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Research Article

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Convolutions of harmonic right half-plane mappings

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Abstract: We first prove that the convolution of a normalized right half-plane mapping with another subclass of normalized right half-plane mappings with the dilatation $-z(a+z)/(1+az)$ is CHD (convex in the horizontal direction) provided $a = 1$ or $-1 \leq a \leq 0$. Secondly, we give a simply method to prove the convolution of two special subclasses of harmonic univalent mappings in the right half-plane is CHD which was proved by Kumar et al. [1, Theorem 2.2]. In addition, we derive the convolution of harmonic univalent mappings involving the generalized harmonic right half-plane mappings is CHD. Finally, we present two examples of harmonic mappings to illuminate our main results.

Keywords: Harmonic univalent mappings, Harmonic convolution, Generalized right half-plane mappings

MSC: 30C45, 58E20

1 Introduction and main results

Assume that $f = u + iv$ is a complex-valued harmonic function defined on the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real harmonic functions in $\mathbb{U}$. Such function can be expressed as $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $\mathbb{U}$, h and g are known as the analytic part and co-analytic part of f, respectively. A harmonic mapping $f = h + \overline{g}$ is locally univalent and sense-preserving if and only if the dilatation of f defined by $\omega(z) = g'(z)/h'(z)$, satisfies $|\omega(z)| < 1$ for all $z \in \mathbb{U}$.

We denote by $\mathcal{S}_H$ the class of all harmonic, sense-preserving and univalent mappings $f = h + \overline{g}$ in $\mathbb{U}$, which are normalized by the conditions $h(0) = g(0) = 0$ and $h'(0) = 1$. Let $\mathcal{S}_H^0$ be the subset of all $f \in \mathcal{S}_H$ in which $g'(0) = 0$. Further, let $\mathcal{K}_H, \mathcal{C}_H$ (resp. $\mathcal{K}_H^0, \mathcal{C}_H^0$) be the subclass of $\mathcal{S}_H$ (resp. $\mathcal{S}_H^0$) whose images are convex and close-to-convex domains. A domain Ω is said to be convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with Ω.

For harmonic univalent functions

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n$$

and

$$F(z) = H(z) + \overline{G(z)} = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z}^n,$$

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the convolution (or Hadamard product) of them is given by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n} \overline{z}^n.$$

Many research papers in recent years have studied the convolution or Hadamard product of planar harmonic mappings, see [2–12]. However, corresponding questions for the class of univalent harmonic mappings seem to be difficult to handle as can be seen from the recent investigations of the authors [13–17]. In [1], Kumar et al. constructed the harmonic functions $f_a = h_a + \overline{g_a} \in \mathcal{K}_H$ in the right half-plane, which satisfy the conditions $h_a + g_a = z/(1-z)$ with $\omega_a(z) = (a-z)/(1-az)$ $(-1 < a < 1)$. By using the technique of shear construction (see [18]), we have

$$h_a(z) = \frac{\frac{1}{1+a}z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad g_a(z) = \frac{\frac{a}{1+a}z - \frac{1}{2}z^2}{(1-z)^2}. \tag{1}$$

Obviously, for $a = 0$, $f_0(z) = h_0(z) + \overline{g_0(z)} \in \mathcal{K}_H^0$ is the standard right half-plane mapping, where

$$h_0(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \quad \text{and} \quad g_0(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2}. \tag{2}$$

Recently Dorff et al. studied the convolution of harmonic univalent mappings in the right half-plane (cf. [14, 15]). They proved that:

Theorem A ([14, Theorem 5]). *Let $f_1 = h_1 + \overline{g_1}$, $f_2 = h_2 + \overline{g_2} \in \mathcal{S}_H^0$ with $h_i + g_i = z/(1-z)$ for $i = 1, 2$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in \mathcal{S}_H^0$ and convex in the horizontal direction.*

Theorem B ([15, Theorem 3]). *Let $f_n = h + \overline{g} \in \mathcal{S}_H^0$ with $h + g = z/(1-z)$ and $\omega(z) = g'(z)/h'(z) = e^{i\theta} z^n$ $(\theta \in \mathbb{R}, n \in \mathbb{N}^+)$. If $n = 1, 2$, then $f_0 * f_n \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

Theorem C ([15, Theorem 4]). *Let $f = h + \overline{g} \in \mathcal{K}_H^0$ with $h(z) + g(z) = z/(1-z)$ and $\omega(z) = (z+a)/(1+az)$ with $a \in (-1, 1)$. Then $f_0 * f \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

We now begin to state the elementary result concerning the convolutions of f_0 with the other special subclass harmonic mappings.

Theorem 1.1. *Let $f = h + \overline{g} \in \mathcal{S}_H^0$ with $h(z) + g(z) = z/(1-z)$ and $\omega(z) = -z(z+a)/(1+az)$, then $f_0 * f \in \mathcal{S}_H^0$ and is convex in the horizontal direction for $a = 1$ or $-1 \le a \le 0$.*

The following generalized right half-plane harmonic univalent mappings were introduced by Muir [7]:

$$L_c(z) = H_c(z) + \overline{G_c(z)} = \frac{1}{1+c}\left[\frac{z}{1-z} + \frac{cz}{(1-z)^2}\right] + \overline{\frac{1}{1+c}\left[\frac{z}{1-z} - \frac{cz}{(1-z)^2}\right]} \quad (z \in \mathbb{U};\ c > 0). \tag{3}$$

Clearly, $L_1(z) = f_0(z)$, it was proved in [7] that $L_c(\mathbb{U}) = \{\text{Re}(\omega) > -1/(1+c)\}$ for each $c > 0$. Moreover, if $f = h + \overline{g} \in \mathcal{S}_H$, then the above representation gives that

$$L_c * f = \frac{h + czh'}{1+c} + \overline{\frac{g - czg'}{1+c}}. \tag{4}$$

The following *Cohn's Rule* is helpful in proving our main results.

