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Prime, weakly prime and almost prime elements in multiplication lattice modules

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Abstract: In this paper, we study multiplication lattice modules. We establish a new multiplication over elements of a multiplication lattice module. With this multiplication, we characterize idempotent element, prime element, weakly prime element and almost prime element in multiplication lattice modules.

Keywords: Multiplication lattice module, Prime element, Weakly prime element and Almost prime element

MSC: 16F10, 16F05

1 Introduction

In 1962, R. P. Dilworth began a study of the ideal theory of commutative rings in an abstract setting in [1]. Since the investigation was to be purely ideal-theoretic, he chose to study a lattice with a commutative multiplication. Then he introduced the concept of the multiplicative lattice. By a *multiplicative lattice*, R. P. Dilworth meant a complete but not necessarily modular lattice L on which there is defined a completely join distributive product. In the study, he denoted the greatest element by 1_L (least element 0_L) and assumed that the 1_L is a compact multiplicative identity. In addition, he introduced the notion of a *principal element* as a generalization to the notion of a *principal ideal* and defined the *Noether lattice* (see [1], Definition 3.1).

Let L be a multiplicative lattice. An element $a \in L$ is said to be *proper* if $a < 1_L$. If a, b belong to L , $(a :_L b)$ is the join of all $c \in L$ such that $cb \leq a$. Dilworth defined a *meet (join) principal* and a *principal element* of a multiplicative lattice as follows. An element e of L is called *meet principal* if $a \wedge be = ((a :_L e) \wedge b)e$ for all $a, b \in L$. An element e of L is called *join principal* if $((ae \vee b) :_L e) = a \vee (b :_L e)$ for all $a, b \in L$. If e is meet principal and join principal, $e \in L$ is said to be *principal*. An element $p < 1_L$ in L is said to be *prime* if $ab \leq p$ implies either $a \leq p$ or $b \leq p$ for all $a, b \in L$. For any $a \in L$, he defined \sqrt{a} as $\vee \{x \in L : x^n \leq a \text{ for some integer } n\}$. An element a of L is called *idempotent* if $a^2 = a$. An element a of L is called *compact* if $a \leq \vee_{\alpha_i \in \Delta} b_{\alpha_i}$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \dots \vee b_{\alpha_n}$ such that $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \Delta$, where Δ is an index set. If each element of L is a join of principal (compact) elements of L , then L is called a *principally generated lattice*, briefly *PG-lattice* (*compactly generated lattice*, briefly *CG-lattice*). By a *C-lattice*, we mean a (not necessarily modular) complete multiplicative lattice, with the least element 0_L and the compact greatest element 1_L (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. For various characterizations of lattice, the reader is referred to [2].

Then in [3], F. Callialp, C. Jayaram and U. Tekir defined weakly prime and almost prime as follows: An element $p < 1_L$ in L is said to be *weakly prime* if $0_L \neq ab \leq p$ implies $a \leq p$ or $b \leq p$ for all $a, b \in L$. An element $p < 1_L$ in L is said to be *almost prime* if $ab \leq p$ and $ab \not\leq p^2$ imply $a \leq p$ or $b \leq p$ for all $a, b \in L$.

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In 1970, E. W. Johnson and J. A. Johnson introduced and studied Noetherian lattice modules in [4, 5]. Hence most of Dilworth's ideas and methods were extended. Then in [2], Anderson defined lattice module as follows:

Let M be a complete lattice. Recall that M is a *lattice module* over the multiplicative lattice L , or simply an L -module in case there is a multiplication between elements of L and M , denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties for all l, l_α, b in L and for all B, B_β in M :

- (1) $(lb)B = l(bB)$;
- (2) $(\bigvee_\alpha l_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha, \beta} l_\alpha B_\beta$;
- (3) $1_L B = B$;
- (4) $0_L B = 0_M$.

