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Research Article

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Some congruences for 3-component multipartitions

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Abstract: Let $p_3(n)$ denote the number of 3-component multipartitions of n. Recently, using a 3-dissection formula for the generating function of $p_3(n)$, Baruah and Ojah proved that for $n \geq 0$, $p_3(9n+5) \equiv 0 \pmod{3^3}$ and $p_3(9n+8) \equiv 0 \pmod{3^4}$. In this paper, we prove several congruences modulo powers of 3 for $p_3(n)$ by using some theta function identities. For example, we prove that for $n \geq 0$, $p_3(243n+233) \equiv p_3(729n+638) \equiv 0 \pmod{3^{10}}$.

Keywords: Congruences, Multipartitions, Theta functions

MSC: 11P83, 05A17

1 Introduction

The objective of this paper is to prove several congruences modulo powers of 3 for 3-component multipartitions by employing theta function identities.

Recall that the k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers λ_i $(i = 1, 2, \ldots, k)$ is a partition of the natural number n if $n = \sum_{i=1}^{k} \lambda_i$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The partition function $p(n)$ is defined to be the number of partitions of n. A multipartition of n with r-components, as called by Andrews [1], also referred to as an r-colored partition, see, for example [2, 3], is an r-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ of partitions whose weights sum to n. Let $p_r(n)$ denote the number of r-component multipartitions of n. Multipartitions arise in combinatorics, representation theory and physics, see, for example Bouwknegt [4] and Fayers [5]. As usual, set $p_r(0) = 1$. The generating function of $p_r(n)$ is

$$\sum_{n=0}^{\infty} p_r(n) q^n = \frac{1}{(q;q)_\infty^r}, \tag{1}$$

where

$$(a;q)_\infty = \prod_{n=1}^{\infty} (1 - aq^n). \tag{2}$$

A number of congruences satisfied by $p_r(n)$ were discovered, see, for example Andrews [1], Eichhorn and Ono [3], Atkin [6], Baruah and Ojah [7], Boylan [8], Cheema and Haskell [9], Gordon [10], Kiming and Olsson [11], Newman [12], Sinick [13], Treneer [14], Xia [15] and Yao [16]. Recently, Baruah and Ojah [7] established a 3-dissection formula for the generating function for $p_3(n)$ and proved that for $n \geq 0$,

$$p_3(9n+5) \equiv 0 \pmod{3^3} \tag{3}$$

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and

$$p_3(9n+8) \equiv 0 \pmod{3^4}. \tag{4}$$

In this paper, we prove several congruences modulo powers of 3 for $p_3(n)$. The main results of this paper can be stated as follows.

Theorem 1.1. *For $n \geq 0$,*

$$p_3(27n+17) \equiv 0 \pmod{3^5}, \tag{5}$$

$$p_3(27n+26) \equiv 0 \pmod{3^7}, \tag{6}$$

$$p_3(81n+44) \equiv 0 \pmod{3^6}, \tag{7}$$

$$p_3(81n+71) \equiv 0 \pmod{3^7}, \tag{8}$$

$$p_3(243n+152) \equiv 0 \pmod{3^8}, \tag{9}$$

$$p_3(243n+233) \equiv 0 \pmod{3^{10}}, \tag{10}$$

$$p_3(729n+395) \equiv 0 \pmod{3^9}, \tag{11}$$

$$p_3(729n+638) \equiv 0 \pmod{3^{10}}. \tag{12}$$

2 Some lemmas

In this section, we collect three lemmas which are needed to prove the main results of this paper.

Lemma 2.1. *The following 3-dissection formulas are true:*

$$a(q) = a(q^3) + 6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \tag{13}$$

and

$$b(q) = a(q^3) - 3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \tag{14}$$

where $a(q)$ and $b(q)$ are defined by

$$a(q) = \sum_{m,\,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{15}$$

and

$$b(q) = \sum_{m,\,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \qquad \omega = \exp(2\pi i/3). \tag{16}$$

Lemma 2.1 was proved by Borwein, Borwein and Garvan [18].

Lemma 2.2. *We have*

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty}. \tag{17}$$

Lemma 2.2 was also proved by Borwein, Borwein and Garvan [18].