Cohn's Rule ([19, p.375]). *Given a polynomial*

$$p(z) = p_0(z) = a_{n,0}z^n + a_{n-1,0}z^{n-1} + \cdots + a_{1,0}z + a_{0,0} \quad (a_{n,0} \neq 0) \tag{5}$$

of degree n, let

$$p^*(z) = p_0^*(z) = z^n \overline{p(1/\overline{z})} = \overline{a_{n,0}} + \overline{a_{n-1,0}}z + \cdots + \overline{a_{1,0}}z^{n-1} + \overline{a_{0,0}}z^n \tag{6}$$

Denote by r and s the number of zeros of $p(z)$ inside the unit circle and on it, respectively. If $|a_{0,0}| < |a_{n,0}|$, *then*

$$p_1(z) = \frac{\overline{a_{n,0}}p(z) - a_{0,0}p^*(z)}{z}$$

is of degree $n-1$ *with* $r_1 = r-1$ *and* $s_1 = s$ *the number of zeros of* $p_1(z)$ *inside the unit circle and on it, respectively.*

It should be remarked that *Cohn's Rule* and *Schur-Cohn's algorithm* [19, p. 378] are important tools to prove harmonic mappings are locally univalent and sense-preserving. Some related works have been done on these topics, one can refer to [1, 3, 6, 13, 15, 16]. In [1], the authors proved the following result.

Theorem D. *Let* $f_a = h_a + \overline{g_a}$ *be given by* (1). *If* $f_n = h + \overline{g}$ *is the right half-plane mapping given by* $h + g = z/(1-z)$ *with* $\omega(z) = e^{i\theta}z^n$ $(\theta \in \mathbb{R}, n \in \mathbb{N}^+)$, *then* $f_a * f_n \in \mathcal{S}_H$ *is CHD for* $a \in [\frac{n-2}{n+2}, 1)$.

In this paper, we will use a new method to prove the above theorem. The main difference of our work from [1] is that we construct a sequence of functions for finding all zeros of polynomials which are in $\overline{\mathbb{U}}$, and we will show that the dilatation of $f_a * f_n$ satisfies $|\widetilde{\omega}_1(z)| = |(g_a * g)'/(h_a * h)'| < 1$ by using mathematical induction, which greatly simplifies the calculation compared with the proof of Theorem 2.2 in [1]. We also show that $L_c * f_a$ is univalent and convex in the horizontal direction for $0 < c \le 2(1+a)/(1-a)$, and derive the following theorem.

Theorem 1.2. *Let* $L_c = H_c + \overline{G_c}$ *be harmonic mappings given by* (3). *If* $f_a = h_a + \overline{g}_a$ *is the right half-plane mappings given by* (1), *then* $L_c * f_a$ *is univalent and convex in the horizontal direction for* $0 < c \le 2(1+a)/(1-a)$.

Recently, Liu and Li [6] defined a subclass of harmonic mappings defined by

$$P_c(z) = H_c(z) - \overline{G_c(z)} = \frac{1}{1+c}\left[\frac{cz}{(1-z)^2} + \frac{z}{1-z}\right] + \frac{1}{1+c}\overline{\left[\frac{cz}{(1-z)^2} - \frac{z}{1-z}\right]} \quad (z \in \mathbb{U};\ c > 0). \tag{7}$$

They proved the following result.

Theorem E. ([6, Theorem 7]) *Let* $P_c(z)$ *be harmonic mappings defined by* (7) *and* $f_n = h + \overline{g}$ *with* $h - g = z/(1-z)$ *and dilatation* $\omega(z) = e^{i\theta}z^n(\theta \in \mathbb{R}, n \in \mathbb{N}^+)$. *Then* $P_c * f_n$ *is univalent and convex in the horizontal direction for* $0 < c \le 2/n$.

Similar to the approach used in the proof of Theorem E, we get the following result.

Theorem 1.3. *Let* $L_c = H_c + \overline{G_c}$ *be harmonic mappings given by* (3) *and* $f_n = h + \overline{g}$ *with* $h + g = z/(1-z)$ *and dilatation* $\omega(z) = e^{i\theta}z^n(\theta \in \mathbb{R}, n \in \mathbb{N}^+)$. *Then* $L_c * f_n$ *is univalent and convex in the horizontal direction for* $0 < c \le 2/n$.

Remark 1.4. *In Theorem 1.2 and Theorem 1.3, let* $c = 1$, *clearly,* $L_1 = f_0$, *so Theorem 1.2 and Theorem 1.3 are generalization of Theorem C and Theorem B, respectively. It also explains why Theorem B does not hold for* $n \ge 3$, *since* $c = 1$, *according to Theorem 1.3, it follows that* $0 < c \le 2/n$ *hold for* $n = 1, 2$.

2 Preliminary results

In this section, we will give the following lemmas which play an important role in proving the main results.

Lemma 2.1 ([15, Eq.(6)]). *If* $f = h + \overline{g} \in \mathcal{S}_H^0$ *with* $h + g = z/(1-z)$ *and dilatation* $\omega(z) = g'(z)/h'(z)$, *then the dilatation of* $f_0 * f$ *is given by*

$$\widetilde{\omega}(z) = -z\frac{\omega^2 + \left[\omega - \frac{1}{2}\omega' z\right] + \frac{1}{2}\omega'}{1 + \left[\omega - \frac{1}{2}\omega' z\right] + \frac{1}{2}\omega' z^2}. \tag{8}$$

