □

Proposition 5.2. *Every directed colimit of a directed system of GP-po-flat right S-posets, is GP-po-flat.*

Proof. Suppose that $(A_i, \phi_{i,j})$ is a directed system of GP-po-flat right S-posets over a directed index set I with directed colimit (A, α_i). Let $as \le a's$ in A_S, for $a, a' \in A$ and $s \in S$. In view of [15, Proposition 2.6 (3)], there exist $i, j \in I$, $a_i \in A_i$, $a_j \in A_j$ such that $a = \alpha_i(a_i)$ and $a' = \alpha_j(a_j)$. Since I is directed, from [15, Proposition 2.6 (4)] we obtain $k \ge i, j$ with $\phi_{i,k}(a_i)s \le \phi_{j,k}(a_j)s$ in A_k. Further, since A_k is GP-po-flat, Lemma 2.4 implies that there exist $m, n \in \mathbb{N}$, $a_1, a_2, \cdots, a_m \in A_k$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{array}{ll} \phi_{i,k}(a_i) \le a_1 s_1 & \\ a_1 t_1 \le a_2 s_2 & s_1 s^n \le t_1 s^n \\ \cdots & \cdots \\ a_m t_m \le \phi_{j,k}(a_j) & s_m s^n \le t_m s^n. \end{array}$$

Acting each inequality in the left hand column of the above scheme by α_k, we can establishes that

$$\alpha_k(\phi_{i,k}(a_i)) \otimes s^n \le \alpha_k(\phi_{j,k}(a_j)) \otimes s^n$$

in $A \otimes_S Ss^n$. Therefore, we can deduce that

$$a \otimes s^n = \alpha_i(a_i) \otimes s^n = \alpha_k \phi_{i,k}(a_i) \otimes s^n \le \alpha_k \phi_{j,k}(a_j) \otimes s^n = \alpha_j(a_j) \otimes s^n = a' \otimes s^n$$

in $A \otimes_S Ss^n$. This shows that A_S is GP-po-flat. □

The following will be used frequently in this section, and its proof is straightforward.

Lemma 5.3. *Let $\{A_i\}_{i \in I}$ be a family of right S-posets and ${}_S B$ be a left S-poset. If $(a_i) \otimes b \le (a'_i) \otimes b'$ in $(\prod_{i \in I} A_i) \otimes_S B$ for any $(a_i), (a'_i) \in \prod_{i \in I} A_i$ and $b, b' \in B$, then $a_i \otimes b \le a'_i \otimes b'$ in $A_i \otimes_S B$ for each $i \in I$.*

Proposition 5.4. *For any family $\{A_i\}_{i \in I}$ of right S-posets, if $\prod_{i \in I} A_i$ is GP-po-flat, then A_i is GP-po-flat for every $i \in I$.*

Proof. Let $a_j s \leq a'_j s$ for $s \in S$ and $a_j, a'_j \in A_j$, $j \in I$. For each $i \neq j$ in I, choose $b_i \in A_i$. Then we define

$$c_i = \begin{cases} b_i, \text{ if } i \neq j, \\ a_j, \text{ if } i = j, \end{cases} \quad \text{and } c'_i = \begin{cases} b_i, \text{ if } i \neq j, \\ a'_j, \text{ if } i = j. \end{cases}$$

This implies that $(c_i)s \leq (c'_i)s$ in $\prod_{i \in I} A_i$, so by assumption, $(c_i) \otimes s^n \leq (c'_i) \otimes s^n$ in $(\prod_{i \in I} A_i) \otimes_S Ss^n$ for some $n \in \mathbb{N}$. The result now follows by Lemma 5.3. □

Corollary 5.5. *For any family $\{A_i\}_{i \in I}$ of right S-posets, if $\prod_{i \in I} A_i$ is principally weakly po-flat, then A_i is principally weakly po-flat for every $i \in I$.*

Proof. Apply Proposition 5.4 for $n = 1$. □

Observing Proposition 5.4 (Corollary 5.5), we remark that pomonoids S need no condition for transferring GP-po-flatness (principal weak po-flatness) from products to their components. However, [10, Example 2.9] shows that direct products do not necessarily preserve these two properties.

Bearing in mind the above, a question naturally arises: when is GP-po-flatness of S-posets preserved under direct products? We first consider the case of finite direct products for this question.

The following is an easy consequence of Lemma 4.9.

