

Lamei Yuan and Henan Wu*

Structures of $W(2, 2)$ Lie conformal algebra

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Abstract: The purpose of this paper is to study $W(2, 2)$ Lie conformal algebra, which has a free $\mathbb{C}[\partial]$ -basis $\{L, M\}$ such that $[L_\lambda L] = (\partial + 2\lambda)L$, $[L_\lambda M] = (\partial + 2\lambda)M$, $[M_\lambda M] = 0$. In this paper, we study conformal derivations, central extensions and conformal modules for this Lie conformal algebra. Also, we compute the cohomology of this Lie conformal algebra with coefficients in its modules. In particular, we determine its cohomology with trivial coefficients both for the basic and reduced complexes.

Keywords: Conformal derivation, Central extension, Conformal module, Cohomology

MSC: 17B10, 17B65, 17B68

1 Introduction

A *Lie conformal algebra* is a $\mathbb{C}[\partial]$ -module \mathcal{R} equipped with a λ -bracket $[\cdot_\lambda \cdot]$ which is a \mathbb{C} -bilinear map from $\mathcal{R} \otimes \mathcal{R}$ to $\mathbb{C}[\lambda] \otimes \mathcal{R}$, such that the following axioms hold for all $a, b, c \in \mathcal{R}$:

$$[\partial a_\lambda b] = -\lambda[a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b] \quad (\text{conformal sesquilinearity}), \quad (1)$$

$$[a_\lambda b] = -[b_{-\lambda-\partial} a] \quad (\text{skew-symmetry}), \quad (2)$$

$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]] \quad (\text{Jacobi identity}). \quad (3)$$

In practice, the λ -brackets arise as generating functions for the singular part of the operator product expansion in two-dimensional conformal field theory [1]. Lie conformal algebras are closely related to vertex algebras and infinite-dimensional Lie (super)algebras satisfying the locality property [2]. In the past few years, semisimple Lie conformal algebras have been intensively studied. In particular, the classification of all finite semisimple Lie conformal (super)algebras were given in [3, 4]. Finite irreducible conformal modules over the Virasoro, the current and the Neveu-Schwarz conformal algebras were classified in [5]. The cohomology theory was developed by Bakalov, Kac, and Voronov in [6], where explicit computations of cohomologies of the Virasoro and the current conformal algebras were given. The aim of this paper is to study structures including derivations, central extensions, conformal modules and cohomologies of a non-semisimple Lie conformal algebra associated to the W -algebra $W(2, 2)$. The method here is based on the theory of the Virasoro conformal modules [5] and the techniques developed in [3, 6]. Our discussions on the cohomology may be useful to do the same thing for the other non-semisimple Lie conformal algebras, such as the Schrödinger-Virasoro, the loop Virasoro and the loop Heisenberg-Virasoro conformal algebras studied in [7–9].

The Lie conformal algebra which we consider in this paper, denoted by \mathcal{W} , is a free $\mathbb{C}[\partial]$ -module of rank 2 generated by L, M , satisfying

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda M] = (\partial + 2\lambda)M, \quad [M_\lambda M] = 0. \quad (4)$$

Lamei Yuan: Academy of Fundamental and Interdisciplinary Sciences, Harbin Institute of Technology, Harbin 150080, China,
E-mail: lmyuan@hit.edu.cn

***Corresponding Author: Henan Wu:** School of Mathematical Sciences, Shanxi University, Taiyuan, 030006, China,
E-mail: wuhenan@sxu.edu.cn

The formal distribution Lie algebra corresponding to \mathcal{W} is the centerless W -algebra $W(2, 2)$ introduced in [10]. Besides, this Lie conformal algebra can be seen as a special case of a more general $W(a, b)$ Lie conformal algebra studied in [11]. Obviously, the Lie conformal algebra \mathcal{W} contains the Virasoro conformal algebra Vir as a subalgebra, namely,

$$\text{Vir} = \mathbb{C}[\partial]L, \quad [L_\lambda L] = (\partial + 2\lambda)L. \quad (5)$$

And it has a nontrivial abelian conformal ideal generated by M . Thus it is not semisimple.

The rest of the paper is organized as follows. In Sect. 2, we study conformal derivations of the Lie conformal algebra \mathcal{W} . It turns out that all the conformal derivations of \mathcal{W} are inner. In Sect. 3, we discuss central extensions of \mathcal{W} and show that \mathcal{W} has a unique nontrivial universal central extension. In Sect. 4, we determine all free nontrivial conformal \mathcal{W} -modules of rank one. In Sect. 5, we compute cohomologies of \mathcal{W} with coefficients in \mathcal{W} -modules \mathbb{C}_a , \mathbb{C}_a and $M_{\Delta, \alpha}$, respectively. Consequently, we have the basic and reduced cohomologies for all $q \geq 0$ determined.

Throughout this paper, all vector spaces, linear maps and tensor products are over the complex field \mathbb{C} . We use notations \mathbb{Z} for the set of integers and \mathbb{Z}_+ for the set of nonnegative integers.

2 Conformal derivation

Let \mathcal{C} denote the ring $\mathbb{C}[\partial]$ of polynomials in the indeterminate ∂ .

Definition 2.1. Let V and W be two \mathcal{C} -modules. A \mathbb{C} -linear map $\phi : V \rightarrow \mathcal{C}[\lambda] \otimes_{\mathcal{C}} W$, denoted by $\phi_\lambda : V \rightarrow W$, is called a conformal linear map, if

$$\phi(\partial v) = (\partial + \lambda)(\phi v), \text{ for } v \in V.$$

Denote by $\text{Chom}(V, W)$ the space of conformal linear maps from \mathcal{C} -modules V to W . It can be made into an \mathcal{C} -module via

$$(\partial \phi)_\lambda v = -\lambda \phi_\lambda v, \text{ for } v \in V.$$

Definition 2.2. Let \mathcal{A} be a Lie conformal algebra. A conformal linear map $d_\lambda : \mathcal{A} \rightarrow \mathcal{A}$ is called a conformal derivation if

$$d_\lambda([a_\mu b]) = [(d_\lambda a)_{\lambda+\mu} b] + [a_\mu (d_\lambda b)], \text{ for all } a, b \in \mathcal{A}.$$

Denote by $\text{CDer}(\mathcal{A})$ the space of all conformal derivations of \mathcal{A} . For any $a \in \mathcal{A}$, one can define a linear map $(\text{ad } a)_\lambda : \mathcal{A} \rightarrow \mathcal{A}$ by $(\text{ad } a)_\lambda b = [a_\lambda b]$ for all $b \in \mathcal{A}$. It is easy to check that $(\text{ad } a)_\lambda$ is a conformal derivation of \mathcal{A} . Any conformal derivation of this kind is called an *inner derivation*. The space of all inner derivations is denoted by $\text{CIInn}(\mathcal{A})$.

For the Lie conformal algebra \mathcal{W} , we have the following result.

Proposition 2.3. Every conformal derivation of \mathcal{W} is inner, namely, $\text{CDer}(\mathcal{W}) = \text{CIInn}(\mathcal{W})$.