Lemma 2.2 ([1, Lemma 2.1]). *Let $f_a = h_a + \overline{g_a}$ be defined by (1) and $f = h + \overline{g} \in \mathcal{S}_H^0$ be right half-plane mapping, where $h+g = z/(1-z)$ with dilatation $\omega(z) = g'(z)/h'(z)$ $(h'(z) \neq 0, z \in \mathbb{U})$. Then $\widetilde{\omega}_1$, the dilatation of $f_a * f$, is given by*

$$\widetilde{\omega}_1(z) = \frac{2(a-z)\omega(1+\omega) + (a-1)\omega' z(1-z)}{2(1-az)(1+\omega) + (a-1)\omega' z(1-z)}. \tag{9}$$

Lemma 2.3. *Let $L_c = H_c + \overline{G_c}$ be defined by (3) and $f = h + \overline{g} \in \mathcal{S}_H^0$ be right half-plane mapping, where $h + g = z/(1-z)$ with dilatation $\omega(z) = g'(z)/h'(z)$ $(h'(z) \neq 0, z \in \mathbb{U})$. Then $\widetilde{\omega}_2$, the dilatation of $L_c * f$, is given by*

$$\widetilde{\omega}_2(z) = \frac{[(1-c)-(1+c)z]\omega(1+\omega) - c\,\omega' z(1-z)}{[(1+c)-(1-c)z](1+\omega) - c\,\omega' z(1-z)}. \tag{10}$$

Proof. By (4), we know that

$$\widetilde{\omega}_2(z) = \frac{(g - czg')'}{(h + czh')'}.$$

Similar calculation as in the proof of [6, Lemma 7] gives (10). □

Lemma 2.4 ([20, Theorem 5.3]). *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{U}$ is a univalent mapping of $\mathbb{U}$ onto a domain convex in the horizontal direction if and only if $h - g$ is a conformal univalent mapping of $\mathbb{U}$ onto a domain convex in the horizontal direction.*

Lemma 2.5 (See [21]). *Let f be an analytic function in $\mathbb{U}$ with $f(0) = 0$ and $f'(0) \neq 0$, and let*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta_1})(1 + ze^{i\theta_2})}, \tag{11}$$

where $\theta_1, \theta_2 \in \mathbb{R}$. If

$$\mathrm{Re}\left(\frac{zf'(z)}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

Then f is convex in the horizontal direction.

Lemma 2.6. *Let $L_c = H_c + \overline{G_c}$ be a mapping given by (3) and $f = h + \overline{g}$ be the right half-plane mapping with $h + g = z/(1-z)$. If $L_c * f$ is locally univalent, then $L_c * f \in \mathcal{S}_H^0$ and is convex in the horizontal direction.*

Proof. Recalling that $L_c = H_c + \overline{G_c}$ and

$$H_c + G_c = \frac{2z}{(1+c)(1-z)}, \qquad h + g = \frac{z}{1-z}.$$

Hence

$$\begin{aligned} h - g &= \frac{1+c}{2}(H_c + G_c) * (h - g) = \frac{1+c}{2}(H_c * h - H_c * g + G_c * h - G_c * g), \\ H_c - G_c &= (H_c - G_c) * (h + g) = (H_c * h + H_c * g - G_c * h - G_c * g). \end{aligned}$$

Thus

$$H_c * h - G_c * g = \frac{1}{2}\left[\frac{2}{1+c}(h-g) + (H_c - G_c)\right]. \tag{12}$$

Next, we will show that $\frac{2}{1+c}(h-g) + (H_c - G_c)$ is convex in the horizontal direction. Letting $\varphi(z) = z/(1-z)^2 \in \mathcal{S}^*$, we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{z}{\varphi}\left[\frac{2}{1+c}(h' - g') + (H_c' - G_c')\right]\right\} &= \mathrm{Re}\left\{\frac{z\left[\frac{2}{1+c}(h'+g')\left(\frac{h'-g'}{h'+g'}\right) + (H_c' + G_c')\left(\frac{H_c' - G_c'}{H_c' + G_c'}\right)\right]}{\varphi}\right\} \\ &= \mathrm{Re}\left\{\frac{z\left[\frac{2}{1+c}(h'+g')\left(\frac{1-\omega}{1+\omega}\right) + (H_c' + G_c')\left(\frac{1-\omega_c}{1+\omega_c}\right)\right]}{\varphi}\right\} \end{aligned}$$

$$=\frac{2}{1+c}\operatorname{Re}\left\{\frac{\frac{z}{(1-z)^2}[r(z)+r_c(z)]}{\frac{z}{(1-z)^2}}\right\}=\frac{2}{1+c}\operatorname{Re}\{r(z)+r_c(z)\}>0,$$

where $r(z)=\frac{1-\omega(z)}{1+\omega(z)}, r_c(z)=\frac{1-\omega_c(z)}{1+\omega_c(z)}$. Therefore, by Lemma 2.5 and the equation (12), we know that $H_c * h - G_c * g$ is convex in the horizontal direction.

Finally, since we assumed that $L_c * f$ is locally univalent and $H_c * h - G_c * g$ is convex in the horizontal direction, we apply Lemma 2.4 to obtain that $L_c * f = H_c * h + \overline{G_c * g}$ is convex in the horizontal direction. □

3 Proofs of theorems

Proof of Theorem 1.1. By Theorem A, we know that $f_0 * f$ is convex in the horizontal direction. Now we need to establish that $f_0 * f$ is locally univalent.

Substituting $\omega(z) = -z(z+a)/(1+az)$ into (8) and simplifying, yields

$$\widetilde{\omega}(z)=z\frac{z^3+\frac{2+3a}{2}z^2+(1+a)z+\frac{a}{2}}{1+\frac{2+3a}{2}z+(1+a)z^2+\frac{a}{2}z^3}=z\frac{q(z)}{q^*(z)}=z\frac{(z-A)(z-B)(z-C)}{(1-\overline{A}z)(1-\overline{B}z)(1-\overline{C}z)}, \tag{13}$$

If $a=1$, then $q(z)=z^3+\frac{5}{2}z^2+2z+\frac{1}{2}=\frac{1}{2}(1+z)^2(1+2z)$ has all its three zeros in $\overline{\mathbb{U}}$. By Cohn's Rule, so $|\widetilde{\omega}(z)|<1$ for all $z\in\mathbb{U}$.