Corollary 5.6. *For any left PSF pomonoid, the following statements are equivalent.*
(1) $\prod_{i=1}^n A_i$ *is GP-po-flat.*
(2) *For any $s \in S$ and $a_i, a'_i \in A_i$, $1 \leq i \leq n$, if $(a_1, \cdots, a_n)s \leq (a'_1, \cdots, a'_n)s$ in $\prod_{i=1}^n A_i$, then there exist $m \in \mathbb{N}$ and $u \in S$ such that $us^m = s^m$ and $(a_1, \cdots, a_n)u \leq (a'_1, \cdots, a'_n)u$.*

It follows the same outline as the corresponding result of [10].

Applying [6, Theorem 3.13], the following is an evident result for principal weak po-flatness.

Corollary 5.7. *For any left PSF pomonoid, the following statements are equivalent.*
(1) $\prod_{i=1}^n A_i$ *is principally weakly po-flat.*
(2) *For any $s \in S$ and $a_i, a'_i \in A_i$, $1 \leq i \leq n$, if $(a_1, \cdots, a_n)s \leq (a'_1, \cdots, a'_n)s$ in $\prod_{i=1}^n A_i$, then there exists $u \in S$ such that $us = s$ and $(a_1, \cdots, a_n)u \leq (a'_1, \cdots, a'_n)u$.*

It is shown in [10] that, for a left PSF monoid, $\prod_{i=1}^n A_i$ is GP-flat if and only if A_i is GP-flat, $1 \leq i \leq n$. For S-posets, the corresponding statement is also valid.

Proposition 5.8. *Let S be a left PSF pomonoid. Then $\prod_{i=1}^n A_i$ is GP-po-flat if and only if A_i is GP-po-flat, $1 \leq i \leq n$.*

Proof. Applying Proposition 5.4, and a similar argument as for acts it can easily be proved. □

Specifically, the following corollary generalizes a result of [12], which says that for any left PSF pomonoid S the S-poset S^n is principally weakly po-flat for each $n \in \mathbb{N}$.

Corollary 5.9. *Let S be a left PSF pomonoid. Then $\prod_{i=1}^n A_i$ is principally weakly po-flat if and only if A_i is principally weakly po-flat for every $1 \leq i \leq n$.*

Proof. This follows from Corollary 5.5 and [6, Theorem 3.13]. □

Our next task is to discuss the case of infinite products for the question mentioned above. The inspiration for some of the following results comes from [11].

By virtue of Lemma 2.4, the following is now immediate.

Lemma 5.10. *Let $A_S = \prod_{i \in I} A_i$ for a family $\{A_i\}_{i \in I}$ of right S-posets. If the S-poset S^I is GP-po-flat and $(u_i)s \leq (v_i)s$ for $u_i, v_i, s \in S$, then for each $i \in I$ and $a_i \in A_i$, $(a_i u_i) \otimes s \leq (a_i v_i) \otimes s$ in $A \otimes_S Ss$.*

Now we intend to present the main results of this paper.

Theorem 5.11. *The following statements are equivalent for a pomonoid S.*
(1) *The direct product of every nonempty family of GP-po-flat right S-posets is GP-po-flat.*
(2) (a) *S^I is GP-po-flat for each nonempty set I, and*
(b) *for each $s \in S$, there exist $m, n \in \mathbb{N}$ such that for every GP-po-flat right S-poset A_S, if $as \leq a's$ for any $a, a' \in A$ then $l_s(a, a') \leq m$ and $d_s(a, a') \leq n$.*

Proof. (1) $\Rightarrow$ (2). Part (a) is obvious. Now we prove part (b) by contradiction. Assume there is $s \in S$ such that, (i) for each $i \in \mathbb{N}$ there exists a GP-po-flat S-posets $(A_i)_S$ such that $a_i s \leq b_i s$ and $l_s(a_i, b_i) > i$ for some $a_i, b_i \in A_i$, or (ii) for each $j \in \mathbb{N}$ there exists a GP-po-flat S-posets $(A_j)_S$ such that $a_j s \leq b_j s$ and $d_s(a_j, b_j) > j$ for some $a_j, b_j \in A_j$. If case (i) holds, then by (1), $\prod_{i=1}^{\infty} A_i$ is GP-po-flat. Thereby, $(a_i)s \leq (b_i)s$ in $\prod_{i=1}^{\infty} A_i$ implies the existence of a scheme of length m (and degree k) in $\prod_{i=1}^{\infty} A_i \times Ss^k$. That is, for each $i \in \mathbb{N}$, a scheme of length m connecting (a_i, s^k) to (b_i, s^k) in $A_i \times Ss^k$, which contradicts $l_s(a_{m+1}, b_{m+1}) > m+1$. Case (ii) can be disposed of similarly.