Proof. Let d_λ be a conformal derivation of \mathcal{W} . Then

$$d_\lambda L = f_1(\lambda, \partial)L + f_2(\lambda, \partial)M, \quad d_\lambda M = h_1(\lambda, \partial)L + h_2(\lambda, \partial)M, \quad (6)$$

where $f_i(\lambda, \partial)$ and $h_i(\lambda, \partial)$ for $i = 1, 2$ are polynomials in $\mathbb{C}[\lambda, \partial]$. Applying d_λ to $[L_\mu L] = (\partial + 2\mu)L$, we have

$$\begin{aligned} d_\lambda[L_\mu L] &= [(d_\lambda L)_{\lambda+\mu} L] + [L_\mu (d_\lambda L)] \\ &= (\partial + 2\lambda + 2\mu)f_1(\lambda, -\lambda - \mu)L + (\partial + 2\lambda + 2\mu)f_2(\lambda, -\lambda - \mu)M \\ &\quad + (\partial + 2\mu)f_1(\lambda, \partial + \mu)L + (\partial + 2\mu)f_2(\lambda, \partial + \mu)M \\ &= d_\lambda((\partial + 2\mu)L) \end{aligned}$$

$$= (\partial + \lambda + 2\mu)(f_1(\lambda, \partial)L + f_2(\lambda, \partial)M).$$

Comparing the coefficients of the similar terms gives

$$(\partial + \lambda + 2\mu)f_i(\lambda, \partial) - (\partial + 2\mu)f_i(\lambda, \partial + \mu) = (\partial + 2\lambda + 2\mu)f_i(\lambda, -\lambda - \mu), \text{ for } i = 1, 2. \quad (7)$$

Write $f_i(\lambda, \partial) = \sum_{j=0}^n a_{i,j}(\lambda)\partial^j$ with $a_{i,n}(\lambda) \neq 0$. Assume $n > 1$. Equating the coefficients of ∂^n in (7) gives $(\lambda - n\mu)a_{i,n}(\lambda) = 0$. Thus $a_{i,n}(\lambda) = 0$, a contradiction. Therefore,

$$f_i(\lambda, \partial) = a_{i,0}(\lambda) + a_{i,1}(\lambda)\partial, \text{ for } i = 1, 2. \quad (8)$$

By replacing d_λ by $d_\lambda - \text{ad}(a_{1,1}(-\partial)L)_\lambda - \text{ad}(a_{2,1}(-\partial)M)_\lambda$, we can suppose $a_{1,1}(\lambda) = a_{2,1}(\lambda) = 0$. Then plugging $f_i(\lambda, \partial) = a_{i,0}(\lambda)$ into (7) gives $a_{i,0}(\lambda) = 0$ for $i = 1, 2$. Thus $d_\lambda(L) = 0$ by (6). Furthermore, applying d_λ to $[L_\mu M] = (\partial + 2\mu)M$, we have

$$(\partial + \lambda + 2\mu)h_i(\lambda, \partial) = (\partial + 2\mu)h_i(\lambda, \partial + \mu), \text{ for } i = 1, 2. \quad (9)$$

Comparing the coefficients of highest degree of λ in (9) gives $h_i(\lambda, \partial) = 0$ for $i = 1, 2$. Hence $d_\lambda(M) = 0$ by (6). This concludes the proof. \square

Remark 2.4. Proposition 2.3 is equivalent to $H^1(\mathcal{W}, \mathcal{W}) = 0$.

3 Central extension

An *extension* of a Lie conformal algebra \mathcal{A} by an abelian Lie conformal algebra \mathfrak{a} is a short exact sequence of Lie conformal algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \hat{\mathcal{A}} \longrightarrow \mathcal{A} \longrightarrow 0.$$

In this case $\hat{\mathcal{A}}$ is also called an extension of \mathcal{A} by \mathfrak{a} . The extension is said to be *central* if

$$\mathfrak{a} \subseteq Z(\hat{\mathcal{A}}) = \{x \in \hat{\mathcal{A}} \mid [x_\lambda y]_{\hat{\mathcal{A}}} = 0 \text{ for all } y \in \hat{\mathcal{A}}\}, \text{ and } \partial\mathfrak{a} = 0.$$

Consider the central extension $\hat{\mathcal{A}}$ of \mathcal{A} by the trivial module \mathbb{C} . This means $\hat{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}\mathfrak{c}$, and

$$[a_\lambda b]_{\hat{\mathcal{A}}} = [a_\lambda b]_{\mathcal{A}} + f_\lambda(a, b)\mathfrak{c}, \text{ for } a, b \in \mathcal{A},$$

where $f_\lambda : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}[\lambda]$ is a bilinear map. The axioms (1)–(3) imply the following properties of the 2-cocycle $f_\lambda(a, b)$:

$$f_\lambda(a, b) = -f_{-\lambda-\partial}(b, a), \quad (10)$$

$$f_\lambda(\partial a, b) = -\lambda f_\lambda(a, b) = -f_\lambda(a, \partial b), \quad (11)$$

$$f_{\lambda+\mu}([a_\lambda b], c) = f_\lambda(a, [b_\mu c]) - f_\mu(b, [a_\lambda c]), \quad (12)$$

for all $a, b, c \in \mathcal{A}$. For any linear function $f : \mathcal{A} \rightarrow \mathbb{C}$, the map

$$\psi_f(a, b) = f([a_\lambda b]), \text{ for } a, b \in \mathcal{A}, \quad (13)$$

defines a trivial 2-cocycle. Let $a'_\lambda(a, b) = a_\lambda(a, b) + \psi_f(a, b)$. The equivalent 2-cocycles $a'_\lambda(a, b)$ and $a_\lambda(a, b)$ define isomorphic extensions.

In the following we determine the central extension $\widehat{\mathcal{W}}$ of \mathcal{W} by $\mathbb{C}\mathfrak{c}$, i.e., $\widehat{\mathcal{W}} = \mathcal{W} \oplus \mathbb{C}\mathfrak{c}$, and the relations in (4) are replaced by

$$[L_\lambda L] = (\partial + 2\lambda)L + a_\lambda(L, L)\mathfrak{c}, \quad [L_\lambda M] = (\partial + 2\lambda)M + a_\lambda(L, M)\mathfrak{c}, \quad [M_\lambda M] = a_\lambda(M, M)\mathfrak{c}, \quad (14)$$

and the others can be obtained by skew-symmetry. Applying the Jacobi identity for (L, L, L) , we have

$$(\lambda + 2\mu)a_\lambda(L, L) - (\mu + 2\lambda)a_\mu(L, L) = (\lambda - \mu)a_{\lambda+\mu}(L, L). \quad (15)$$

Write $a_\lambda(L, L) = \sum_{i=0}^n a_i \lambda^i \in \mathbb{C}[\lambda]$ with $a_n \neq 0$. Then, assuming $n > 1$ and equating the coefficients of λ^n in (15), we get $2\mu a_n = (n-1)\mu a_n$ and thus $n = 3$. Then

$$a_\lambda(L, L) = a_0 + a_1 \lambda + a_2 \lambda^2 + a_3 \lambda^3.$$

Plugging this in (15) and comparing the similar terms, we obtain $a_0 = a_2 = 0$. Thus

$$a_\lambda(L, L) = a_1 \lambda + a_3 \lambda^3. \quad (16)$$

To compute $a_\lambda(L, M)$, we apply the Jacobi identity for (L, L, M) and obtain

$$(\lambda + 2\mu)a_\lambda(L, M) - (\mu + 2\lambda)a_\mu(L, M) = (\lambda - \mu)a_{\lambda+\mu}(L, M).$$

By doing similar discussions as those in the process of computing $a_\lambda(L, L)$, we have

$$a_\lambda(L, M) = b_1 \lambda + b_3 \lambda^3, \text{ for some } b_1, b_3 \in \mathbb{C}. \quad (17)$$

Finally, applying the Jacobi identity for (L, M, M) yields $(\lambda - \mu)a_{\lambda+\mu}(M, M) = -(2\lambda + \mu)a_\mu(M, M)$, which implies

$$a_\lambda(M, M) = 0. \quad (18)$$

From the discussions above, we obtain the following results.