If $a=0$, it is clearly that $|\widetilde{\omega}(z)|=|z^2|<1$ for all $z\in\mathbb{U}$.

If $-1\le a<0$, we apply *Cohn's Rule* to $q(z)=z^3+\frac{2+3a}{2}z^2+(1+a)z+\frac{a}{2}$. Note that $|\frac{a}{2}|<1$, thus we get

$$Q(z)=\frac{\overline{a_3}q(z)-a_0q^*(z)}{z}=\frac{4-a^2}{4}\left(z^2+\frac{2(2+2a-a^2)}{4-a^2}z+\frac{4+2a-3a^2}{4-a^2}\right):=\frac{4-a^2}{4}q_1(z).$$

Since

$$\left|\frac{4+2a-3a^2}{4-a^2}\right|=\left|1+\frac{2a(1-a)}{4-a^2}\right|<1 \text{ (as } -1\le a<0),$$

we use Cohn's Rule on $q_1(z)$ again, we get

$$q_2(z)=\frac{q_1(z)-\frac{4+2a-3a^2}{4-a^2}q_1^*(z)}{z}=\frac{4a(a-1)(4+a-2a^2)}{(4-a^2)^2}\left(z+\frac{2+2a-a^2}{4+a-2a^2}\right).$$

Clearly, $q_2(z)$ has one zero at

$$z_0=-\frac{2+2a-a^2}{4+a-2a^2}.$$

We show that $|z_0|\le 1$, or equivalently,

$$|2+2a-a^2|^2-|4+a-2a^2|^2=(2+2a-a^2)^2-(4+a-2a^2)^2=3(4-a^2)(a^2-1)\le 0.$$

Therefore, by *Cohn's Rule*, $q(z)$ has all its three zeros in $\overline{\mathbb{U}}$, that is $A,B,C\in\overline{\mathbb{U}}$ and so $|\widetilde{\omega}(z)|<1$ for all $z\in\mathbb{U}$. □

A new proof of Theorem D. By Theorem A, it suffices to show that the dilatation of $f_a * f_n$ satisfies $|\widetilde{\omega}_1(z)|<1$ for all $z\in\mathbb{U}$. Setting $\omega(z)=e^{i\theta}z^n$ in (9), we have

$$\widetilde{\omega}_1(z)=-z^ne^{2i\theta}\left[\frac{z^{n+1}-az^n+\frac{1}{2}(2+an-n)e^{-i\theta}z+\frac{1}{2}(n-2a-an)e^{-i\theta}}{1-az+\frac{1}{2}(2+an-n)e^{i\theta}z^n+\frac{1}{2}(n-2a-an)e^{i\theta}z^{n+1}}\right]=-z^ne^{2i\theta}\frac{p(z)}{p^*(z)}, \tag{14}$$

where

$$p(z)=z^{n+1}-az^n+\frac{1}{2}(2+an-n)e^{-i\theta}z+\frac{1}{2}(n-2a-an)e^{-i\theta}, \tag{15}$$

and

$$p^*(z)=z^{n+1}\overline{p(1/\overline{z})}.$$

Firstly, we will show that $|\widetilde{\omega}_1(z)| < 1$ for $a = \frac{n-2}{n+2}$. In this case, substituting $a = \frac{n-2}{n+2}$ into (14), we have

$$\begin{aligned}|\widetilde{\omega}_1(z)| &= \left|-z^n e^{2i\theta}\frac{z^{n+1} - \frac{n-2}{n+2}z^n - \frac{n-2}{n+2}e^{-i\theta}z + e^{-i\theta}}{1 - \frac{n-2}{n+2}z - \frac{n-2}{n+2}e^{i\theta}z^n + e^{i\theta}z^{n+1}}\right| \\ &= \left|-z^n e^{i\theta}\frac{e^{i\theta}z^{n+1} - \frac{n-2}{n+2}e^{i\theta}z^n - \frac{n-2}{n+2}z + 1}{1 - \frac{n-2}{n+2}z - \frac{n-2}{n+2}e^{i\theta}z^n + e^{i\theta}z^{n+1}}\right| \\ &= |-z^n e^{i\theta}| < 1.\end{aligned}$$

Next, we will show that $|\widetilde{\omega}_1(z)| < 1$ for $\frac{n-2}{n+2} < a < 1$. Obviously, if z_0 is a zero of $p(z)$, then $1/\overline{z_0}$ is a zero of $p^*(z)$. Hence, if $A_1, A_2, \cdots, A_{n+1}$ are the zeros of $p(z)$ (not necessarily distinct), then we can write

$$\widetilde{\omega}_1(z) = -z^n e^{2i\theta}\frac{(z-A_1)}{(1-\overline{A_1}z)}\frac{(z-A_2)}{(1-\overline{A_2}z)}\cdots\frac{(z-A_{n+1})}{(1-\overline{A_{n+1}}z)}.$$

Now for $|A_j| \le 1$, $\frac{z-A_j}{1-\overline{A_j}z}$ $(j = 1, 2, \cdots, n+1)$ maps $\mathbb{U}$ onto $\mathbb{U}$. It suffices to show that all zeros of (15) lie on $\overline{\mathbb{U}}$ for $\frac{n-2}{n+2} < a < 1$. Since $|a_{0,0}| = |\frac{1}{2}(n - 2a - an)e^{-i\theta}| < |a_{n+1,0}| = 1$, then by using *Cohn's Rule* on $p(z)$, we get

$$\begin{aligned}p_1(z) &= \frac{\overline{a_{n+1,0}}p(z) - a_{0,0}p^*(z)}{z} = \frac{p(z) - \frac{1}{2}(n-2a-an)e^{-i\theta}p^*(z)}{z} \\ &= \frac{(1-a)(2+n)[2(1+a)-(1-a)n]}{4}\left(z^n - \frac{n}{n+2}z^{n-1} + \frac{2}{n+2}e^{-i\theta}\right).\end{aligned}$$