(2) $\Rightarrow$ (1). Suppose that $\{A_i\}_{i \in I}$ is a family of GP-po-flat right S-posets and $A_S = \prod_{i \in I} A_i$. Let $(a_i)s \leq (a'_i)s$ in A_S. Then $a_i s \leq a'_i s$ for each $i \in I$, and by (b), there exist $m, n \in \mathbb{N}$ such that (a_i, s^n) and (a'_i, s^n) are connected by a scheme of length m and degree n in $A_i \times Ss^n$ for every $i \in I$. This implies that for every $i \in I$, there exists a scheme of the form

$$\begin{array}{rl} a_i \leq a_{1i}s_{1i} & \\ a_{1i}t_{1i} \leq a_{2i}s_{2i} & s_{1i}s^n \leq t_{1i}s^n \\ \cdots & \cdots \\ a_{mi}t_{mi} \leq a'_i & s_{mi}s^n \leq t_{mi}s^n, \end{array}$$

where $a_{1i}, \cdots, a_{mi} \in A_i$, and $s_{1i}, \cdots, s_{mi}, t_{1i}, \cdots, t_{mi} \in S$. From the right-hand of the above schemes, we get $(s_{ji})s^n \leq (t_{ji})s^n$ in S^I for all $1 \leq j \leq m$. Further, in light of Lemma 5.10, we see that for every $1 \leq j \leq m$,

$$(a_{ji}s_{ji}) \otimes s^{nk} \leq (a_{ji}t_{ji}) \otimes s^{nk}$$

in $A \otimes_S Ss^{nk}$ for some $k \in \mathbb{N}$. Therefore, we may compute that

$$\begin{aligned} (a_i) \otimes s^{nk} &\leq (a_{1i}t_{1i}) \otimes s^{nk} \leq (a_{1i}t_{1i}) \otimes s^{nk} \leq (a_{2i}t_{2i}) \otimes s^{nk} \leq (a_{2i}t_{2i}) \otimes s^{nk} \\ &\leq \cdots \leq (a_{mi}t_{mi}) \otimes s^{nk} \leq (a_{mi}t_{mi}) \otimes s^{nk} \leq (a'_i) \otimes s^{nk} \end{aligned}$$

in $A \otimes_S Ss^{nk}$, and the proof is complete. □

Observing the proof of Theorem 5.11, when $|I| < \infty$, we can readily obtain the condition (b) of the part (2) in Theorem 5.11. Thereby, we have the following.

Corollary 5.12. *For $n \in \mathbb{N}$, S^n is GP-po-flat right S-poset if and only if $\prod_{i=1}^{n} A_i$ is GP-po-flat where A_i, $1 \leq i \leq n$, are GP-po-flat right S-posets.*

In order to make Theorem 5.11 more specific, we give a description of pomonoids S over which S^I is GP-po-flat for each nonempty set I.

Proposition 5.13. *The following statements are equivalent for a pomonoid S.*
(1) *S^I is GP-po-flat for each nonempty set I.*
(2) *For any $s \in S$, there exist $m, n \in \mathbb{N}$ and $(s_1, t_1), \cdots, (s_m, t_m) \in D(S)$ such that*
(a) *$s_i s^n \leq t_i s^n$ for all $1 \leq i \leq m$, and*
(b) *if $us \leq vs$ for some $u, v \in S$, then there exist $u_1, \cdots, u_m \in S$ such that*

$$u \leq u_1 s_1$$

$$u_1t_1 \leq u_2s_2$$
$$\cdots$$
$$u_{m-1}t_{m-1} \leq u_ms_m$$
$$u_mt_m \leq v.$$

Proof. (1) $\Rightarrow$ (2). Let $L = \{(u,v) \in D(S) | us \leq vs\}$, and index L by $L = \{(u_i, v_i) | i \in I\}$. Then we see $(u_i)s \leq (v_i)s$ in S^I. Since S^I is GP-po-flat, from Lemma 2.4 we obtain $m, n \in \mathbb{N}$, $(u_{1i}), \cdots, (u_{mi}) \in S^I$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that

$$\begin{array}{cc} (u_i) \leq (u_{1i})s_1 & \\ (u_{1i})t_1 \leq (u_{2i})s_2 & s_1s^n \leq t_1s^n \\ \cdots & \cdots \\ (u_{mi})t_m \leq (v_i) & s_ms^n \leq t_ms^n. \end{array}$$

Hence we see that we have reached the desired conclusion.