Lemma 2.3. *The following 3-dissection formula holds:*

$$\frac{1}{(q;q)_\infty^3} = \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{10}} \left(a^2(q^3) + 3qa(q^3)\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} + 9q^2 \frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2} \right). \tag{18}$$

Proof. It is easy to check that for any positive integer n,

$$(1-q^n)(1-(\omega q)^n)(1-(\omega^2 q)^n) = \begin{cases} (1-q^n)^3, & \text{if } 3|n \\ (1-q^{3n}), & \text{if } 3 \nmid n. \end{cases} \tag{19}$$

By (17) and (19),

$$\begin{aligned} b(q)b(\omega q)b(\omega^2 q) &= \frac{(q;q)_\infty^3 (\omega q;\omega q)_\infty^3 (\omega^2 q;\omega^2 q)_\infty^3}{(q^3;q^3)_\infty^3} \\ &= \frac{1}{(q^3;q^3)_\infty^3} \left(\prod_{n=1}^{\infty} (1-q^n)(1-(\omega q)^n)(1-(\omega^2 q)^n) \right)^3 \\ &= \frac{1}{(q^3;q^3)_\infty^3} \left(\prod_{3\nmid n} (1-q^{3n}) \prod_{3|n} (1-q^n)^3 \right)^3 \\ &= \frac{1}{(q^3;q^3)_\infty^3} \left((q^3;q^9)_\infty (q^6;q^9)_\infty (q^3;q^3)_\infty^3 \right)^3 = \frac{(q^3;q^3)_\infty^9}{(q^9;q^9)_\infty^3}. \end{aligned} \tag{20}$$

In view of (14), we find that

$$\begin{aligned} b(\omega q)b(\omega^2 q) &= \left(a(q^3) - 3\omega q \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \right) \left(a(q^3) - 3\omega^2 q \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \right) \\ &= a^2(q^3) + 3q \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} a(q^3) + 9q^2 \frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}. \end{aligned} \tag{21}$$

It follows from (17), (20) and (21) that

$$\begin{aligned} \frac{1}{(q;q)_\infty^3} &= \frac{1}{(q^3;q^3)_\infty b(q)} = \frac{b(\omega q)b(\omega^2 q)}{(q^3;q^3)_\infty b(q)b(\omega q)b(\omega^2 q)} \\ &= \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{10}} \left(a^2(q^3) + 3qa(q^3) \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} + 9q^2 \frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2} \right). \end{aligned} \tag{22}$$

This completes the proof. □

Remark. *Baruah and Ojah [7] also deduced the following 3-dissection formula for* $\frac{1}{(q;q)_\infty^3}$*, which is different from* (18)*:*

$$\frac{1}{(q;q)_\infty^3} = \frac{(q^9;q^9)_\infty^9}{(q^3;q^3)_\infty^{12}} \left(\frac{1}{w^2(q^3)} + \frac{3q}{w(q^3)} + 9q^2 + 8q^3 w(q^3) + 12q^4 w^2(q^3) + 16q^6 w^4(q^3) \right), \tag{23}$$

where

$$w(q) = \frac{(q;q)_\infty (q^6;q^6)_\infty^3}{(q^2;q^2)_\infty (q^3;q^3)_\infty^3}. \tag{24}$$

3 Proof of Theorem 1.1

Setting $r = 3$ in (1) and using (18), we obtain

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty^{10}} \left(a^2(q^3) + 3qa(q^3) \frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} + 9q^2 \frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2} \right), \tag{25}$$

which yields

$$\sum_{n=0}^{\infty} p_3(3n+2)q^n = 9\frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}}. \tag{26}$$