Since $\frac{n-2}{n+2} < a < 1$, we have $\frac{(1-a)(2+n)[2(1+a)-(1-a)n]}{4} > 0$. Let $q_1(z) = z^n - \frac{n}{n+2}z^{n-1} + \frac{2}{n+2}e^{-i\theta}$, since $|a_{0,1}| = |\frac{2}{n+2}e^{-i\theta}| < 1 = |a_{n,1}|$, by using *Cohn's Rule* on $q_1(z)$ again, we obtain

$$\begin{aligned}p_2(z) &= \frac{\overline{a_{n,1}}q_1(z) - a_{0,1}q_1^*(z)}{z} = \frac{q_1(z) - \frac{2}{n+2}e^{-i\theta}q_1^*(z)}{z} \\ &= \frac{n(n+4)}{(n+2)^2}\left(z^{n-1} - \frac{n+2}{n+4}z^{n-2} + \frac{2}{n+4}e^{-i\theta}\right).\end{aligned}$$

Let $q_2(z) = z^{n-1} - \frac{n+2}{n+4}z^{n-2} + \frac{2}{n+4}e^{-i\theta}$, then $|a_{0,2}| = \frac{2}{n+4} < 1 = |a_{n-1,2}|$, we get

$$\begin{aligned}p_3(z) &= \frac{\overline{a_{n-1,2}}q_2(z) - a_{0,2}q_2^*(z)}{z} = \frac{q_2(z) - \frac{2}{n+4}e^{-i\theta}q_2^*(z)}{z} \\ &= \frac{(n+2)(n+6)}{(n+4)^2}\left(z^{n-2} - \frac{n+4}{n+6}z^{n-3} + \frac{2}{n+6}e^{-i\theta}\right).\end{aligned}$$

Let $q_3(z) = z^{n-2} - \frac{n+4}{n+6}z^{n-3} + \frac{2}{n+6}e^{-i\theta}$, then $|a_{0,3}| = \frac{2}{n+6} < 1 = |a_{n-2,3}|$, we obtain

$$\begin{aligned}p_4(z) &= \frac{\overline{a_{n-1,3}}q_3(z) - a_{0,3}q_3^*(z)}{z} = \frac{q_3(z) - \frac{2}{n+6}e^{-i\theta}q_3^*(z)}{z} \\ &= \frac{(n+4)(n+8)}{(n+6)^2}\left(z^{n-3} - \frac{n+6}{n+8}z^{n-4} + \frac{2}{n+8}e^{-i\theta}\right).\end{aligned}$$

By using this manner, we claim that

$$p_k(z) = \frac{[n+2(k-2)](n+2k)}{[n+2(k-1)]^2}\left(z^{n-k+1} - \frac{n+2(k-1)}{n+2k}z^{n-k} + \frac{2}{n+2k}e^{-i\theta}\right). \tag{16}$$

where $k = 2, 3, \cdots, n$.

To prove the equation (16) is correct for all $k \in \mathbb{N}^+ (k \ge 2)$, it suffices to show

$$p_{k+1}(z) = \frac{[n+2(k-1)][n+2(k+1)]}{(n+2k)^2}\left(z^{n-k} - \frac{n+2k}{n+2(k+1)}z^{n-(k+1)} + \frac{2}{n+2(k+1)}e^{-i\theta}\right). \tag{17}$$

Let $q_k(z) = z^{n-k+1} - \frac{n+2(k-1)}{n+2k} z^{n-k} + \frac{2}{n+2k} e^{-i\theta}$, then $q_k^*(z) = z^{n-k+1}\overline{q_k(1/\overline{z})} = 1 - \frac{n+2(k-1)}{n+2k} z + \frac{2}{n+2k} e^{-i\theta} z^{n-k+1}$. Since $|a_{0,k}| = |\frac{2}{n+2k} e^{-i\theta}| = \frac{2}{n+2k} < 1 = |a_{n-k+1,k}|$, by using *Cohn's Rule* on $q_k(z)$, we deduce that

$$\begin{aligned} p_{k+1}(z) &= \frac{\overline{a_{n-k+1,k}} q_k(z) - a_{0,k} q_k^*(z)}{z} = \frac{q_k(z) - \frac{2}{n+2k} e^{-i\theta} q_k^*(z)}{z} \\ &= \frac{[n+2(k-1)][n+2(k+1)]}{(n+2k)^2} \left(z^{n-k} - \frac{n+2k}{n+2(k+1)} z^{n-(k+1)} + \frac{2}{n+2(k+1)} e^{-i\theta} \right). \end{aligned}$$

Setting $n = k (k \geq 2)$ in (16), we have

$$p_n(z) = \frac{3n(3n-4)}{(3n-2)^2} \left(z - \frac{3n-2-2e^{-i\theta}}{3n} \right).$$

Then $z_0 = \frac{3n-2-2e^{-i\theta}}{3n}$ is a zero of $p_n(z)$, and

$$|z_0| = \left| \frac{3n-2-2e^{-i\theta}}{3n} \right| \leq \frac{|3n-2| + |2e^{-i\theta}|}{3n} = \frac{3n-2+2}{3n} = 1.$$

So z_0 lies inside or on the unit circle $|z| = 1$, by Lemma 1, we know that all zeros of (15) lie on $\overline{\mathbb{U}}$. This completes the proof. □