(2) $\Rightarrow$ (1). Let $I \neq \emptyset$, and let $(u_i), (v_i) \in S^I$ be such that $(u_i) \otimes s \leq (v_i) \otimes s$ in $S^I \otimes_S S$. Then we see $(u_i)s \leq (v_i)s$ in S^I. By (2), there exist $m, n \in \mathbb{N}$ and $(s_1, t_1), \cdots, (s_m, t_m) \in D(S)$ such that $s_js^n \leq t_js^n$ for all $1 \leq j \leq m$, and there exist $u_{1i}, \cdots, u_{mi}$ for all $i \in I$ such that

$$u_i \leq u_{1i}s_1$$
$$u_{1i}t_1 \leq u_{2i}s_2$$
$$\cdots$$
$$u_{mi}t_m \leq v_i.$$

Then we can compute

$$\begin{aligned} (u_i) \otimes s^n &\leq (u_{1i})s_1 \otimes s^n = (u_{1i}) \otimes s_1s^n \leq (u_{1i}) \otimes t_1s^n \\ &= (u_{1i})t_1 \otimes s^n \leq (u_{2i})s_2 \otimes s^n \leq \cdots \leq (u_{mi})t_m \otimes s^n \leq (v_i) \otimes s^n \end{aligned}$$

in $S^I \otimes_S Ss^n$, and this shows that S^I is GP-po-flat, as required. □

In Lemma 2.4, particularly when $n = 1$, we have the following result.

Lemma 5.14. *A right S-poset A_S is principally weakly po-flat if and only if for any $a, a' \in A$ and $s \in S$, $as \leq a's$ in A_S implies that there exist $m \in \mathbb{N}$, $a_1, a_2, \cdots, a_m \in A$ and $s_1, t_1, \cdots, s_m, t_m \in S$ such that*

$$\begin{array}{cc} a \leq a_1s_1 & \\ a_1t_1 \leq a_2s_2 & s_1s \leq t_1s \\ \cdots & \cdots \\ a_mt_m \leq a' & s_ms \leq t_ms. \end{array}$$

In the above lemma, we define $d'_s(a, a')$ to be the minimum length of the existing schemes connecting (a, s) to (a', s).

Now by a similar argument as in the proof of Theorem 5.11, for principal weak po-flatness we have the following result, which extends Proposition 2.4 of [12].

Corollary 5.15. *The following statements are equivalent for a pomonoid S.*

(1) *The direct product of every nonempty family of principally weakly po-flat right S-posets is principally weakly po-flat.*

(2) (a) *S^I is principally weakly po-flat for each nonempty set I, and*

  (b) *for each $s \in S$, there exists $n \in \mathbb{N}$ such that for every principally weakly po-flat right S-poset A_S, if $as \leq a's$ for any $a, a' \in A$ then $d'_s(a, a') \leq n$.*

We pointed that in Section 1, for GP-po-flatness and principal weak po-flatness, the direct product case is different from that of Conditions (P), (E) and (P_w). In other words, we need to identify that the two conditions in Theorem 5.11(2) or Corollary 5.15(2) are independent. Indeed, from [11, Example 2.13] we can see that the condition (a) in Theorem 5.11(2) (resp., Corollary 5.15(2)) does not imply the condition (b). Also, the following example shows that the converse is not true.

Example 5.16 ([11, Examples 2.12]). *Let $S = \langle x_1, x_2, \cdots | x_{n+1}x_n = x_{n+1} = x_n x_{n+1}\rangle$, where the order of S is discrete. It is not hard to see, that S is a commutative pomonoid consisting of the elements of the form 1 and x_i^k $(k \in \mathbb{N})$. Then we could directly apply Examples 2.12 from [11] and obtain that, for every principally weakly po-flat S-poset A_S, if $as \leq a's$ for $a, a' \in A$ and $s \in S$ then there exists an S-tossing of length 1 connecting the pairs (a, s) and (a', s) in $A \times Ss$. This shows that S satisfies the condition (b) in Corollary 5.15, and so it also satisfies the condition (b) of Theorem 5.11 (it suffices in Theorem 5.11 to take $m = 1$.) On the other hand, we assume S^2 is GP-po-flat. Then, according the above statement, for $(1, x_1)x_2 = (x_1, x_1)x_2$ in S^2, there exists an S-tossing of length 1 and degree 1 connecting the pairs $((1, x_1), x_2)$ and $((x_1, x_1), x_2)$ in $S^2 \times Sx_2$, but this is impossible. Thus S^2 is not GP-po-flat, and so it is not principally weakly flat.*