Theorem 3.1.

- (i) For any $a, b \in \mathbb{C}$ with $(a, b) \neq (0, 0)$, there exists a unique nontrivial universal central extension of the Lie conformal algebra \mathcal{W} by $\mathbb{C}\mathfrak{c}$, such that

$$[L_\lambda L] = (\partial + 2\lambda)L + a\lambda^3 \mathfrak{c}, \quad [L_\lambda M] = (\partial + 2\lambda)M + b\lambda^3 \mathfrak{c}, \quad [M_\lambda M] = 0. \quad (19)$$

- (ii) There exists a unique nontrivial universal central extension of \mathcal{W} by $\mathbb{C}\mathfrak{c} \oplus \mathbb{C}\mathfrak{c}'$, satisfying

$$[L_\lambda L] = (\partial + 2\lambda)L + \lambda^3 \mathfrak{c}, \quad [L_\lambda M] = (\partial + 2\lambda)M + \lambda^3 \mathfrak{c}', \quad [M_\lambda M] = 0. \quad (20)$$

Proof. (i) By (16)–(18), replacing L, M respectively by $L - \frac{1}{2}a_1 \mathfrak{c}$, $M - \frac{1}{2}b_1 \mathfrak{c}$ and noticing that $\partial \mathfrak{c} = 0$, we can suppose $a_1 = b_1 = 0$. This shows (19). The universality of the extension follows from [12] and the fact that \mathcal{W} is perfect, namely, $[\mathcal{W}_\lambda \mathcal{W}] = \mathcal{W}[\lambda]$.

- (ii) This follows from the proof of (i). □

Remark 3.2. Theorem 3.1 (ii) implies that $\dim H^2(\mathcal{W}, \mathbb{C}) = 2$.

4 Conformal module

Let us first recall the notion of a conformal module given in [5].

Definition 4.1. A (conformal) module V over a Lie conformal algebra \mathcal{A} is a $\mathbb{C}[\partial]$ -module endowed with a bilinear map $\mathcal{A} \otimes V \rightarrow V[\lambda]$, $a \otimes v \mapsto a_\lambda v$ satisfying the following axioms for $a, b \in \mathcal{A}$, $v \in V$:

$$\begin{aligned} a_\lambda(b_\mu v) - b_\mu(a_\lambda v) &= [a_\lambda b]_{\lambda+\mu} v, \\ (\partial a)_\lambda v &= -\lambda a_\lambda v, \quad a_\lambda(\partial v) = (\partial + \lambda)a_\lambda v. \end{aligned}$$

An \mathcal{A} -module V is called finite if it is finitely generated over $\mathbb{C}[\partial]$.

The vector space \mathbb{C} can be seen as a module (called the *trivial module*) over any conformal algebra \mathcal{A} with both the action of ∂ and the action of \mathcal{A} being zero. For a fixed nonzero complex constant a , there is a natural $\mathbb{C}[\partial]$ -module \mathbb{C}_a , which is the one-dimensional vector space \mathbb{C} such that $\partial v = av$ for $v \in \mathbb{C}_a$. Then \mathbb{C}_a becomes an \mathcal{A} -module, where \mathcal{A} acts as zero.

For the Virasoro conformal algebra Vir , it is known from [3] that all the free nontrivial Vir -modules of rank one over $\mathbb{C}[\partial]$ are the following ones ($\Delta, \alpha \in \mathbb{C}$):

$$M_{\Delta, \alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v. \quad (21)$$

The module $M_{\Delta, \alpha}$ is irreducible if and only if $\Delta \neq 0$. The module $M_{0, \alpha}$ contains a unique nontrivial submodule $(\partial + \alpha)M_{0, \alpha}$ isomorphic to $M_{1, \alpha}$. Moreover, the modules $M_{\Delta, \alpha}$ with $\Delta \neq 0$ exhaust all finite irreducible nontrivial Vir -modules.

The following result describes the free nontrivial \mathcal{W} -modules of rank one. Similar result for the more general $W(a, b)$ Lie conformal algebra was given in [11]. We aim to consider it in details in the $W(2, 2)$ case.

Proposition 4.2. *All free nontrivial \mathcal{W} -modules of rank one over $\mathbb{C}[\partial]$ are the following ones:*

$$M_{\Delta, \alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta\lambda)v, \quad M_\lambda v = 0, \quad \text{for some } \Delta, \alpha \in \mathbb{C}.$$

Proof. Suppose that $L_\lambda v = f(\partial, \lambda)v$, $M_\lambda v = g(\partial, \lambda)v$, where $f(\partial, \lambda), g(\partial, \lambda) \in \mathbb{C}[\lambda, \partial]$. By the result of Vir -modules, we have

$$f(\partial, \lambda) = \partial + \alpha + \Delta\lambda, \quad \text{for some } \alpha, \Delta \in \mathbb{C}.$$

On the other hand, it follows from $M_\lambda(M_\mu v) = M_\mu(M_\lambda v)$ that

$$g(\partial, \lambda)g(\partial + \lambda, \mu) = g(\partial, \mu)g(\partial + \mu, \lambda).$$

This implies $\deg_\lambda g(\partial, \lambda) + \deg_\partial g(\partial, \lambda) = \deg_\lambda g(\partial, \lambda)$, where the notation $\deg_\lambda g(\partial, \lambda)$ stands for the highest degree of λ in $g(\partial, \lambda)$. Thus $\deg_\partial g(\partial, \lambda) = 0$ and so $g(\partial, \lambda) = g(\lambda)$ for some $g(\lambda) \in \mathbb{C}[\lambda]$. Finally, $[L_\lambda M]_{\lambda+\mu} v = (\lambda - \mu)M_{\lambda+\mu} v$ gives $(\lambda - \mu)g(\lambda + \mu) = -\mu g(\mu)$, which yields $g(\lambda) = 0$. This proves the result. \square

5 Cohomology

For completeness, we recall the following definition from [6]:

Definition 5.1. *An n -cochain ($n \in \mathbb{Z}_+$) of a Lie conformal algebra \mathcal{A} with coefficients in an \mathcal{A} -module V is a \mathbb{C} -linear map*

$$\gamma : \mathcal{A}^{\otimes n} \rightarrow V[\lambda_1, \dots, \lambda_n], \quad a_1 \otimes \dots \otimes a_n \mapsto \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n)$$

satisfying the following conditions:

- (i) $\gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, \partial a_i, \dots, a_n) = -\lambda_i \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n)$ (conformal antilinearity),
- (ii) γ is skew-symmetric with respect to simultaneous permutations of a_i 's and λ_i 's (skew-symmetry).

As usual, let $\mathcal{A}^{\otimes 0} = \mathbb{C}$, so that a 0-cochain is an element of V . Denote by $\tilde{C}^n(\mathcal{A}, V)$ the set of all n -cochains. The differential d of an n -cochain γ is defined as follows:

$$\begin{aligned} (d\gamma)_{\lambda_1, \dots, \lambda_{n+1}}(a_1, \dots, a_{n+1}) \\ = \sum_{i=1}^{n+1} (-1)^{i+1} a_{i\lambda_i} \gamma_{\lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_{n+1}}(a_1, \dots, \hat{a}_i, \dots, a_{n+1}) \\ + \sum_{\substack{i, j=1 \\ i < j}}^{n+1} (-1)^{i+j} \gamma_{\lambda_i + \lambda_j, \lambda_1, \dots, \hat{\lambda}_i, \dots, \hat{\lambda}_j, \dots, \lambda_{n+1}}([a_{i\lambda_i}, a_{j\lambda_j}], a_1, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_{n+1}), \end{aligned} \quad (22)$$

where γ is linearly extended over the polynomials in λ_i . In particular, if $\gamma \in V$ is a 0-cochain, then $(d\gamma)_\lambda(a) = a_\lambda \gamma$.