Proof of Theorem 1.2. In view of Lemma 2.6, it suffices to show that $L_c * f_a$ is locally univalent and sense-preserving. Substituting $\omega(z) = \omega_a(z) = (a-z)/(1-az)$ into (10), we have

$$\begin{aligned} \widetilde{\omega}_2(z) &= \frac{[(1-c)-(1+c)z]\left(\frac{a-z}{1-az}\right)(1+\frac{a-z}{1-az}) - \frac{c(a^2-1)}{(1-az)^2} z(1-z)}{[(1+c)-(1-c)z](1+\frac{a-z}{1-az}) - \frac{c(a^2-1)}{(1-az)^2} z(1-z)} \\ &= \frac{[(1-c)-(1+c)z][(a-z)(1-az)+(a-z)^2] - (a^2-1)cz(1-z)}{[(1+c)-(1-c)z][(1-az)^2+(a-z)(1-az)] - (a^2-1)cz(1-z)} \\ &= -\frac{z^3 - \frac{2+a-c+2ac}{1+c} z^2 + \frac{1+2a-2c+ac}{1+c} z - \frac{a(1-c)}{1+c}}{1 - \frac{2+a-c+2ac}{1+c} z + \frac{1+2a-2c+ac}{1+c} z^2 - \frac{a(1-c)}{1+c} z^3} \end{aligned} \tag{18}$$

Next we just need to show that $|\widetilde{\omega}_2(z)| < 1$ for $0 < c \leq 2(1+a)/(1-a)$, where $-1 < a < 1$. We shall consider the following two cases.

Case 1. *Suppose that $a = 0$. Then substituting $a = 0$ into* (18) *yields*

$$\widetilde{\omega}_2(z) = -z \frac{z^2 - \frac{2-c}{1+c} z + \frac{1-2c}{1+c}}{1 - \frac{2-c}{1+c} z + \frac{1-2c}{1+c} z^2} = -z \frac{(z-1)\left(z - \frac{1-2c}{1+c}\right)}{(1-z)\left(1 - \frac{1-2c}{1+c} z\right)}.$$

Then two zeros $z_1 = 1$ and $z_2 = (1-2c)/(1+c)$ of the above numerator lie in or on the unit circle for all $0 < c \leq 2$, so we have $|\widetilde{\omega}_2(z)| < 1$.

Case 2. *Suppose that $a \neq 0$. From* (18)*, we can write*

$$\widetilde{\omega}_2(z) = -\frac{z^3 - \frac{2+a-c+2ac}{1+c} z^2 + \frac{1+2a-2c+ac}{1+c} z - \frac{a(1-c)}{1+c}}{1 - \frac{2+a-c+2ac}{1+c} z + \frac{1+2a-2c+ac}{1+c} z^2 - \frac{a(1-c)}{1+c} z^3} = -\frac{p(z)}{p^*(z)} = -\frac{(z-A)(z-B)(z-C)}{(1-\overline{A}z)(1-\overline{B}z)(1-\overline{C}z)}.$$

We will show that $A, B, C \in \overline{\mathbb{U}}$ for $0 < c \leq 2(1+a)/(1-a)$. Applying Lemma 1 to

$$p(z) = z^3 - \frac{2+a-c+2ac}{1+c} z^2 + \frac{1+2a-2c+ac}{1+c} z - \frac{a(1-c)}{1+c}.$$

Note that $|-\frac{a(1-c)}{1+c}|<1$ *for* $c>0$ *and* $-1<a<1$, *we get*

$$
\begin{aligned}
p_1(z) &= \frac{\overline{a_3}p(z)-a_0p^*(z)}{z} = \frac{p(z)+\frac{a(1-c)}{1+c}p^*(z)}{z} \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}z^2 + \frac{-2-c-6ac+c^2+2a^2-a^2c-a^2c^2}{(1+c)^2}z \\
&\quad + \frac{1-c+6ac-2c^2-a^2-a^2c+2a^2c^2}{(1+c)^2} \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}\left(z^2+\frac{-2+c-2a-ac}{1+c+a-ac}z+\frac{1-2c+a+2ac}{1+c+a-ac}\right) \\
&= \frac{(1+c+a-ac)(1+c-a+ac)}{(1+c)^2}(z-1)\left(z-\frac{1+a-2c(1-a)}{1+a+c(1-a)}\right).
\end{aligned}
$$

So $p_1(z)$ *has two zeros* $z_1^*=1$ *and* $z_2^*=\frac{1+a-2c(1-a)}{1+a+c(1-a)}$ *which are in or on the unit circle for* $0<c\le 2(1+a)/(1-a)$. *Thus, by Lemma 1, all zeros of* $p(z)$ *lie on* $\overline{\mathbb{U}}$, *that is* $A,B,C\in\overline{\mathbb{U}}$ *and so* $|\widetilde{\omega}_2(z)|<1$ *for all* $z\in\mathbb{U}$. □

4 Examples

In this section, we give interesting examples resulting from Theorem 1.1 and Theorem 1.2.

Example 4.1. *In Theorem 1.1, note that* $f=h+\overline{g}\in\mathcal{S}_H^0$ *with* $h+g=z/(1-z)$ *and the dilatation* $\omega(z)=-z(a+z)/(1+az)$. *By shearing*

$$h'(z)+g'(z)=\frac{1}{(1-z)^2} \quad \textit{and} \quad g'(z)=\omega(z)h'(z).$$

Solving these equations, we get

$$h'(z)=\frac{1+az}{(1-z)^3(1+z)} \quad \textit{and} \quad g'(z)=\frac{-z(a+z)}{(1-z)^3(1+z)}.$$

Integration gives

$$h(z)=\frac{1}{2}\frac{z}{1-z}+\frac{1+a}{4}\frac{z}{(1-z)^2}+\frac{1-a}{8}\log\left(\frac{1+z}{1-z}\right), \tag{19}$$

and then,

$$g(z)=\frac{1}{2}\frac{z}{1-z}-\frac{1+a}{4}\frac{z}{(1-z)^2}-\frac{1-a}{8}\log\left(\frac{1+z}{1-z}\right). \tag{20}$$