Recall that a pomonoid S is called *left PP* if the S-subposet Sx is projective for all $x \in S$. (Note, however, that Sx may not be an ideal of S in the ordered sense.) According to [7, Proposition 4.8], a pomonoid S is left PP if and only if for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and $us \leq vs$ implies $ue \leq ve$ for $u, v \in S$. Further, using Proposition 3.3 and Corollary 3.7 of [20], it is straightforward to prove that for every $x \in S$, Sx is projective if and only if $[1]_{ker\rho_x}$ contains a right zero, where $ker\rho_x = \{(u, v) \in S \times S |\ ux = vx\}$.

Note that, every left PP pomonoid is left PSF, and but the converse is not true. The next proposition shows that, for a commutative pomonoid this intervening gap between these two classes of pomonoids can be filled by GP-po-flatness.

Proposition 5.17. *The following are equivalent for a commutative pomonoid S.*
(1) *S is a left PP pomonoid.*
(2) *S is a left PSF pomonoid and S^I is principally weakly po-flat for any nonempty set I.*
(3) *S is a left PSF pomonoid and S^I is GP-po-flat for any nonempty set I.*

Proof. (1) $\Rightarrow$ (2). Let S be a left PP pomonoid. Then S is left PSF. To show that S^I is principally weakly po-flat, assume that $(s_i)s \leq (t_i)s$ in S^I. Since S is left PP, there exists $e^2 = e \in S$ such that $es = s$ and $s_i e \leq t_i e$ for each $i \in I$. This means that $(s_i)e \leq (t_i)e$ in S^I, and the desired result is readily obtained.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Suppose $s \in S$ for a left PSF pomonoid S. Based on the above discussion, it is enough to find a right zero in $[1]_{\ker\rho_s}$. Assume that $[1]_{\ker\rho_s}$ is represented by an index set I as $[1]_{\ker\rho_s} = \{u_i | i \in I\}$. Then $(u_i)s = s$ in S^I. Since S^I is principally weakly po-flat, by Lemma 4.9, we obtain $u, w \in S$ with $us = s$, $u_i u \leq u$, and $ws = s$, $w \leq u_i w$ for any $i \in I$. Further, since S is commutative, we can compute that for each $i \in I$,

$$u_i uw \leq uw \leq uu_i w = u_i uw,$$

that is, $u_i uw = uw$. Therefore, $uw \in [1]_{\ker\rho_s}$ is a right zero. □

From Proposition 5.8, we remark that, for any left PSF pomonoid S, S^n is GP-po-flat for each $n \in \mathbb{N}$. However, the example below shows that the converse is not true in general. Indeed, let S denote the monoid $\{0, x, 1\}$ in which $x^2 = 0$. The order of S is discrete. We can verify that S^2 is GP-po-flat. On the other hand, note that $0 \cdot x = x \cdot x$, there are no elements $r \in S$ such that $r \cdot x = x$ and $0 \cdot r = x \cdot r$. Hence S is not a left PSF pomonoid.

It is natural to ask when GP-po-flatness of S^n implies that S is a left PSF pomonoid. To reach this target, we need to introduce a corresponding notion, known as left $P(P)$ monoids, for S-posets.

Definition 5.18. *We call a pomonoid S left $P(P)$ if every principal left ideal of S satisfies Condition (P).*

It can be readily checked that a pomonoid S is left $P(P)$ if and only if $us \le vs$ for $u, v, s \in S$, implies the existence of $u', v' \in S$ such that $uu' \le vv'$ and $u's = v's = s$. Clearly, every left PSF pomonoid is left $P(P)$. But, from [22, Example 2.4] we see that the converse is not true in general.

Now we can establish one of our main results.

Theorem 5.19. *Let S be a pomonoid and 1 the identity of S, in which 1 is isolated. Then the following conditions on pomonoids are equivalent.*
(1) *S is a left PSF pomonoid.*
(2) *S is a left $P(P)$ pomonoid and S^n is principally weakly po-flat for each $n \in \mathbb{N}$.*
(3) *S is a left $P(P)$ pomonoid and S^n is GP-po-flat for each $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2). It follows directly from [12, Proposition 2.3].

