Research Article

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Pochhammer symbol with negative indices. A new rule for the method of brackets

Abstract: The method of brackets is a method of integration based upon a small number of heuristic rules. Some of these have been made rigorous. An example of an integral involving the Bessel function is used to motivate a new evaluation rule.

1 Introduction

The evaluation of definite integrals is connected, in a surprising manner, to many topics in Mathematics. The last author has described in [15] and [17] some of these connections. Many of these evaluations appear in the classical table [13] and proofs of these entries have appeared in a series of papers starting with [16] and the latest one is [1]. An interesting new method of integration, developed in [10] in the context of integrals coming from Feynman diagrams, is illustrated here in the evaluation of and entry from the classical table mentioned above. Each of these entries have an assigned number and the one discussed here is 6.671.7 in [13]:

$$I := \int_0^\infty J_0(ax) \sin(bx) \, dx = \begin{cases} 0, & \text{if } 0 < b < a, \\ 1/\sqrt{b^2 - a^2}, & \text{if } 0 < a < b. \end{cases} \tag{1}$$

Here J_0 is the Bessel function. The reader is referred to [3] for an introduction to these classical functions.

The new integration procedure has been named *the method of brackets*. It is based upon a small number of heuristic rules. This method is close in spirit to the classical *umbral method*, developed in the Combinatorial world by G. C. Rota. The reader will find in [18] an elementary introduction to this topic. The application of this method to Special Functions may be found, for instance, in a series of papers by G. Dattoli and collaborators, such as [4], [5] and [6]. The connections between the umbral method and the method of brackets is the subject of current work and results will be reported elsewhere.

The rules for the method of brackets are described in the next section. Sections 3 and 4 present the evaluation of (1). It turns out that a *naive limiting process* leads to a conflict between these two evaluations. The issue comes from a limit computation involving the Pochhammer symbol $(a)_k$. It is the goal of this paper to clarify this conflict. This is presented in Section 5 leading to the proposal of a new rule dealing with the extension of the Pochhammer symbol to negative integer indices.

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2 The method of brackets

A method to evaluate integrals over the half line $[0, \infty)$, based on a small number of rules has been developed in [10–12]. This method of brackets is described next. The heuristic rules are currently being made rigorous in [2] and [14]. The reader will find in [7–9] a large collection of evaluations of definite integrals that illustrate the power and flexibility of this method.

For $a \in \mathbb{C}$, the symbol

$$\langle a \rangle \mapsto \int_0^\infty x^{a-1}\, dx \tag{2}$$

is the *bracket* associated to the (divergent) integral on the right. The symbol

$$\phi_n := \frac{(-1)^n}{\Gamma(n+1)} \tag{3}$$

is called the *indicator* associated to the index n. The notation $\phi_{i_1 i_2 \cdots i_r}$, or simply $\phi_{12\cdots r}$, denotes the product $\phi_{i_1} \phi_{i_2} \cdots \phi_{i_r}$.

Rules for the production of bracket series

Rule P_1. If the function f is given by the formal power series

$$f(x) = \sum_{n=0}^\infty a_n x^{\alpha n + \beta - 1}, \tag{4}$$

then the improper integral of f over the positive real line is formally written as the *bracket series*

$$\int_0^\infty f(x) dx = \sum_n a_n \langle \alpha n + \beta \rangle. \tag{5}$$

Rule P_2. For $\alpha \in \mathbb{C}$, the multinomial power $(a_1 + a_2 + \cdots + a_r)^\alpha$ is assigned the r-dimension bracket series

$$\sum_{n_1} \sum_{n_2} \cdots \sum_{n_r} \phi_{n_1 n_2 \cdots n_r} a_1^{n_1} \cdots a_r^{n_r} \frac{\langle -\alpha + n_1 + \cdots + n_r \rangle}{\Gamma(-\alpha)}. \tag{6}$$

Rules for the evaluation of a bracket series

Rule E_1. The one-dimensional bracket series is assigned the value

$$\sum_n \phi_n f(n) \langle an + b \rangle = \frac{1}{|a|} f(n^*) \Gamma(-n^*), \tag{7}$$

where n^* is obtained from the vanishing of the bracket; that is, n^* solves $an + b = 0$.

The next rule provides a value for multi-dimensional bracket series where the number of sums is equal to the number of brackets.