It is known from [6] that the operator d preserves the space of cochains and $d^2 = 0$. Thus the cochains of a Lie conformal algebra \mathcal{A} with coefficients in its module V form a complex, which is denoted by

$$\tilde{C}^\bullet(\mathcal{A}, V) = \bigoplus_{n \in \mathbb{Z}_+} \tilde{C}^n(\mathcal{A}, V), \quad (23)$$

and called the *basic complex*. Moreover, define a (left) $\mathbb{C}[\partial]$ -module structure on $\tilde{C}^\bullet(\mathcal{A}, V)$ by

$$(\partial\gamma)_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n) = (\partial_V + \sum_{i=1}^n \lambda_i) \gamma_{\lambda_1, \dots, \lambda_n}(a_1, \dots, a_n),$$

where ∂_V denotes the action of ∂ on V . Then $d\partial = \partial d$ and thus $\partial\tilde{C}^\bullet(\mathcal{A}, V) \subset \tilde{C}^\bullet(\mathcal{A}, V)$ forms a subcomplex. The quotient complex

$$C^\bullet(\mathcal{A}, V) = \tilde{C}^\bullet(\mathcal{A}, V) / \partial\tilde{C}^\bullet(\mathcal{A}, V) = \bigoplus_{n \in \mathbb{Z}_+} C^n(\mathcal{A}, V)$$

is called the *reduced complex*.

Definition 5.2. The basic cohomology $\tilde{H}^\bullet(\mathcal{A}, V)$ of a Lie conformal algebra \mathcal{A} with coefficients in an \mathcal{A} -module V is the cohomology of the basic complex $\tilde{C}^\bullet(\mathcal{A}, V)$ and the (reduced) cohomology $H^\bullet(\mathcal{A}, V)$ is the cohomology of the reduced complex $C^\bullet(\mathcal{A}, V)$.

For a q -cochain $\gamma \in \tilde{C}^q(\mathcal{A}, V)$, we call γ a q -cocycle if $d(\gamma) = 0$; a q -coboundary if there exists a $(q-1)$ -cochain $\phi \in \tilde{C}^{q-1}(\mathcal{A}, V)$ such that $\gamma = d(\phi)$. Two cochains γ_1 and γ_2 are called *equivalent* if $\gamma_1 - \gamma_2$ is a coboundary. Denote by $\tilde{D}^q(\mathcal{A}, V)$ and $\tilde{B}^q(\mathcal{A}, V)$ the spaces of q -cocycles and q -boundaries, respectively. By Definition 5.2,

$$\tilde{H}^q(\mathcal{A}, V) = \tilde{D}^q(\mathcal{A}, V) / \tilde{B}^q(\mathcal{A}, V) = \{\text{equivalent classes of } q\text{-cocycles}\}.$$

The main results of this section are the following.

Theorem 5.3. For the Lie conformal algebra \mathcal{W} , the following statements hold.

(i) For the trivial module \mathbb{C} ,

$$\dim \tilde{H}^q(\mathcal{W}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 4, 5, 6, \\ 2 & \text{if } q = 3, \\ 0 & \text{otherwise,} \end{cases} \quad (24)$$

and

$$\dim H^q(\mathcal{W}, \mathbb{C}) = \begin{cases} 1 & \text{if } q = 0, 6, \\ 2 & \text{if } q = 2, 4, 5 \\ 3 & \text{if } q = 3, \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

(ii) If $a \neq 0$, then $\dim H^q(\mathcal{W}, \mathbb{C}_a) = 0$, for $q \geq 0$.

(iii) If $\alpha \neq 0$, then $\dim H^q(\mathcal{W}, M_{\Delta, \alpha}) = 0$, for $q \geq 0$.

Proof. (i) For any $\gamma \in \tilde{C}^0(\mathcal{W}, \mathbb{C}) = \mathbb{C}$, we have $(d\gamma)_\lambda(X) = X_\lambda \gamma = 0$ for $X \in \mathcal{W}$. This means $\tilde{D}^0(\mathcal{W}, \mathbb{C}) = \mathbb{C}$ and $\tilde{B}^0(\mathcal{W}, \mathbb{C}) = 0$. Thus $\tilde{H}^0(\mathcal{W}, \mathbb{C}) = \mathbb{C}$ and $H^0(\mathcal{W}, \mathbb{C}) = \mathbb{C}$ since $\partial\mathbb{C} = 0$.

Let $\gamma \in \tilde{C}^1(\mathcal{W}, \mathbb{C})$ be such that $d\gamma \in \partial\tilde{C}^2(\mathcal{W}, \mathbb{C})$, namely, there is $\phi \in \tilde{C}^2(\mathcal{W}, \mathbb{C})$ such that

$$\gamma_{\lambda_1 + \lambda_2}([X_{\lambda_1} Y]) = -(d\gamma)_{\lambda_1, \lambda_2}(X, Y) = -(\partial\phi)_{\lambda_1, \lambda_2}(X, Y) = -(\lambda_1 + \lambda_2)\phi_{\lambda_1, \lambda_2}(X, Y), \quad (26)$$

for $X, Y \in \{L, M\}$. By (26) and (4),

$$(\lambda_1 - \lambda_2)\gamma_{\lambda_1 + \lambda_2}(X) = -(\lambda_1 + \lambda_2)\phi_{\lambda_1, \lambda_2}(L, X), \quad X \in \{L, M\}. \quad (27)$$

Letting $\lambda = \lambda_1 + \lambda_2$ in (27) gives

$$(\lambda - 2\lambda_2)\gamma_\lambda(X) = -\lambda\phi_{\lambda_1, \lambda_2}(L, X), \quad X \in \{L, M\}, \quad (28)$$

which implies that $\gamma_\lambda(X)$ is divisible by λ . Define

$$\gamma'_\lambda(X) = \lambda^{-1}\gamma_\lambda(X), \quad X \in \{L, M\}.$$

Clearly, $\gamma' \in \tilde{C}^1(\mathcal{W}, \mathbb{C})$ and $\gamma = \partial\gamma' \in \partial\tilde{C}^1(\mathcal{W}, \mathbb{C})$. Thus $H^1(\mathcal{W}, \mathbb{C}) = 0$. If γ is a 1-cocycle (this means $\phi = 0$), then (28) gives $\gamma = 0$. Hence, $\tilde{H}^1(\mathcal{W}, \mathbb{C}) = 0$.