Let M be an L -lattice module. The greatest (least) element of M is denoted by 1_M (0_M). An element $N \in M$ is said to be *proper* if $N < 1_M$. If N, K belong to M , $(N :_L K)$ is the join of all $a \in L$ such that $aK \leq N$. Especially, $(0_M :_L 1_M)$ is denoted by $\text{ann}(M)$. In addition, if $\text{ann}(M) = 0_L$ then M is called a faithful lattice module. If $a \in L$ and $N \in M$, then $(N :_M a)$ is the join of all $H \in M$ such that $aH \leq N$. An element N of M is called *meet principal* if $(b \wedge (B :_L N))N = bN \wedge B$ for all $b \in L$ and for all $B \in M$. An element N of M is called *join principal* if $b \vee (B :_L N) = ((bN \vee B) :_L N)$ for all $b \in L$ and for all $B \in M$. N is said to be *principal* if it is meet principal and join principal. An element N in M is called *compact* if $N \leq \bigvee_{\alpha_i \in \Delta} B_{\alpha_i}$ implies $N \leq B_{\alpha_1} \vee B_{\alpha_2} \vee \dots \vee B_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \Delta$, where Δ is an index set. If each element of M is a join of principal (compact) elements of M , then M is called a *principally generated lattice module*, briefly *PG-lattice module* (*compactly generated lattice*, briefly *CG-lattice module*). For various information on lattice module, one is referred to [6–8].

In 1988, Z. A. El-Bast and P. F. Smith introduced the concept of *multiplication module* in [9]. There are many studies on multiplication modules [10–13]. With the help of the concept of multiplication module, in 2011, F. Callialp and U. Tekir defined *multiplication lattice modules* in [14] (see, Definition 5). They characterized multiplication lattice modules with the help of principal elements of lattice modules. In addition, they examined maximal and prime elements of lattice modules. Then in 2014, F. Callialp, U. Tekir and E. Aslankarayigit proved Nakayama Lemma for multiplication lattice modules ([15], Theorem 1. 19). Moreover in the study, the authors obtained some characterizations of maximal, prime and primary elements in multiplication lattice modules.

In this study, we continue to examine multiplication lattice modules. Our aim is to extend the concepts of almost prime ideals and idempotent ideals of commutative rings to non-modular multiplicative lattices. So, we introduce almost prime element and idempotent element in lattice modules. To define the above-mentioned elements, we use the studies [16–19]. Then we obtain the relationship between the prime (weakly prime and almost prime, respectively) element of L -module M and the prime (weakly prime and almost prime, respectively) element of L (see, Theorem 3.6–Theorem 3.8). In addition, we define a new multiplication over multiplication lattice modules (see, Definition 3.9). With the help of the multiplication, we characterize idempotent element, prime element, weakly prime element and almost prime element in Theorem 3.11–Theorem 3.14, respectively.

Throughout this paper, L denotes a multiplicative lattice and M denotes a complete lattice. Moreover, L_* , M_* denote the set of all compact elements of L , M , respectively.

2 Some definitions and properties

Definition 2.1 ([6], Definition 3.1). *Let M be an L -lattice module and N be a proper element of M . N is called a prime element of M , if whenever $a \in L$, $X \in M$ such that $aX \leq N$, then $X \leq N$ or $a \leq (N :_L 1_M)$.*

Especially, M is said to be prime L -lattice module if 0_M is prime element of M .

Definition 2.2 ([8], Definition 2.1). *Let M be an L -lattice module and N be a proper element of M . N is called a weakly prime element of M , if whenever $a \in L$, $X \in M$ such that $0_M \neq aX \leq N$, then $X \leq N$ or $a \leq (N :_L 1_M)$.*

Definition 2.3. Let M be an L -lattice module and N be a proper element of M . N is called an almost prime element of M , if whenever $a \in L$, $X \in M$ such that $aX \leq N$ and $aX \not\leq (N :_L 1_M)N$, then $X \leq N$ or $a \leq (N :_L 1_M)$.

Clearly, any prime element is weakly prime and weakly prime element is almost prime. However, any weakly prime element may not be prime, see the following example:

Example 2.4. Let M be a non-prime L -lattice module. The zero element 0_M is weakly prime, which is not prime.

For an almost prime element which is not weakly prime, we consider the following example:

Example 2.5. Let Z_{24} be Z -module. Assume that $\langle k \rangle$ denotes the cyclic ideal of Z generated by $k \in Z$ and $\langle \bar{i} \rangle$ denotes the cyclic submodule of Z -module Z_{24} by $\bar{i} \in Z_{24}$.

Suppose that $L = L(Z)$ is the set of all ideals of Z and $M = L(Z_{24})$ is the set of all submodules of Z -module Z_{24} . There is a multiplication between elements of L and M , for every $\langle k_i \rangle \in L$ and $\langle \bar{t}_j \rangle \in M$ denoted by $\langle k_i \rangle \langle \bar{t}_j \rangle = \langle \bar{k}_i \bar{t}_j \rangle$, where $k_i, t_j \in Z$. Then M is a lattice module over L .