Substituting (18) into (26), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(3n+2)q^n =& 9\frac{(q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^{31}}\left(a^2(q^3)+3qa(q^3)\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^4\\ \equiv& 9\frac{(q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^{31}}a^8(q^3)+108q\frac{(q^9;q^9)_\infty^{15}}{(q^3;q^3)_\infty^{32}}a^7(q^3)+810q^2\frac{(q^9;q^9)_\infty^{18}}{(q^3;q^3)_\infty^{33}}a^6(q^3)\\ &+3888q^3\frac{(q^9;q^9)_\infty^{21}}{(q^3;q^3)_\infty^{34}}a^5(q^3)+13851q^4\frac{(q^9;q^9)_\infty^{24}}{(q^3;q^3)_\infty^{35}}a^4(q^3)\\ &+34992q^5\frac{(q^9;q^9)_\infty^{27}}{(q^3;q^3)_\infty^{36}}a^3(q^3)+6561q^6\frac{(q^9;q^9)_\infty^{30}}{(q^3;q^3)_\infty^{37}}a^2(q^3)\\ &+19683q^7\frac{(q^9;q^9)_\infty^{33}}{(q^3;q^3)_\infty^{38}}a(q^3) \pmod{3^{10}},\end{aligned} \tag{27}$$

which yields

$$\sum_{n=0}^{\infty} p_3(9n+8)q^n \equiv 810\frac{(q^3;q^3)_\infty^{18}}{(q;q)_\infty^{33}}a^6(q)+34992q\frac{(q^3;q^3)_\infty^{27}}{(q;q)_\infty^{36}}a^3(q) \pmod{3^{10}}. \tag{28}$$

Substituting (13) and (18) into (28), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(9n+8)q^n \equiv& 810\frac{(q^9;q^9)_\infty^{33}}{(q^3;q^3)_\infty^{92}}\left(a^2(q^3)+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}a(q^3)+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^{11}\\ &\times\left(a(q^3)+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right)^6\\ &+34992q\frac{(q^9;q^9)_\infty^{36}}{(q^3;q^3)_\infty^{93}}\left(a^2(q^3)+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}a(q^3)+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^{12}\\ &\times\left(a(q^3)+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right)^3\\ \equiv& 810\frac{(q^9;q^9)_\infty^{33}}{(q^3;q^3)_\infty^{92}}a^{28}(q^3)+31833q\frac{(q^9;q^9)_\infty^{36}}{(q^3;q^3)_\infty^{93}}a^{27}(q^3)+50301q^2\frac{(q^9;q^9)_\infty^{39}}{(q^3;q^3)_\infty^{94}}a^{26}(q^3)\\ &+52488q^3\frac{(q^9;q^9)_\infty^{42}}{(q^3;q^3)_\infty^{95}}a^{25}(q^3)+39366q^4\frac{(q^9;q^9)_\infty^{45}}{(q^3;q^3)_\infty^{96}}a^{24}(q^3) \pmod{3^{10}},\end{aligned} \tag{29}$$

which implies that

$$\sum_{n=0}^{\infty} p_3(27n+17)q^n \equiv 131\times 3^5\frac{(q^3;q^3)_\infty^{36}}{(q;q)_\infty^{93}}a^{27}(q)+2\times 3^9 q\frac{(q^3;q^3)_\infty^{45}}{(q;q)_\infty^{96}}a^{24}(q) \pmod{3^{10}} \tag{30}$$

and

$$\sum_{n=0}^{\infty} p_3(27n+26)q^n \equiv 23\times 3^7\frac{(q^3;q^3)_\infty^{39}}{(q;q)_\infty^{94}}a^{26}(q) \pmod{3^{10}}. \tag{31}$$

Congruences (5) and (6) follow from (30) and (31).

Substituting (13) and (18) into (30), we get

$$\sum_{n=0}^{\infty} p_3(27n+17)q^n \equiv 131\times 3^5 \frac{(q^9;q^9)_\infty^{93}}{(q^3;q^3)_\infty^{274}} \left(a^2(q^3)+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}a(q^3)+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^{31}$$
$$\times\left(a(q^3)+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right)^{27}$$
$$+2\times 3^9 q\frac{(q^9;q^9)_\infty^{96}}{(q^3;q^3)_\infty^{275}}\left(a^2(q^3)+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}a(q^3)+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^{32}$$
$$\times\left(a(q^3)+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right)^{24}$$
$$\equiv 31833\frac{(q^9;q^9)_\infty^{93}}{(q^3;q^3)_\infty^{274}}a^{89}(q^3)+8019q\frac{(q^9;q^9)_\infty^{96}}{(q^3;q^3)_\infty^{275}}a^{88}(q^3)$$
$$+30618q^2\frac{(q^9;q^9)_\infty^{99}}{(q^3;q^3)_\infty^{276}}a^{87}(q^3)+52488q^3\frac{(q^9;q^9)_\infty^{102}}{(q^3;q^3)_\infty^{277}}a^{86}(q^3)$$
$$+39366q^4\frac{(q^9;q^9)_\infty^{105}}{(q^3;q^3)_\infty^{278}}a^{85}(q^3) \pmod{3^{10}}. \tag{32}$$