Let ψ be a 2-cocycle. For $X \in \mathcal{W}$, we have

$$\begin{aligned} 0 &= (d\psi)_{\lambda_1, \lambda_2, \lambda_3}(X, L, L) \\ &= -(\lambda_1 - \lambda_2)\psi_{\lambda_1 + \lambda_2, \lambda_3}(X, L) + (\lambda_1 - \lambda_3)\psi_{\lambda_1 + \lambda_3, \lambda_2}(X, L) - (\lambda_2 - \lambda_3)\psi_{\lambda_2 + \lambda_3, \lambda_1}(L, X). \end{aligned}$$

Letting $\lambda_3 = 0$ and $\lambda_1 + \lambda_2 = \lambda$ gives $(\lambda - 2\lambda_2)\psi_{\lambda, 0}(X, L) = \lambda\psi_{\lambda_1, \lambda_2}(X, L)$. Hence, $\psi_{\lambda, 0}(X, L)$ is divisible by λ . Define a 1-cochain f by

$$f_{\lambda_1}(L) = \lambda_1^{-1}\psi_{\lambda_1, \lambda}(L, L)|_{\lambda=0}, \quad f_{\lambda_1}(M) = \lambda_1^{-1}\psi_{\lambda_1, \lambda}(M, L)|_{\lambda=0}. \quad (29)$$

Set $\gamma = \psi + df$, which is also a 2-cocycle. By (29),

$$\gamma_{\lambda_1, \lambda}(L, L)|_{\lambda=0} = \psi_{\lambda_1, \lambda}(L, L)|_{\lambda=0} - \lambda_1 f_{\lambda_1}(L) = 0, \quad (30)$$

$$\gamma_{\lambda_1, \lambda}(M, L)|_{\lambda=0} = \psi_{\lambda_1, \lambda}(M, L)|_{\lambda=0} - \lambda_1 f_{\lambda_1}(M) = 0. \quad (31)$$

By (30), we have

$$\begin{aligned} 0 &= (d\gamma)_{\lambda_1, \lambda_2, \lambda}(L, L, L)|_{\lambda=0} \\ &= -\gamma_{\lambda_1 + \lambda_2, \lambda}([L_{\lambda_1} L], L)|_{\lambda=0} + \gamma_{\lambda_1 + \lambda, \lambda_2}([L_{\lambda_1} L], L)|_{\lambda=0} - \gamma_{\lambda_2 + \lambda, \lambda_1}([L_{\lambda_2} L], L)|_{\lambda=0} \\ &= \lambda_1 \gamma_{\lambda_1, \lambda_2}(L, L) - \lambda_2 \gamma_{\lambda_2, \lambda_1}(L, L) \\ &= (\lambda_1 + \lambda_2) \gamma_{\lambda_1, \lambda_2}(L, L). \end{aligned}$$

Thus $\gamma_{\lambda_1, \lambda_2}(L, L) = 0$. Similarly, by (31),

$$0 = (d\gamma)_{\lambda_1, \lambda_2, \lambda}(L, M, L)|_{\lambda=0} = (\lambda_1 + \lambda_2) \gamma_{\lambda_1, \lambda_2}(L, M),$$

which gives $\gamma_{\lambda_1, \lambda_2}(L, M) = 0$ and so $\gamma_{\lambda_1, \lambda_2}(M, L) = 0$. Finally,

$$0 = (d\gamma)_{\lambda_1, \lambda_2, \lambda}(L, M, M)|_{\lambda=0} = -(\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2, 0}(M, M) + \lambda_1 \gamma_{\lambda_1, \lambda_2}(M, M). \quad (32)$$

Setting $\lambda_1 = 0$ in (32) gives $\gamma_{\lambda_2, 0}(M, M) = 0$ and thus $\gamma_{\lambda_1, \lambda_2}(M, M) = 0$. This shows $\gamma = 0$. Hence $\tilde{H}^2(\mathcal{W}, \mathbb{C}) = 0$. According to Theorem 3.1 (ii), $\dim H^2(\mathcal{W}, \mathbb{C}) = 2$.

To determine higher dimensional cohomologies (for $q \geq 3$), we define an operator $\tau : \tilde{C}^q(\mathcal{W}, \mathbb{C}) \rightarrow \tilde{C}^{q-1}(\mathcal{W}, \mathbb{C})$ by

$$(\tau\gamma)_{\lambda_1, \dots, \lambda_{q-1}}(X_1, \dots, X_{q-1}) = (-1)^{q-1} \frac{\partial}{\partial \lambda} \gamma_{\lambda_1, \dots, \lambda_{q-1}, \lambda}(X_1, \dots, X_{q-1}, L)|_{\lambda=0}, \quad (33)$$

for $X_1, \dots, X_{q-1} \in \{L, M\}$. By (22), (33) and skew-symmetry of γ ,

$$\begin{aligned} &((d\tau + \tau d)\gamma)_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \\ &= (-1)^q \frac{\partial}{\partial \lambda} \sum_{i=1}^q (-1)^{i+q+1} \gamma_{\lambda_i + \lambda, \lambda_1, \dots, \hat{\lambda}_i, \dots, \lambda_q}([X_i \lambda_i L], X_1, \dots, \hat{X}_i, \dots, X_q)|_{\lambda=0} \\ &= \frac{\partial}{\partial \lambda} \sum_{i=1}^q \gamma_{\lambda_1, \dots, \lambda_{i-1}, \lambda_i + \lambda, \lambda_{i+1}, \dots, \lambda_q}(X_1, \dots, X_{i-1}, [X_i \lambda_i L], X_{i+1}, \dots, X_q)|_{\lambda=0}. \end{aligned} \quad (34)$$

By the fact that $[X_i \lambda_i L] = (\partial + 2\lambda_i)X_i$ and conformal antilinearity of γ , $[X_i \lambda_i L]$ can be replaced by $(\lambda_i - \lambda)X_i$ in (34). Thus, equality (34) can be rewritten as

$$\begin{aligned} & ((d\tau + \tau d)\gamma)_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \\ &= \frac{\partial}{\partial \lambda} \sum_{i=1}^q (\lambda_i - \lambda) \gamma_{\lambda_1, \dots, \lambda_{i-1}, \lambda_i + \lambda, \lambda_{i+1}, \dots, \lambda_q}(X_1, \dots, X_{i-1}, X_i, X_{i+1}, \dots, X_q)|_{\lambda=0} \\ &= (\deg \gamma - q) \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q), \end{aligned} \quad (35)$$

where $\deg \gamma$ is the total degree of γ in $\lambda_1, \dots, \lambda_q$. As it was explained in [6], only those homogeneous cochains, whose degree as a polynomial is equal to their degree as a cochain, contribute to the cohomology of $\tilde{C}^\bullet(\mathcal{W}, \mathbb{C})$. Without loss of generality, we always assume that the first k variables are L and the last $q - k$ variables are M in $\gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q)$, so that $\gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q)$ as a polynomial in $\lambda_1, \dots, \lambda_q$ is skew-symmetric in $\lambda_1, \dots, \lambda_k$ and also skew-symmetric in $\lambda_{k+1}, \dots, \lambda_q$. Therefore, it is divisible by

$$\prod_{1 \leq i < j \leq k} (\lambda_i - \lambda_j) \times \prod_{k+1 \leq i < j \leq q} (\lambda_i - \lambda_j),$$

whose polynomial degree is $k(k-1)/2 + (q-k)(q-k-1)/2$. Consider the quadratic inequality $k(k-1)/2 + (q-k)(q-k-1)/2 \leq q$, whose discriminant is $-4k^2 + 12k + 9$. Since $-4k^2 + 12k + 9 \geq 0$ has $k = 0, 1, 2$ and 3 as the only integral solutions, we have

$$q = \begin{cases} 0, 1, 2, 3, & \text{if } k = 0, \\ 1, 2, 3, 4, & \text{if } k = 1, \\ 2, 3, 4, 5, & \text{if } k = 2, \\ 3, 4, 5, 6, & \text{if } k = 3. \end{cases} \quad (36)$$

Thus $\tilde{H}^q(\mathcal{W}, \mathbb{C}) = 0$ for $q \geq 7$. It remains to compute $\tilde{H}^q(\mathcal{W}, \mathbb{C})$ for $q = 3, 4, 5, 6$.