Let N be the cyclic submodule of M generated by $\bar{8}$. Then clearly $N = (N :_L 1_M)N$ and so N is an almost prime element. In contrast,

$0_M = \langle \bar{0} \rangle \neq \langle \bar{4} \rangle \leq N = \langle \bar{8} \rangle$ with $\langle \bar{4} \rangle \not\leq N$ and $\langle \bar{4} \rangle \not\leq (N :_L 1_M)$ and so N is not weakly prime.

Definition 2.6. Let M be an L -lattice module and N be an element of M . N is called an idempotent element of M , if $N = (N :_L 1_M)N$.

Thus, any proper idempotent element of M is almost prime.

Definition 2.7 ([14], Definition 4). An L -lattice module M is called a multiplication lattice module if for every $N \in M$, there exists $a \in L$ such that $N = a1_M$.

To achieve comprehensiveness in this study, we state the following Proposition.

Proposition 2.8 ([14], Proposition 3). Let M be an L -lattice module. Then M is a multiplication lattice module if and only if $N = (N :_L 1_M)1_M$ for all $N \in M$.

We recall $M/N = \{B \in M : N \leq B\}$ is an L -lattice module with multiplication $c \circ D = cD \vee N$ for every $c \in L$ and for every $N \leq D \in M$, [1].

Proposition 2.9. Let M be an L -lattice module and N be a proper element of M . Then N is an almost prime element in M if and only if N is a weakly prime element in $M/(N :_L 1_M)N$.

Proof. \implies : Suppose N is almost prime in M . Let $r \in L$ and $X \in M/(N :_L 1_M)N$, such that $0_{M/(N :_L 1_M)N} \neq r \circ X \leq N$. Then we have two cases:

Case 1: Suppose, on the contrary, that $(N :_L 1_M)N = N$. Then $N = 0_{M/(N :_L 1_M)N}$. Since $r \circ X \leq N$, we have $N \leq rX \vee N \leq rX \vee (N :_L 1_M)N = r \circ X \leq N$, that is, $r \circ X = N$. But then $0_{M/(N :_L 1_M)N} \neq r \circ X = N = 0_{M/(N :_L 1_M)N}$, a contradiction.

Case 2: Suppose that $(N :_L 1_M)N < N$. As $r \circ X \leq N$, we get $rX \leq N$. Moreover, since $0_{M/(N :_L 1_M)N} \neq r \circ X = rX \vee (N :_L 1_M)N$, then we have $rX \not\leq (N :_L 1_M)N$. Indeed, if $rX \leq (N :_L 1_M)N$, then we get $r \circ X = rX \vee (N :_L 1_M)N = (N :_L 1_M)N = 0_{M/(N :_L 1_M)N}$, a contradiction. As $rX \leq N$, $rX \not\leq (N :_L 1_M)N$ and N is almost prime in M , then we have $X \leq N$ or $r \leq (N :_L 1_M) = (N :_L 1_{M/(N :_L 1_M)N})$. Thus, N is weakly prime in $M/(N :_L 1_M)N$.

\impliedby : Suppose N is weakly prime in $M/(N :_L 1_M)N$. Let $r \in L$ and $X \in M$ such that $rX \leq N$ and $rX \not\leq (N :_L 1_M)N$. Since $rX \not\leq (N :_L 1_M)N$ and $r \circ X = rX \vee (N :_L 1_M)N$, we have $r \circ X \neq (N :_L 1_M)N$, i.e., $0_{M/(N :_L 1_M)N} \neq r \circ X$. Moreover, as $rX \leq N$ then we get $r \circ X \leq N$. Since N is weakly prime in

$M/(N :_L 1_M)N$, we obtain $X \leq N$ or $r \leq (N :_L 1_{M/(N :_L 1_M)N}) = (N :_L 1_M)$. Thus, N is an almost prime element in M . \square

Theorem 2.10. *Let N be an almost prime element of an L -lattice module M . If K is an element of M with $K \leq N$, then N is an almost prime element of M/K .*

Proof. Let $r \in L$ and $X \in M/K$ such that $r \circ X \leq N$ and $r \circ X \not\leq (N :_L 1_{M/K}) \circ N$. Firstly, we show $rX \not\leq (N :_L 1_M)N$. Assume that $rX \leq (N :_L 1_M)N$. Then we have $rX \vee K \leq (N :_L 1_M)N \vee K$, i.e., $r \circ X \leq (N :_L 1_M) \circ N = (N :_L 1_{M/K}) \circ N$, which is a contradiction. Thus we get $rX \not\leq (N :_L 1_M)N$. Moreover, as $r \circ X \leq N$, then we obtain $rX \leq N$. Since N is an almost prime element in M , we get $X \leq N$ or $r \leq (N :_L 1_M) = (N :_L 1_{M/K})$. Consequently, N is an almost prime element in M/K . \square