Rule E_2. Assume the matrix $A = (a_{ij})$ is non-singular, then the assignment is

$$\sum_{n_1} \cdots \sum_{n_r} \phi_{n_1 \cdots n_r} f(n_1, \cdots, n_r) \langle a_{11} n_1 + \cdots + a_{1r} n_r + c_1 \rangle \cdots \langle a_{r1} n_1 + \cdots + a_{rr} n_r + c_r \rangle$$

$$= \frac{1}{|\det(A)|} f(n_1^*, \cdots n_r^*) \Gamma(-n_1^*) \cdots \Gamma(-n_r^*)$$

where $\{n_i^*\}$ is the (unique) solution of the linear system obtained from the vanishing of the brackets. There is no assignment if A is singular.

Rule E_3. Each representation of an integral by a bracket series has associated an *index of the representation* via

$$\text{index} = \text{number of sums} - \text{number of brackets}. \tag{8}$$

It is important to observe that the index is attached to a specific representation of the integral and not just to integral itself. The experience obtained by the authors using this method suggests that, among all representations of an integral as a bracket series, the one with *minimal index* should be chosen.

    The value of a multi-dimensional bracket series of positive index is obtained by computing all the contributions of maximal rank by Rule E_2. These contributions to the integral appear as series in the free parameters. Series converging in a common region are added and divergent series are discarded.

Note 2.1. *The rule E_1 is a version of the Ramanujan's Master Theorem. A complete justification of this rule is provided in [2]. The justification of the remaining rules is the subject of current work.*

Note 2.2. *A systematic procedure in the simplification of the series obtained by this procedure has been used throughout the literature: express factorials in terms of the gamma function and the transform quotients of gamma terms into Pochhammer symbol, defined by*

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{9}$$

Any presence of a Pochhammer with a negative index k is transformed by the rule

$$(a)_{-k} = \frac{(-1)^k}{(1-a)_k}. \tag{10}$$

The example discussed in the next two section provides motivation for an additional evaluation rule for the method of brackets.

3 A first evaluation of entry 6.671.7 in Gradshteyn and Ryzhik

The evaluation of (1) uses the series

$$J_0(ax) = \sum_{m=0}^{\infty} \phi_m \frac{a^{2m}}{\Gamma(m+1)2^{2m}} x^{2m} \tag{11}$$

and

$$\sin(bx) = \sum_{n=0}^{\infty} \phi_n \frac{\Gamma(n+1)}{\Gamma(2n+2)} b^{2n+1} x^{2n+1}. \tag{12}$$

Therefore the integral in (1) is given by

$$I = \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1} \Gamma(n+1)}{2^{2m} \Gamma(m+1)\Gamma(2n+2)} \langle 2m+2n+2 \rangle. \tag{13}$$

The duplication formula for the gamma function transforms this expression to

$$I = \frac{\sqrt{\pi}}{2} \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m+2n} \Gamma(m+1)\Gamma(n+3/2)} \langle 2m+2n+2 \rangle. \tag{14}$$

Eliminating the parameter n using Rule E_1 gives $n^* = -m - 1$ and produces

$$I = \frac{\sqrt{\pi}}{b} \sum_{m=0}^{\infty} \phi_m \frac{1}{\Gamma\left(-m+\frac{1}{2}\right)} \left(\frac{a}{b}\right)^{2m}$$

$$= \frac{1}{b} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{\left(\frac{a^2}{b^2}\right)^m}{\left(\frac{1}{2}\right)_{-m}}$$

$$= \frac{1}{b} \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)_m \frac{1}{m!} \left(\frac{a^2}{b^2}\right)^m$$

$$= \frac{1}{\sqrt{b^2 - a^2}}.$$

The condition $|b| > |a|$ is imposed to guarantee the convergence of the series on the third line of the previous argument.

The series obtained by eliminating the parameter m by $m^* = -n - 1$ vanishes because of the factor $\Gamma(m + 1)$ in the denominator. The formula has been established.

4 An alternative evaluation

A second evaluation of (1) begins with

$$\int_0^{\infty} J_0(ax) \sin(bx)\, dx = \sum_{m,n} \phi_{m,n} \frac{a^{2m} b^{2n+1}}{2^{2m} m! (2n+1)!} \langle 2m + 2n + 2 \rangle. \tag{15}$$

The evaluation of the bracket series is described next.

Case 1. Choose n as the free parameter. Then $m^* = -n - 1$ and the contribution to the integral is

$$I_1 = \frac{1}{2} \sum_{n=0}^{\infty} \phi_n \left(\frac{a}{2}\right)^{-2n-2} b^{2n+1} \frac{1}{\Gamma(-n)} \frac{n!}{(2n+1)!} \Gamma(n+1). \tag{16}$$

Each term in the sum vanishes because the gamma function has a pole at the negative integers.