By the convolutions, we have

$$f_0*f=h_0*h+\overline{g_0*g}=\frac{h(z)+zh'(z)}{2}+\overline{\left(\frac{g(z)-zg'(z)}{2}\right)}.$$

So that

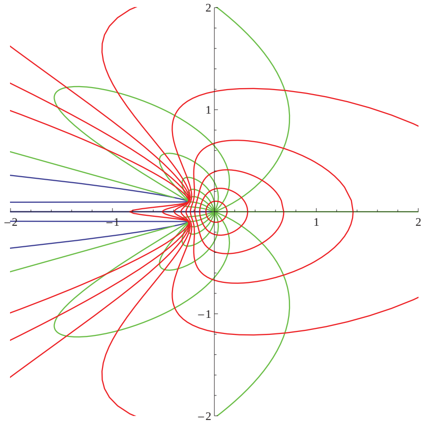
$$h_0*h=\frac{1}{4}\frac{z}{1-z}+\frac{1+a}{8}\frac{z}{(1-z)^2}+\frac{1-a}{16}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{2}\frac{z(1+az)}{(1-z)^3(1+z)}$$

and

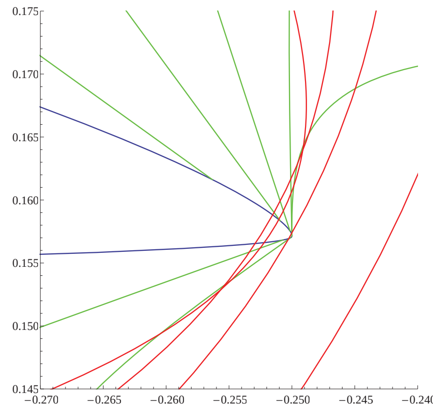
$$g_0*g=\frac{1}{4}\frac{z}{1-z}-\frac{1+a}{8}\frac{z}{(1-z)^2}-\frac{1-a}{16}\log\left(\frac{1+z}{1-z}\right)+\frac{1}{2}\frac{z^2(a+z)}{(1-z)^3(1+z)}.$$

Images of $\mathbb{U}$ *under* f_0*f *for certain values of* a *are drawn in Figure 1 and Figure 2 (a)-(d) by using Mathematica. By Theorem 1.1, it follows that* f_0*f *is locally univalent and convex in the horizontal direction for* $a=-1,-0.5,0,1$, *but it is not locally univalent for* $a=0.2$.

Fig. 1. Images of $\mathbb{U}$ under $f_0 * f$ for $a = 0.2$.

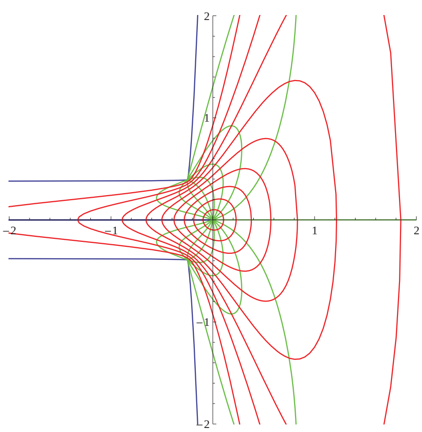


(a) $a = 0.2$

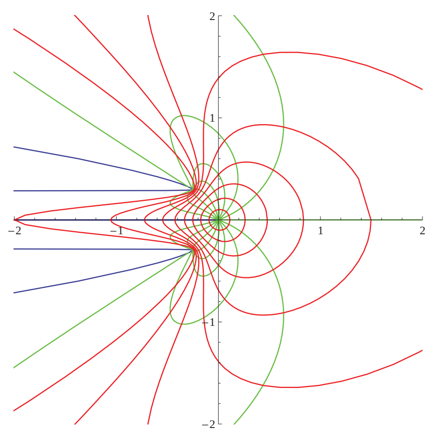


(b) Partial enlarged view for $a = 0.2$

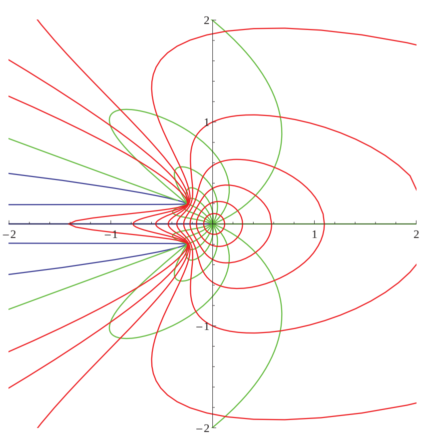
Fig. 2. Images of $\mathbb{U}$ under $f_0 * f$ for various values a.