Dilworth in Lemma 4.2 of [1] proved that N is a prime element of M if and only if N is a prime element of M/K , for any element $K \leq N$. In the previous Theorem, we prove Lemma 4.2's one part for an almost prime case. The other part may not be true; see the following example:

Example 2.11. *For any non-almost prime element N of L -lattice module M . Then we always know that $0_{M/N}$ is a weakly prime element of M/N . Hence $0_{M/N} = N$ is a weakly prime (and so almost prime) element of M/N . However by our assumption, N is not almost prime. Consequently, N is an almost prime element of M/N , but N is not an almost prime element of M .*

3 Some characterizations

In this part, we obtain several characterizations of some elements in Lattice Modules under special conditions.

Lemma 3.1. *Let M be a C -lattice L -module. Let $N_1, N_2 \in M$. Suppose $B \in M$ satisfies the following properties:*

(*) *If $H \in M$ is compact with $H \leq B$, then either $H \leq N_1$ or $H \leq N_2$.*

Then either $B \leq N_1$ or $B \leq N_2$.

Proof. Assume that $B \not\leq N_1$ and $B \not\leq N_2$. Then since B is a join of compact elements, we can find compact elements $H_1 \leq B$ and $H_2 \leq B$ such that $H_1 \not\leq N_1$ and $H_2 \not\leq N_2$. Since $H = H_1 \vee H_2 \leq B$ is compact, then by hypothesis (*) we have $H \leq N_1$ or $H \leq N_2$, a contradiction. Consequently, we have either $B \leq N_1$ or $B \leq N_2$. \square

Theorem 3.2. *Let L be a C -lattice, M be a C -lattice L -module and N be an element of M . Then the following statements are equivalent:*

- (1) *N is weakly prime in M .*
- (2) *For any $a \in L$ such that $a \not\leq (N :_L 1_M)$, either $(N :_M a) = N$ or $(N :_M a) = (0_M :_M a)$.*
- (3) *For every $a \in L_*$ and every $K \in M_*$, $0_M \neq aK \leq N$ implies either $a \leq (N :_L 1_M)$ or $K \leq N$.*

Proof. (1) \implies (2) Suppose (1) holds. Let H be a compact element of M such that $H \leq B = (N :_M a)$ and $a \not\leq (N :_L 1_M)$. Then $aH \leq N$. We have two cases:

Case 1: Let $aH = 0_M$. Then $H \leq (0_M :_M a)$.

Case 2: Let $aH \neq 0_M$. Since $aH \leq N$, $a \not\leq (N :_L 1_M)$ and N is weakly prime, it follows that $H \leq N$.

Hence by Lemma 3.1, either $(N :_M a) \leq (0_M :_M a)$ or $(N :_M a) \leq N$. Consequently, either $(N :_M a) = (0_M :_M a)$ or $(N :_M a) = N$.

(2) \implies (3) Suppose (2) holds. Let $0_M \neq aK \leq N$ and $a \not\leq (N :_L 1_M)$ for $a \in L_*$ and $K \in M_*$. We will show that $K \leq N$. Since $aK \leq N$, it follows that $K \leq (N :_M a)$. If $(N :_M a) = N$, then $K \leq N$. If $(N :_M a) = (0_M :_M a)$, then $aK = 0_M$. This is a contradiction. Consequently, $K \leq N$.

(3) \implies (1) Suppose (3) holds. Let $aK \leq N$, $a \not\leq (N :_L 1_M)$ and $K \not\leq N$ for some $a \in L$ and $K \in M$. Choose $x_1 \in L_*$ and $Y_1 \in M_*$ such that $x_1 \leq a$, $x_1 \not\leq (N :_L 1_M)$, $Y_1 \leq K$ and $Y_1 \not\leq N$. Let $x_2 \leq a$ and $Y_2 \leq K$ be any two compact elements of L, M , respectively. Then by our assumption (3), we have $(x_2 \vee x_1)(Y_2 \vee Y_1) = 0_M$ and so $x_2 Y_2 = 0_M$. Therefore $aK = 0_M$. This shows that N is weakly prime in M . \square

Theorem 3.3. *Let L be a C -lattice, M be a C -lattice L -module and N be an element of M . Then the following statements are equivalent:*

- (1) N is almost prime in M .
- (2) For any $a \in L$ such that $a \not\leq (N :_L 1_M)$, either $(N :_M a) = N$ or $(N :_M a) = ((N :_L 1_M)N :_M a)$.
- (3) For every $a \in L_*$ and every $K \in M_*$, $aK \leq N$ and $aK \not\leq (N :_L 1_M)N$ implies either $a \leq (N :_L 1_M)$ or $K \leq N$.