It follows from (32) that

$$\sum_{n=0}^{\infty} p_3(81n+44)q^n \equiv 11\times 3^6\frac{(q^3;q^3)_\infty^{96}}{(q;q)_\infty^{275}}a^{88}(q)+2\times 3^9 q\frac{(q^3;q^3)_\infty^{105}}{(q;q)_\infty^{278}}a^{85}(q) \pmod{3^{10}} \tag{33}$$

and

$$\sum_{n=0}^{\infty} p_3(81n+71)q^n \equiv 14\times 3^7\frac{(q^3;q^3)_\infty^{99}}{(q;q)_\infty^{276}}a^{87}(q) \pmod{3^{10}}. \tag{34}$$

Congruences (7) and (8) follow from (33) and (34).

By the binomial theorem,

$$\frac{(q^3;q^3)_\infty^{90}}{(q;q)_\infty^{270}} \equiv 1 \pmod{27}. \tag{35}$$

Combining (34) and (35), we have

$$\sum_{n=0}^{\infty} p_3(81n+71)q^n \equiv 14\times 3^7\frac{(q^3;q^3)_\infty^{9}}{(q;q)_\infty^{6}}a^{87}(q) \pmod{3^{10}}. \tag{36}$$

Substituting (13) and (18) into (36), we find that

$$\sum_{n=0}^{\infty} p_3(81n+71)q^n \equiv 14\times 3^7\frac{(q^9;q^9)_\infty^{6}}{(q^3;q^3)_\infty^{11}}\left(a^2(q^3)+3q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}a(q^3)+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^{2}$$
$$\times\left(a(q^3)+6q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}\right)^{87}$$
$$\equiv 14\times 3^7\frac{(q^9;q^9)_\infty^{6}}{(q^3;q^3)_\infty^{11}}a^{91}(q^3)+7\times 3^8 q\frac{(q^9;q^9)_\infty^{9}}{(q^3;q^3)_\infty^{12}}a^{90}(q^3) \pmod{3^{10}}. \tag{37}$$

Congruences (9) and (10) follow from (37).

Congruence (37) also implies that

$$\sum_{n=0}^{\infty} p_3(243n+152)q^n \equiv 7\times 3^8 \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}} a^{90}(q) \pmod{3^{10}}. \tag{38}$$

By (13) and the binomial theorem,

$$a^3(q) \equiv a^3(q^3) \pmod{9}. \tag{39}$$

Thanks to (38) and (39),

$$\sum_{n=0}^{\infty} p_3(243n+152)q^n \equiv 7\times 3^8 \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^{12}} a^{90}(q^3) \pmod{3^{10}}. \tag{40}$$

Substituting (18) into (40), we see that

$$\begin{aligned}\sum_{n=0}^{\infty} p_3(243n+152)q^n &\equiv 7\times 3^8 a^{90}(q^3)\frac{(q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^{31}}\left(a^2(q^3)+3qa(q^3)\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty}+9q^2\frac{(q^9;q^9)_\infty^6}{(q^3;q^3)_\infty^2}\right)^4\\ &\equiv 7\times 3^8 \frac{(q^9;q^9)_\infty^{12}}{(q^3;q^3)_\infty^{31}} a^{98}(q^3) + 3^9 q\frac{(q^9;q^9)_\infty^{15}}{(q^3;q^3)_\infty^{32}} a^{97}(q^3) \pmod{3^{10}}.\end{aligned} \tag{41}$$

Congruences (11) and (12) follow from (41). This completes the proof of Theorem 1.1.

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