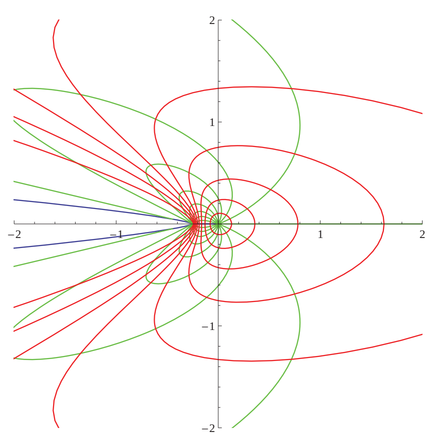
(a) $a = -1$



(b) $a = -0.5$



(c) $a = 0$



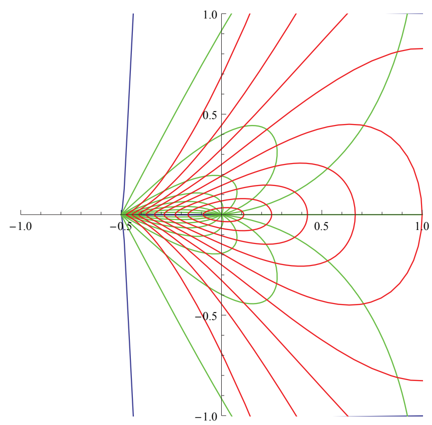
(d) $a = 1$

Example 4.2. *In Theorem 1.2, by* (1) *and* (4)*, we have*

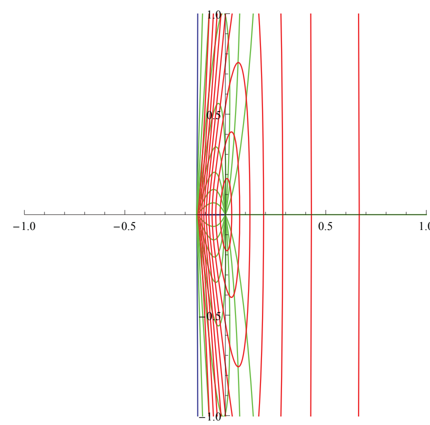
$$\begin{aligned}
L_c * f_a &= \frac{1}{1+c}\left[h_a(z) + czh_a'(z)\right] + \frac{1}{1+c}\overline{\left[g_a(z) - czg_a'(z)\right]}\\
&= \frac{1}{1+c}\left[\frac{\frac{1}{1+a}z - \frac{1}{2}z^2}{(1-z)^2} + \frac{cz(1-az)}{(1+a)(1-z)^3}\right] + \frac{1}{1+c}\overline{\left[\frac{\frac{a}{1+a}z - \frac{1}{2}z^2}{(1-z)^2} - \frac{cz(a-z)}{(1+a)(1-z)^3}\right]}\\
&= \operatorname{Re}\left\{\frac{z}{(1+c)(1-z)} + \frac{c(1-a)z(1+z)}{(1+c)(1+a)(1-z)^3}\right\} + i\operatorname{Im}\left\{\frac{\left(\frac{1-a}{1+a}+c\right)z}{(1+c)(1-z)^2}\right\}.
\end{aligned}$$

If we take $a = 0.5, c = 6$*, in view of Theorem 1.2, we know that* $L_6 * f_{0.5}$ *is univalent and convex in the horizontal direction. The image of* $\mathbb{U}$ *under* $f_{0.5}, L_6$ *and* $L_6 * f_{0.5}$ *are shown in Figure 3 (a)-(c), respectively.*

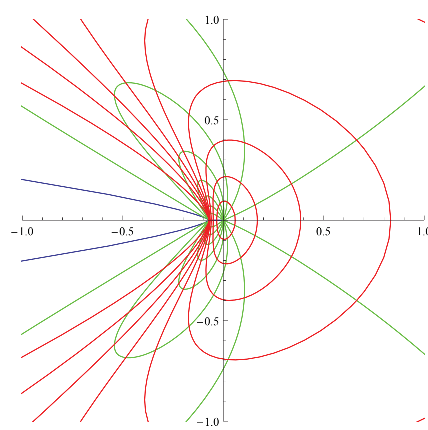
Fig. 3. Images of $\mathbb{U}$ under $f_{0.5}, L_6$ and $L_6 * f_{0.5}$.



(a) $f_{0.5}$



(b) L_6



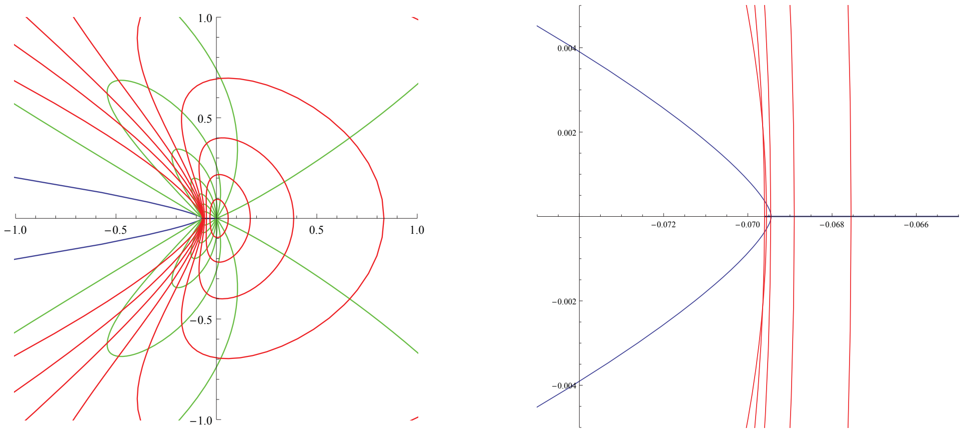
(c) $L_6 * f_{0.5}$

If we take $a = 0.5, c = 6.2$*, then*

$$L_{6.2} * f_{0.5} = \operatorname{Re}\left\{\frac{1}{7.2}\frac{z}{1-z} + \frac{3.1}{10.8}\frac{z(1+z)}{(1-z)^3}\right\} + i\operatorname{Im}\left\{\frac{\frac{1}{3}+6.2}{7.2}\frac{z}{(1-z)^2}\right\}.$$

The image of $\mathbb{U}$ *under* $L_{6.2} * f_{0.5}$ *is shown in Figure 4 (a). Figure 4 (b) is a partial enlarged view of Figure 4 (a) showing that the images of two outer most concentric circles in* $\mathbb{U}$ *are intersecting and so* $L_{6.2} * f_{0.5}$ *is not univalent.*

Fig. 4. Images of the unit disk $\mathbb{U}$ under $L_{6.2} * f_{0.5}$.



(a) $L_{6.2} * f_{0.5}$ (b) Partial enlarged view of $L_{6.2} * f_{0.5}$

Remark 4.3. *This example shows that the condition* $0 < c \le 2(1+a)/(1-a)$ *in Theorem 1.2 is sharp.*

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