Proof. (1) \implies (2) Suppose (1) holds. Let H be a compact element of M such that $H \leq B = (N :_M a)$ and $a \not\leq (N :_L 1_M)$. Then $aH \leq N$. We have two cases:

Case 1: If $aH \leq (N :_L 1_M)N$, then $H \leq ((N :_L 1_M)N :_M a)$.

Case 2: If $aH \not\leq (N :_L 1_M)N$, since $aH \leq N$, $a \not\leq (N :_L 1_M)$ and N is almost prime, it follows that $H \leq N$.

Hence by Lemma 3.1, we prove that either $(N :_M a) \leq ((N :_L 1_M)N :_M a)$ or $(N :_M a) \leq N$. One can see, as $(N :_L 1_M)N \leq N$, we get $((N :_L 1_M)N :_M a) \leq (N :_M a)$. Moreover, always $N \leq (N :_M a)$. Consequently, either $(N :_M a) = ((N :_L 1_M)N :_M a)$ or $(N :_M a) = N$.

(2) \implies (3) Suppose (2) holds. Let $aK \leq N$ and $aK \not\leq (N :_L 1_M)N$ for $a \in L_*$ and $K \in M_*$. Assume that $a \not\leq (N :_L 1_M)$. We show that $K \leq N$. Since $aK \leq N$, it follows that $K \leq (N :_M a)$. If $(N :_M a) = N$, then $K \leq N$. If $(N :_M a) = ((N :_L 1_M)N :_M a)$, then $K \leq ((N :_L 1_M)N :_M a)$. So we have $aK \leq (N :_L 1_M)N$, a contradiction. Thus $K \leq N$.

(3) \implies (1) Suppose (3) holds. Let $aK \leq N$, $aK \not\leq (N :_L 1_M)N$ for some $a \in L$ and $K \in M$. Assume that $a \not\leq (N :_L 1_M)$ and $K \not\leq N$. Choose $x_1 \in L_*$ and $Y_1 \in M_*$ such that $x_1 \leq a$, $x_1 \not\leq (N :_L 1_M)$, $Y_1 \leq K$ and $Y_1 \not\leq N$. As L and M are C -lattices, there exist two compact elements of $x_2 \in L$ and $Y_2 \in M$ such that $x_2 \leq a$ and $Y_2 \leq K$. Moreover, as $x_1, x_2 \in L_*$ and $Y_1, Y_2 \in M_*$, we have $x_1 \vee x_2 \in L_*$ and $Y_1 \vee Y_2 \in M_*$. Since $x_1 \leq a$ and $x_2 \leq a$, we have $x_1 \vee x_2 \leq a$. Similarly, we have $Y_1 \vee Y_2 \leq K$. Thus $(x_2 \vee x_1)(Y_2 \vee Y_1) \leq aK \leq N$. In addition, $(x_2 \vee x_1)(Y_2 \vee Y_1) \not\leq (N :_L 1_M)N$. Indeed, assume that $(x_2 \vee x_1)(Y_2 \vee Y_1) \leq (N :_L 1_M)N$. Then we get $x_2 Y_2 \leq (N :_L 1_M)N$. Since $x_2 Y_2 \leq aK$, we can write $aK \leq (N :_L 1_M)N$, for $x_2 Y_2 \in M_*$. But it is a contradiction.

Consequently, as $(x_2 \vee x_1)(Y_2 \vee Y_1) \leq N$ and $(x_2 \vee x_1)(Y_2 \vee Y_1) \not\leq (N :_L 1_M)N$, by our assumption (3), we have $(x_2 \vee x_1) \leq (N :_L 1_M)$ or $(Y_2 \vee Y_1) \leq N$. Then we get $x_1 \leq (N :_L 1_M)$ or $Y_1 \leq N$, a contradiction. This shows that N is almost prime in M . \square

Lemma 3.4. *Let L be a C -lattice and M be a multiplication C -lattice L -module. If N is an almost prime element of M , then $\sqrt{((N :_L 1_M)N :_L 1_M)N} = (N :_L 1_M)N$.*

Proof. We first note that $(N :_L 1_M)^2 \leq ((N :_L 1_M)N :_L 1_M)$. Indeed, since M is a multiplication lattice module, we have $(N :_L 1_M)(N :_L 1_M)1_M = (N :_L 1_M)N$, i.e., $(N :_L 1_M)^2 \leq ((N :_L 1_M)N :_L 1_M)$.