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Let $us \le vs$ for $u, v, s \in S$. Then we see $(1,u)s \le (1,v)s$ in S^2 and by Lemma 2.4, there exists a scheme realizing the inequality $(1,u) \otimes s \le (1,v) \otimes s$ in $S^2 \otimes_S Ss$ of the form

$$\begin{array}{ll} (1,u) \le (x_1,y_1)s_1 & \\ (x_1,y_1)t_1 \le (x_2,y_2)s_2 & s_1 s^n \le t_1 s^n \\ (x_2,y_2)t_2 \le (x_3,y_3)s_3 & s_2 s^n \le t_2 s^n \\ \cdots & \cdots \\ (x_m,y_m)t_m \le (1,v) & s_m s^n \le t_m s^n \end{array}$$

of length m, where $m, n \in \mathbb{N}$, $x_1, \cdots, x_m$, $y_1, \cdots, y_m$, $s_1, \cdots, s_m$, $t_1, \cdots, t_m \in S$. Without loss of generality, suppose that the length m of this scheme is minimal. We claim that $m = 1$ and hence our scheme would be of the form

$$\begin{array}{ll} (1,u) \le (x_1,y_1)s_1 & \\ (x_1,y_1)t_1 \le (1,v) & s_1 s^n \le t_1 s^n, \end{array}$$

thereby from the left hand of the above scheme, we get $1 \le x_1 s_1$ and $x_1 t_1 \le 1$. But 1 is isolated and we obtain $x_1 = s_1 = t_1 = 1$, and then $us_1 \le y_1 s_1 \le vs_1$ and $s_1 s = s$, as desired.

Assume $m > 1$. The inequalities $(x_1,y_1)t_1 \le (x_2,y_2)s_2$ and $(x_2,y_2)t_2 \le (x_3,y_3)s_3$ yield $x_1 t_1 \le x_2 s_2$, $y_1 t_1 \le y_2 s_2$, $x_2 t_2 \le x_3 s_3$ and $y_2 t_2 \le y_3 s_3$. Since S is a left $P(P)$ pomonoid, from the inequality $s_2 s^n \le t_2 s^n$ we obtain $u_1, v_1 \in S$ with $s_2 u_1 \le t_2 v_1$ and $u_1 s^n = v_1 s^n = s^n$. Then we see $x_1 t_1 u_1 \le x_2 s_2 u_1 \le x_2 t_2 v_1 \le x_3 s_3 v_1$ and $y_1 t_1 u_1 \le y_2 s_2 u_1 \le y_2 t_2 v_1 \le y_3 s_3 v_1$. This shows the following is a scheme of length $m-1$ realizing the inequality $(1,u) \otimes s \le (1,v) \otimes s$ in $S^2 \otimes_S Ss$:

$$\begin{array}{ll} (1,u) \le (x_1,y_1)s_1 & \\ (x_1,y_1)t_1 u_1 \le (x_3,y_3)s_3 v_1 & s_1 s^n \le (t_1 u_1) s^n \\ (x_3,y_3)t_3 \le (x_4,y_4)s_4 & (s_3 v_1) s^n \le t_3 s^n \\ \cdots & \cdots \\ (x_m,y_m)t_m \le (1,v) & s_m s^n \le t_m s^n. \end{array}$$

This contradicts the minimality of m. □

Here we prove that the two conditions in the second part and in the third part of Theorem 5.19 are independent from each other. On the one hand, note from [11, Example 2.13] that there is a pomonoid S (the order of S is discrete) over which S^I is principally weakly po-flat (hence S^I is GP-po-flat) for each nonempty set I, but S is not a left PSF pomonoid. Thus in view of Theorem 5.19, principal weak po-flatness (GP-po-flatness) of S^n does not imply that S is a left $P(P)$ pomonoid. On the other hand, from [22, Example 2.4] there is a left $P(P)$ pomonoid which is not a left PSF pomonoid. This shows that being a left $P(P)$ pomonoid does not imply GP-po-flatness (principal weak po-flatness) of S^n.

As the concluding result, we have

Proposition 5.20. *For a right po-cancellative pomonoid S and any family $\{A_i\}_{i\in I}$ of right S-posets, the following statements are equivalent.*
(1) $\prod_{i\in I} A_i$ *is principally weakly po-flat.*
(2) $\prod_{i\in I} A_i$ *is GP-po-flat.*
(3) $\prod_{i\in I} A_i$ *is po-torsion free.*
(4) S *is a po-group.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are obvious.
(3) $\Rightarrow$ (4). It is true by Theorem 4.2. □

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