Case 2. Choose m as a free parameter. Then $2m + 2n + 2 = 0$ gives $n^* = -m - 1$. The contribution to the integral is

$$I_2 = \frac{1}{2} \sum_m \phi_m \left(\frac{a}{2}\right)^{2m} b^{-2m-1} \frac{1}{m!} \cdot \frac{n!}{(2n+1)!}\bigg|_{n=-m-1} \Gamma(m+1). \tag{17}$$

Now write

$$\frac{n!}{(2n+1)!} = \frac{1}{(n+1)_{n+1}} \tag{18}$$

and (17) becomes

$$I_2 = \frac{1}{2b} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{a}{2b}\right)^{2m} \frac{1}{(-m)_{-m}}. \tag{19}$$

Transforming the term $(-m)_{-m}$ by using

$$(x)_{-n} = \frac{(-1)^n}{(1-x)_n} \tag{20}$$

gives

$$(-m)_{-m} = \frac{(-1)^m}{(1+m)_m}. \tag{21}$$

Replacing in (19) produces

$$I_2 = \frac{1}{2b} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \frac{1}{k!} \left(\frac{a^2}{b^2}\right)^k$$

$$= \frac{1}{2} \frac{1}{\sqrt{b^2 - a^2}}.$$

The method of brackets produces *half of the expected answer*. Naturally, it is possible that the entry in [13] is erroneous (this happens once in a while). Some numerical computations and the evaluation in the previous section, should convince the reader that this is not the case. The source of the error is the use of (20) for the evaluation of the term $(-m)_{-m}$. A discussion is presented in Section 5.

5 Extensions of the Pochhammer symbol

Rule E_1 of the method of brackets requires the evaluation of $f(n^*)$. In many instances, this involves the evaluation of the Pochhammer symbol $(x)_m$ for $m \notin \mathbb{N}$. In particular, the question of the value

$$(-m)_{-m} \quad \text{for} \quad m \in \mathbb{N} \tag{22}$$

is at the core of the missing factor of 2 in Section 4.

The first extension of $(x)_m$ to negative values of n comes from the identity

$$(x)_{-m} = \frac{(-1)^m}{(1-x)_m}. \tag{23}$$

This is obtained from

$$(x)_{-m} = \frac{\Gamma(x-m)}{\Gamma(x)} = \frac{\Gamma(x-m)}{(x-1)(x-2)\cdots(x-m)\Gamma(x-m)} \tag{24}$$

and then changing the signs of each of the factors. This is valid as long as x is not a negative integer. The limiting value of the right-hand side in (23) as $x \to -km$, with $k \in \mathbb{N}$, is

$$(-km)_{-m} = \frac{(-1)^m (km)!}{((k+1)m)!}. \tag{25}$$

On the other hand, the limiting value of the left-hand side is

$$
\begin{aligned}
\lim_{\varepsilon \to 0} (-k(m+\varepsilon))_{-(m+\varepsilon)} &= \lim_{\varepsilon \to 0} \frac{\Gamma(-(k+1)m - (k+1)\varepsilon)}{\Gamma(-km - k\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \frac{\Gamma(-(k+1)\varepsilon)(-(k+1)\varepsilon)_{-(k+1)m}}{\Gamma(-k\varepsilon)(-k\varepsilon)_{-km}} \\
&= \lim_{\varepsilon \to 0} \frac{\Gamma(-(k+1)\varepsilon)}{\Gamma(-k\varepsilon)} \frac{(-1)^{(k+1)m}}{(1+(k+1)\varepsilon)_{(k+1)m}} \cdot \frac{(1+k\varepsilon)_{km}}{(-1)^{km}} \\
&= \frac{(-1)^m (km)!}{((k+1)m)!} \cdot \frac{k}{k+1}.
\end{aligned}
$$

Therefore the function $(x)_{-m}$ is discontinuous at $x = -km$, with

$$\frac{\text{Direct } (-km)_{-m}}{\text{Limiting } (-km)_{-m}} = \frac{k+1}{k}. \tag{26}$$

For $k = 1$, this ratio becomes 2. This explains the missing $\frac{1}{2}$ in the calculation in Section 4. Therefore it is the discontinuity of (10) at negative integer values of the variables, what is responsible for the error in the evaluation of the integral (1).

This example suggest that the rules of the method of brackets should be supplemented with an additional one:

Rule E$_4$. Let $k \in \mathbb{N}$ be fixed. In the evaluation of series, the rule

$$(-km)_{-m} = \frac{k}{k+1} \frac{(-1)^m (km)!}{((k+1)m)!} \tag{27}$$

must be used to eliminate Pochhammer symbols with negative index and negative integer base.

A variety of other examples confirm that this heuristic rule leads to correct evaluations.

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