Let a be a compact element in L and $a \leq \sqrt{((N :_L 1_M)N :_L 1_M)}$.

If $a \leq (N :_L 1_M)$, then we have $a(N :_L 1_M) \leq (N :_L 1_M)^2 \leq ((N :_L 1_M)N :_L 1_M)$. Thus we obtain $aN = a(N :_L 1_M)1_M \leq (N :_L 1_M)N :_L 1_M = (N :_L 1_M)N$.

If $a \not\leq (N :_L 1_M)$, then we have either $(N :_M a) = ((N :_L 1_M)N :_M a)$ or $(N :_M a) = N$ by Theorem 3.3(2).

Case 1: Suppose that $(N :_M a) = ((N :_L 1_M)N :_M a)$. Since $N \leq (N :_M a)$, then we have $aN \leq a(N :_M a) = a((N :_L 1_M)N :_M a) \leq (N :_L 1_M)N$.

Case 2: Suppose that $(N :_M a) = N$. Let n be the smallest positive integer such that $a^n \leq ((N :_L 1_M)N :_L 1_M)$. If $n = 1$, then we have $a1_M \leq (N :_L 1_M)N \leq N$, a contradiction.

So, we assume $n \geq 2$. Then $a^n 1_M \leq (N :_L 1_M)N \leq N$ with $a^k 1_M \not\leq (N :_L 1_M)N$ for every $k \leq n - 1$. It follows that $a^{n-1} 1_M \leq (N :_M a) = N$ and $a^{n-1} 1_M \not\leq (N :_L 1_M)N$. If $n = 2$, we also get a contradiction. If

$n \geq 3$, we have $a(a^{n-2}1_M) \leq N$ and $a(a^{n-2}1_M) \not\leq (N :_L 1_M)N$. Thus, since N is an almost prime element, we obtain either $a \leq (N :_L 1_M)$ or $a^{n-2}1_M \leq N$. Continuing this process, we conclude that $a \leq (N :_L 1_M)$, which is a contradiction. Therefore $\sqrt{(N :_L 1_M)N :_L 1_M}N \leq (N :_L 1_M)N$.

For the second part, let a be a compact element in L and $a \leq (N :_L 1_M)$. Then we have $a^k 1_M \leq a 1_M \leq N$ for positive integer k , i.e., $a^k \leq (N :_L 1_M)$. Thus, we obtain $a^{k+1} 1_M \leq a^k N \leq (N :_L 1_M)N$, i.e., $a^{k+1} \leq ((N :_L 1_M)N :_L 1_M)$. Consequently, $a \leq \sqrt{((N :_L 1_M)N :_L 1_M)}$. \square

Lemma 3.5. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. Then we have $(aN :_L 1_M) = a(N :_L 1_M)$ for every element a in L .

Proof. As M is a multiplication lattice module, then we have $a(N :_L 1_M)1_M = aN = (aN :_L 1_M)1_M$. By Theorem 5 in [14], we obtain $a(N :_L 1_M) = (aN :_L 1_M)$. \square

Theorem 3.6. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice L -module. For $1_M \neq N \in M$, the followings are equivalent:

- (1) N is prime.
- (2) $(N :_L 1_M)$ is prime.
- (3) $N = q1_M$ for some prime element q of L .

Proof. The proof can be easily seen with Corollary 3 in [14]. \square

Theorem 3.7. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. For $1_M \neq N \in M$, then the followings are equivalent:

- (1) N is weakly prime.
- (2) $(N :_L 1_M)$ is weakly prime.
- (3) $N = q1_M$ for some weakly prime element q of L .

Proof. (1) \implies (2): Suppose N is weakly prime and $a, b \in L$ such that $0_L \neq ab \leq (N :_L 1_M)$. Then we have $ab1_M \leq N$. Since M is faithful and $0_L \neq ab$, then we obtain $0_M \neq ab1_M$. Now, as N is weakly prime, then we get either $a \leq (N :_L 1_M)$ or $b1_M \leq N$ (and so $b \leq (N :_L 1_M)$). Hence $(N :_L 1_M)$ is a weakly prime element in L .