For $q = 3$, we need to consider four cases for k , i.e., $k = 0, 1, 2, 3$. Let $\gamma \in \tilde{D}^3(\mathcal{W}, \mathbb{C})$ be a 3-cocycle. A direct computation shows that

$$0 = (d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda}(M, M, M, L)|_{\lambda=0} = -(\lambda_1 + \lambda_2 + \lambda_3) \gamma_{\lambda_1, \lambda_2, \lambda_3}(M, M, M).$$

This gives $\gamma_{\lambda_1, \lambda_2, \lambda_3}(M, M, M) = 0$. In the case of $k = 1$, we have

$$\begin{aligned} 0 &= (d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda}(L, M, M, L)|_{\lambda=0} \\ &= (\lambda_1 - \lambda_2) \gamma_{0, \lambda_3, \lambda_1 + \lambda_2}(L, M, M) - (\lambda_1 - \lambda_3) \gamma_{0, \lambda_2, \lambda_1 + \lambda_3}(L, M, M) \\ &\quad - (\lambda_1 + \lambda_2 + \lambda_3) \gamma_{\lambda_1, \lambda_2, \lambda_3}(L, M, M). \end{aligned} \quad (37)$$

Note that $\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, M, M)$ is a homogeneous polynomial of degree 3 and skew-symmetric in λ_2 and λ_3 . Thus it is divisible by $\lambda_2 - \lambda_3$. We can suppose that

$$\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, M, M) = (\lambda_2 - \lambda_3)(a_1 \lambda_1^2 + a_2(\lambda_2^2 + \lambda_3^2) + a_3 \lambda_2 \lambda_3 + a_4 \lambda_1(\lambda_2 + \lambda_3)), \quad (38)$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Plugging (38) into (37) gives $a_4 = 0, a_3 = 2a_2, a_1 = -a_2$. Therefore,

$$\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, M, M) = a_2 \phi_1, \text{ where } \phi_1 = (\lambda_2 - \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3)(-\lambda_1 + \lambda_2 + \lambda_3). \quad (39)$$

Note that ϕ_1 is a coboundary of $\tilde{\gamma}_{\lambda_1, \lambda_2}(M, M) = \lambda_2^2 - \lambda_1^2$. In fact,

$$\begin{aligned} (d\tilde{\gamma})_{\lambda_1, \lambda_2, \lambda_3}(L, M, M) &= -(\lambda_1 - \lambda_2) \gamma_{\lambda_1 + \lambda_2, \lambda_3}(M, M) + (\lambda_1 + \lambda_3) \gamma_{\lambda_1 + \lambda_3, \lambda_2}(M, M) \\ &= -(\lambda_1 - \lambda_2)(\lambda_3^2 - (\lambda_1 + \lambda_2)^2) + (\lambda_1 - \lambda_3)(\lambda_2^2 - (\lambda_1 + \lambda_3)^2) \\ &= \phi_1. \end{aligned}$$

Similarly, suppose that

$$\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)(b_1(\lambda_1^2 + \lambda_2^2) + b_2 \lambda_3^2 + b_3(\lambda_1 + \lambda_2)\lambda_3 + b_4 \lambda_1 \lambda_2), \quad (40)$$

where $b_1, b_2, b_3, b_4 \in \mathbb{C}$. Substituting (40) into the following equality

$$\begin{aligned} 0 &= (d\gamma)_{\lambda_1, \lambda_2, \lambda_3, \lambda}(L, L, M, L)|_{\lambda=0} \\ &= (\lambda_1 - \lambda_2)\gamma_{\lambda_1 + \lambda_2, 0, \lambda_3}(L, L, M) + (\lambda_1 - \lambda_3)\gamma_{\lambda_2, 0, \lambda_1 + \lambda_3}(L, L, M) \\ &\quad - (\lambda_2 - \lambda_3)\gamma_{\lambda_1, 0, \lambda_2 + \lambda_3}(L, L, M) - (\lambda_1 + \lambda_2 + \lambda_3)\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) \end{aligned}$$

gives $b_4 = b_1 + b_2$. Hence,

$$\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)(b_1(\lambda_1^2 + \lambda_2^2) + b_2\lambda_2^2 + (b_1 + b_2)\lambda_1\lambda_2 + b_3(\lambda_1 + \lambda_2)\lambda_3). \quad (41)$$

On the other hand, there is a 2-cochain $\bar{\gamma}_{\lambda_1, \lambda_2}(L, M) = b_1\lambda_1^2 + b_2\lambda_1\lambda_2$ such that

$$(d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) + \gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = -(b_1 + b_2 - b_3)(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)\lambda_3. \quad (42)$$

Thus $\gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, M)$ in (41) is equivalent to a constant factor of $\chi := \chi_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)\lambda_3$, which is not a coboundary. By [6, Theorem 7.1], $\Lambda_3 := \gamma_{\lambda_1, \lambda_2, \lambda_3}(L, L, L) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)$ (up to a constant factor) is a 3-cocycle, but not a coboundary. Therefore, $\dim \tilde{H}^3(\mathcal{W}, \mathbb{C}) = 2$. Specifically, $\tilde{H}^3(\mathcal{W}, \mathbb{C}) = \mathbb{C}\chi \oplus \mathbb{C}\Lambda_3$.

For $q = 4$, three cases (i.e., $k = 1, 2, 3$) should be taken into account. Let $\gamma \in \tilde{D}^4(\mathcal{W}, \mathbb{C})$ be a 4-cocycle. By using the method of undetermined coefficients and doing similar calculations to the case when $q = 3$, we obtain

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, M, M, M) = c(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4), \quad (43)$$

$$\begin{aligned} \gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) &= (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(c_1(\lambda_1^2 + \lambda_2^2) + c_2(\lambda_3 + \lambda_4)^2 \\ &\quad + (c_1 + c_2)\lambda_1\lambda_2 + c_3(\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4)), \end{aligned} \quad (44)$$

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)(e_1(\lambda_1 + \lambda_2 + \lambda_3) + e_2\lambda_4), \quad (45)$$

where $c, c_1, c_2, c_3, e_1, e_2 \in \mathbb{C}$. And there exist three 3-cochains of degree 3

$$\bar{\gamma}_{\lambda_1, \lambda_2, \lambda_3}(M, M, M) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3), \quad (46)$$

$$\bar{\gamma}_{\lambda_1, \lambda_2, \lambda_3}(L, M, M) = (\lambda_2 - \lambda_3)(c_1\lambda_1^2 + c_2\lambda_1(\lambda_2 + \lambda_3)), \quad (47)$$

$$\bar{\gamma}_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) = (\lambda_1 - \lambda_2)(\lambda_1^2 + \lambda_2^2), \quad (48)$$

such that

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, M, M, M) - c(d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, M, M, M) = 0, \quad (49)$$

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) + (d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) = (c_3 - c_1 - c_2)\psi_1, \quad (50)$$

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) + e_2(d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) = (e_1 - e_2)\psi_2, \quad (51)$$

where

$$\psi_1 := \psi_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 + \lambda_4), \quad (52)$$

$$\psi_2 := \psi_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, L, M) = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3). \quad (53)$$

Moreover, $4\psi_1 = -(d\bar{\psi})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M)$ with $\bar{\psi}_{\lambda_1, \lambda_2, \lambda_3}(L, M, M) = (\lambda_2 - \lambda_3)(3\lambda_1^2 - (\lambda_2^2 + \lambda_3^2))$. This, together with (49)–(53), gives $\tilde{H}^4(\mathcal{W}, \mathbb{C}) = \mathbb{C}\psi_2$.

For $q = 5$, we need to consider $k = 2, 3$. Let $\gamma \in \tilde{D}^5(\mathcal{W}, \mathbb{C})$ be a 5-cocycle. We obtain

$$\begin{aligned} \gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M) &= (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_5) \\ &\quad \times (\bar{a}_1(\lambda_1 + \lambda_2) + \bar{a}_2(\lambda_3 + \lambda_4 + \lambda_5)), \end{aligned} \quad (54)$$

$$\begin{aligned} \gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, M, M) &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5) \\ &\quad \times (\bar{b}_1(\lambda_1 + \lambda_2 + \lambda_3) + \bar{b}_2(\lambda_4 + \lambda_5)), \end{aligned} \quad (55)$$

where $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2 \in \mathbb{C}$. On the other hand, there exist two 4-cochains of degree 4

$$\bar{\gamma}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, M, M, M) = \lambda_1(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)(\lambda_2 - \lambda_4),$$

$$\bar{\gamma}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1^2 + \lambda_2^2),$$

such that

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M) + a_1(d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M) = (a_2 - a_1)\varphi_1, \quad (56)$$

$$\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, M, M) + b_1(d\bar{\gamma})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, M, M) = (b_2 - b_1)\varphi_2, \quad (57)$$

where

$$\begin{aligned} \varphi_1 &:= \varphi_{1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M) \\ &= (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_5)(\lambda_3 + \lambda_4 + \lambda_5), \end{aligned} \quad (58)$$

$$\begin{aligned} \varphi_2 &:= \varphi_{2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, M, M) \\ &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)(\lambda_4 - \lambda_5)(\lambda_4 + \lambda_5). \end{aligned} \quad (59)$$

Furthermore, there exists another one 4-cochains of degree 4

$$\bar{\varphi}_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, L, M, M) = (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_1\lambda_2 - \lambda_3\lambda_4),$$

such that $2\varphi_2 = (d\bar{\varphi})_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, L, M, M)$, namely, φ_2 is a coboundary. By (56) and (57), $\dim \tilde{H}^5(\mathcal{W}, \mathbb{C}) = 1$, and $\tilde{H}^5(\mathcal{W}, \mathbb{C}) = \mathbb{C}\varphi_1$.

For $q = 6$, one only needs to consider the case when $k = 3$. One can check that

$$\begin{aligned} \Lambda &:= \gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6}(L, L, L, M, M, M) \\ &= (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_4 - \lambda_5)(\lambda_4 - \lambda_6)(\lambda_5 - \lambda_6) \end{aligned} \quad (60)$$

is a 6-cocycle. It is not a coboundary. Because it can be the coboundary of a 5-cochain of degree 5, which must be a constant factor of $\gamma_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M)$ in (54), whose coboundary is zero. Therefore, $\dim \tilde{H}^6(\mathcal{W}, \mathbb{C}) = 1$ and $\tilde{H}^6(\mathcal{W}, \mathbb{C}) = \mathbb{C}\Lambda$. This proves (24).

According to [6, Proposition 2.1], the map $\gamma \mapsto \partial\gamma$ gives an isomorphism $\tilde{H}^q(\mathcal{W}, \mathbb{C}) \cong H^q(\partial\tilde{C}^\bullet)$ for $q \geq 1$. Therefore

$$H^q(\partial\tilde{C}^\bullet) = \begin{cases} \mathbb{C}(\partial\chi) \oplus \mathbb{C}(\partial\Lambda_3) & \text{if } q = 3, \\ \mathbb{C}(\partial\psi_2) & \text{if } q = 4, \\ \mathbb{C}(\partial\varphi_1) & \text{if } q = 5, \\ \mathbb{C}(\partial\Lambda) & \text{if } q = 6, \\ 0 & \text{otherwise.} \end{cases} \quad (61)$$

It remains to compute $H^\bullet(\mathcal{W}, \mathbb{C})$. This is based on the short exact sequence of complexes

$$0 \longrightarrow \partial\tilde{C}^\bullet \xrightarrow{\iota} \tilde{C}^\bullet \xrightarrow{\pi} C^\bullet \longrightarrow 0 \quad (62)$$

where ι and π are the embedding and the natural projection, respectively. The exact sequence (62) gives the following long exact sequence of cohomology groups (cf. [6]):

$$\begin{aligned} \cdots &\longrightarrow H^q(\partial\tilde{C}^\bullet) \xrightarrow{\iota_q} \tilde{H}^q(\mathcal{W}, \mathbb{C}) \xrightarrow{\pi_q} H^q(\mathcal{W}, \mathbb{C}) \xrightarrow{\omega_q} \\ &\longrightarrow H^{q+1}(\partial\tilde{C}^\bullet) \xrightarrow{\iota_{q+1}} \tilde{H}^{q+1}(\mathcal{W}, \mathbb{C}) \xrightarrow{\pi_{q+1}} H^{q+1}(\mathcal{W}, \mathbb{C}) \longrightarrow \cdots \end{aligned} \quad (63)$$

where ι_q, π_q are induced by ι, π respectively and ω_q is the q -th connecting homomorphism. Given $\partial\gamma \in H^q(\partial\tilde{C}^\bullet)$ with a nonzero element $\gamma \in \tilde{H}^q(\mathcal{W}, \mathbb{C})$, then $\iota_q(\partial\gamma) = \partial\gamma \in \tilde{H}^q(\mathcal{W}, \mathbb{C})$. Since $\deg(\partial\gamma) = \deg(\gamma) + 1 = q + 1$, we have $\partial\gamma = 0 \in \tilde{H}^q(\mathcal{W}, \mathbb{C})$. Then the image of ι_q is zero for any $q \in \mathbb{Z}_+$. Because $\ker(\pi_q) = \text{im}(\iota_q) = \{0\}$ and $\text{im}(\omega_q) = \ker(\iota_{q+1}) = H^{q+1}(\partial\tilde{C}^\bullet)$, we obtain the following short exact sequence

$$0 \longrightarrow \tilde{H}^q(\mathcal{W}, \mathbb{C}) \xrightarrow{\pi_q} H^q(\mathcal{W}, \mathbb{C}) \xrightarrow{\omega_q} H^{q+1}(\partial\tilde{C}^\bullet) \longrightarrow 0. \quad (64)$$

Therefore,

$$\dim H^q(\mathcal{W}, \mathbb{C}) = \dim \tilde{H}^q(\mathcal{W}, \mathbb{C}) + \dim H^{q+1}(\partial \tilde{C}^\bullet), \text{ for all } q \geq 0. \quad (65)$$

Then (25) follows from (65). Moreover, we can give a basis for $H^q(\mathcal{W}, \mathbb{C})$. Indeed, any basis of $H^q(\mathcal{W}, \mathbb{C})$ can be obtained by combining the images of a basis of $\tilde{H}^q(\mathcal{W}, \mathbb{C})$ with the pre-images of a basis of $\tilde{H}^{q+1}(\mathcal{W}, \mathbb{C})$. For a nonzero $\partial\varphi \in H^{q+1}(\partial \tilde{C}^\bullet)$ with φ being a $(q+1)$ -cocycle, (35) gives

$$d(\tau(\partial\varphi)) = (d\tau + \tau d)(\partial\varphi) = (\deg(\partial\varphi) - (q+1))(\partial\varphi) = ((q+2) - (q+1))(\partial\varphi) = \partial\varphi. \quad (66)$$

Thus the pre-image of $\partial\varphi$ under the connecting homomorphism ω_p is $\omega_q^{-1}(\partial\varphi) = \tau(\partial\varphi)$.