(2) \implies (1): Let $(N :_L 1_M)$ be weakly prime in L . Let $r \in L$ and $X \in M$, such that $0_M \neq rX \leq N$. By Lemma 3.5, we have $r(X :_L 1_M) = (rX :_L 1_M) \leq (N :_L 1_M)$. Moreover $r(X :_L 1_M) \neq 0_L$ because otherwise, if $r(X :_L 1_M) = 0_L$, then $rX = r(X :_L 1_M)1_M = 0_L 1_M = 0_M$. As $(N :_L 1_M)$ is weakly prime, then either $r \leq (N :_L 1_M)$ or $(X :_L 1_M) \leq (N :_L 1_M)$. Since M is a multiplication lattice module, we obtain $r \leq (N :_L 1_M)$ or $X = (X :_L 1_M)1_M \leq (N :_L 1_M)1_M = N$. Thus, N is weakly prime in M .

(2) \implies (3): Choose $q = (N :_L 1_M)$.

(3) \implies (2): Suppose that $N = q1_M$ for some weakly prime element q of L . By Lemma 3.5, we have $(N :_L 1_M) = (q1_M :_L 1_M) = q(1_M :_L 1_M) = q$. Thus $q = (N :_L 1_M)$ is a weakly prime element. \square

Theorem 3.8. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. For $1_M \neq N \in M$, then the followings are equivalent:

- (1) N is almost prime.
- (2) $(N :_L 1_M)$ is almost prime.
- (3) $N = q1_M$ for some almost prime element q of L .

Proof. (1) \implies (2): Suppose N is almost prime and $a, b \in L$ such that $ab \leq (N :_L 1_M)$ and $ab \not\leq (N :_L 1_M)^2$. Then we have $ab1_M \leq N$ and $ab1_M \not\leq (N :_L 1_M)N$. Indeed, if $ab1_M \leq (N :_L 1_M)N$, by Lemma 3.5, $ab \leq ((N :_L 1_M)N :_L 1_M) = (N :_L 1_M)(N :_L 1_M) = (N :_L 1_M)^2$, a contradiction. Now, N is almost prime implies that either $a \leq (N :_L 1_M)$ or $b1_M \leq N$ (and so $b \leq (N :_L 1_M)$). Hence $(N :_L 1_M)$ is an almost prime element in L .

(2) \implies (1): Let $r \in L$ and $X \in M$ such that $rX \leq N$ and $rX \not\leq (N :_L 1_M)N$. By Lemma 3.5, we have $r(X :_L 1_M) = (rX :_L 1_M) \leq (N :_L 1_M)$. Moreover $r(X :_L 1_M) \not\leq (N :_L 1_M)^2$. Indeed, if $r(X :_L 1_M) \leq (N :_L 1_M)^2 = ((N :_L 1_M)N :_L 1_M)$, then $rX = r(X :_L 1_M)1_M \leq ((N :_L 1_M)N :_L 1_M)1_M = (N :_L 1_M)N$, a contradiction. As $(N :_L 1_M)$ is almost prime, either $r \leq (N :_L 1_M)$ or $(X :_L 1_M) \leq (N :_L 1_M)$. By Proposition 2.8, we have $X = (X :_L 1_M)1_M \leq (N :_L 1_M)1_M = N$. Thus, we obtain $r \leq (N :_L 1_M)$ or $X \leq N$, i.e., N is almost prime in M .

(2) \implies (3): Choose $q = (N :_L 1_M)$.

(3) \implies (2): Suppose that $N = q1_M$ for some almost prime element q of L . By Lemma 3.5, we have $(N :_L 1_M) = (q1_M :_L 1_M) = q(1_M :_L 1_M) = q$. Thus $q = (N :_L 1_M)$ is an almost prime element. \square

Now, we define a new multiplication over the multiplication lattice modules.

Definition 3.9. If M is a multiplication L -lattice module and $N = a1_M$, $K = b1_M$ are two elements of M , where $a, b \in L$, the product of N and K is defined as $NK = (a1_M)(b1_M) = ab1_M$.

Proposition 3.10. Let M be a multiplication L -lattice module and $N = a1_M$, $K = b1_M$ are two elements of M , where $a, b \in L$. Then the product of N and K is independent of expression of N and K .

Proof. Let $N = a_11_M = a_21_M$ and $K = b_11_M = b_21_M$ for $a_1, a_2, b_1, b_2 \in L$. Then $NK = (a_1b_1)1_M = a_1(b_11_M) = a_1(b_21_M) = b_2(a_11_M) = b_2(a_21_M) = (a_2b_2)1_M$. \square

With the help of the new defined multiplication, we obtain the following results.

Theorem 3.11. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. Then N is an idempotent element in M if and only if $N^2 = N$.

Proof. \implies : Since N is idempotent, then we have $N = (N :_L 1_M)N$. As M is a multiplication lattice module, then we get $N^2 = NN = (N :_L 1_M)1_M(N :_L 1_M)1_M = (N :_L 1_M)^21_M$. By Proposition 2.8 and Lemma 3.5, we obtain $N = (N :_L 1_M)N = ((N :_L 1_M)N :_L 1_M)1_M = (N :_L 1_M)(N :_L 1_M)1_M = (N :_L 1_M)^21_M$. Thus we have $N^2 = (N :_L 1_M)^21_M = N$.

\Leftarrow : Suppose that $N^2 = N$. Following the same steps in the first part of the proof, we obtain $N = N^2 = (N :_L 1_M)^21_M = (N :_L 1_M)N$, i.e., $N = (N :_L 1_M)N$. Consequently, N is idempotent in M . \square

Theorem 3.12. Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. Then $N < 1_M$ is prime in M if and only if whenever X and Y are elements of M such that $XY \leq N$, either $X \leq N$ or $Y \leq N$.

Proof. \implies : Assume that N is prime in M . By Theorem 3.6, we get $(N :_L 1_M)$ is prime in L . Suppose that X and Y are elements of M such that $XY \leq N$, but $X \not\leq N$ and $Y \not\leq N$. By Proposition 2.8, we have $X = (X :_L 1_M)1_M$ and $Y = (Y :_L 1_M)1_M$ and so $XY = (X :_L 1_M)(Y :_L 1_M)1_M$. Since M is a multiplication lattice module, then we have $(X :_L 1_M) \not\leq (N :_L 1_M)$ and $(Y :_L 1_M) \not\leq (N :_L 1_M)$. Indeed, if $(X :_L 1_M) \leq (N :_L 1_M)$ and $(Y :_L 1_M) \leq (N :_L 1_M)$, then we have $(X :_L 1_M)1_M \leq (N :_L 1_M)1_M$ and $(Y :_L 1_M)1_M \leq (N :_L 1_M)1_M$. So, by Proposition 2.8, $X \leq N$ and $Y \leq N$, a contradiction. Hence $(X :_L 1_M) \not\leq (N :_L 1_M)$ and $(Y :_L 1_M) \not\leq (N :_L 1_M)$. Thus, since $(N :_L 1_M)$ is prime, we obtain $(X :_L 1_M)(Y :_L 1_M) \not\leq (N :_L 1_M)$. Moreover, we have $XY = (X :_L 1_M)(Y :_L 1_M)1_M \leq N$, i.e., $(X :_L 1_M)(Y :_L 1_M) \leq (N :_L 1_M)$, a contradiction. Therefore, either $X \leq N$ or $Y \leq N$.

\Leftarrow : We assume that if $XY \leq N$, then $X \leq N$ or $Y \leq N$. To prove that N is prime in M , it is enough, by Theorem 3.6, to prove that $(N :_L 1_M)$ is prime in L . Let $r_1, r_2 \in L$ such that $r_1r_2 \leq (N :_L 1_M)$. Let $X = r_11_M$ and $Y = r_21_M$. Then $XY = r_1r_21_M \leq N$. By assumption, either $r_11_M = X \leq N$ or $r_21_M = Y \leq N$ and so, either $r_1 \leq (N :_L 1_M)$ or $r_2 \leq (N :_L 1_M)$. Hence $(N :_L 1_M)$ is prime in L . Consequently, N is prime in M . \square

The proof of the next Theorem can be shown to be similar to the previous proof with using Proposition 2.8 and Theorem 3.7.

Theorem 3.13. *Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. Then $N < 1_M$ is weakly prime in M if and only if whenever X and Y are elements of M such that $0_M \neq XY \leq N$, either $X \leq N$ or $Y \leq N$.*

Finally, the proof of the following Theorem is obtained, as in the case of Theorem 3.12, by using the proof of Proposition 2.8, Lemma 3.5 and Theorem 3.8.

Theorem 3.14. *Let L be a PG -lattice with 1_L compact and M be a faithful multiplication PG -lattice module with 1_M compact. Then $N < 1_M$ is almost prime in M if and only if whenever X and Y are elements of M such that $XY \leq N$ and $XY \not\leq (N :_L 1_M)N$, either $X \leq N$ or $Y \leq N$.*

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