Finally, we finish our proof by giving a basis of $H^q(\mathcal{W}, \mathbb{C})$ for $q = 2, \dots, 6$. For $q = 2$, we have known that $\tilde{H}^2(\mathcal{W}, \mathbb{C}) = 0$ and $H^3(\partial \tilde{C}^\bullet) = \mathbb{C}(\partial\chi) \oplus \mathbb{C}(\partial\Lambda_3)$. By (33) and (42),

$$\begin{aligned} \bar{\chi} &:= (\tau(\partial\chi))_{\lambda_1, \lambda_2}(L, M) \\ &= (-1)^2 \frac{\partial}{\partial\lambda} (\partial\chi)_{\lambda_1, \lambda_2, \lambda}(L, M, L)|_{\lambda=0} \\ &= -\frac{\partial}{\partial\lambda} (\lambda_1 + \lambda_2 + \lambda)(\lambda_1^2 - \lambda^2)\lambda_2|_{\lambda=0} \\ &= -\lambda_1^2\lambda_2, \\ \bar{\Lambda}_3 &:= (\tau(\partial\Lambda_3))_{\lambda_1, \lambda_2}(L, L) \\ &= (-1)^2 \frac{\partial}{\partial\lambda} (\partial\Lambda_3)_{\lambda_1, \lambda_2, \lambda}(L, L, L)|_{\lambda=0} \\ &= \frac{\partial}{\partial\lambda} (\lambda_1 + \lambda_2 + \lambda)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda)(\lambda_1 - \lambda)|_{\lambda=0} \\ &= -\lambda_1^3 + \lambda_2^3. \end{aligned}$$

This gives $H^2(\mathcal{W}, \mathbb{C}) = \mathbb{C}\bar{\chi} \oplus \mathbb{C}\bar{\Lambda}_3$. For $q = 3$, by (33), (53) and (61),

$$\begin{aligned} \bar{\psi} &:= (\tau(\partial\psi_2))_{\lambda_1, \lambda_2, \lambda_3}(L, L, M) \\ &= (-1)^3 \frac{\partial}{\partial\lambda} (\partial\psi_2)_{\lambda_1, \lambda_2, \lambda_3, \lambda}(L, L, M, L)|_{\lambda=0} \\ &= \frac{\partial}{\partial\lambda} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_1 + \lambda_2 + \lambda)|_{\lambda=0} \\ &= -\lambda_1^4 - \lambda_1^3\lambda_3 + \lambda_2^3(\lambda_2 + \lambda_3). \end{aligned}$$

Hence, $H^3(\mathcal{W}, \mathbb{C}) = \tilde{H}^3(\mathcal{W}, \mathbb{C}) \oplus \mathbb{C}\bar{\psi} = \mathbb{C}\chi \oplus \mathbb{C}\Lambda_3 \oplus \mathbb{C}\bar{\psi}$. By (33), (58), (60) and (61),

$$\begin{aligned} \bar{\varphi} &:= (\tau(\partial\varphi_1))_{\lambda_1, \lambda_2, \lambda_3, \lambda_4}(L, M, M, M) \\ &= (-1)^4 \frac{\partial}{\partial\lambda} (\partial\varphi_1)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda}(L, M, M, M, L)|_{\lambda=0} \\ &= -\frac{\partial}{\partial\lambda} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda)(\lambda_1 - \lambda)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_4 - \lambda_2)(\lambda_2 + \lambda_3 + \lambda_4)|_{\lambda=0} \\ &= (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)(\lambda_2 + \lambda_3 + \lambda_4)^2, \\ \bar{\Lambda} &:= (\tau(\partial\Lambda))_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5}(L, L, M, M, M) \\ &= (-1)^5 \frac{\partial}{\partial\lambda} (\partial\Lambda)_{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda}(L, L, M, M, M, L)|_{\lambda=0} \\ &= \frac{\partial}{\partial\lambda} (\sum_{i=1}^5 \lambda_i + \lambda)(\lambda_1 - \lambda_2)(\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_4 - \lambda_5)(\lambda_4 - \lambda_3)(\lambda_5 - \lambda_3)|_{\lambda=0} \\ &= -(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_5)(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 + (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4 + \lambda_5)) \\ &\equiv \lambda_1\lambda_2(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_5)(\lambda_4 - \lambda_5) \pmod{\partial \tilde{C}^5(\mathcal{W}, \mathbb{C})}. \end{aligned}$$

Therefore, $H^4(\mathcal{W}, \mathbb{C}) = \mathbb{C}\psi_2 \oplus \mathbb{C}\bar{\varphi}$, $H^5(\mathcal{W}, \mathbb{C}) = \mathbb{C}\varphi_1 \oplus \mathbb{C}\bar{\Lambda}$ and $H^6(\mathcal{W}, \mathbb{C}) = \mathbb{C}\Lambda$. Thus (i) is proved.

(ii) Define an operator $\tau_2 : \tilde{C}^q(\mathcal{W}, \mathbb{C}_a) \rightarrow \tilde{C}^{q-1}(\mathcal{W}, \mathbb{C}_a)$ by

$$(\tau_2 \gamma)_{\lambda_1, \dots, \lambda_{q-1}}(X_1, \dots, X_{q-1}) = (-1)^{q-1} \gamma_{\lambda_1, \dots, \lambda_{q-1}, \lambda}(X_1, \dots, X_{q-1}, L)|_{\lambda=0}, \quad (67)$$

for $X_1, \dots, X_{q-1} \in \{L, M\}$. By the fact that $\partial \tilde{C}^q(\mathcal{W}, \mathbb{C}_a) = (a + \sum_{i=1}^q \lambda_i) \tilde{C}^q(\mathcal{W}, \mathbb{C}_a)$, we have

$$\begin{aligned} ((d\tau_2 + \tau_2 d)\gamma)_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) &= (\sum_{i=1}^q \lambda_i) \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \\ &\equiv -a \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \pmod{\partial \tilde{C}^q(\mathcal{W}, \mathbb{C}_a)}. \end{aligned} \quad (68)$$

Let $\gamma \in \tilde{C}^q(\mathcal{W}, \mathbb{C}_a)$ be a q -cochain such that $d\gamma \in \partial \tilde{C}^{q+1}(\mathcal{W}, \mathbb{C}_a)$, namely, there is a $(q+1)$ -cochain ϕ such that $d\gamma = (a + \sum_{i=1}^{q+1} \lambda_i) \phi$. By (67), $\tau_2 d\gamma = (a + \sum_{i=1}^q \lambda_i) \tau_2 \phi \in \partial \tilde{C}^q(\mathcal{W}, \mathbb{C}_a)$. It follows from (68) that $\gamma \equiv -d(a^{-1} \tau_2 \gamma)$ is a reduced coboundary. This proves (ii).

(iii) In this case, $\partial \tilde{C}^q(\mathcal{W}, M_{\Delta, \alpha}) = (\partial + \sum_{i=1}^q \lambda_i) \tilde{C}^q(\mathcal{W}, M_{\Delta, \alpha})$. As in the proof of (ii), we define an operator $\tau_3 : C^q(\mathcal{W}, M_{\Delta, \alpha}) \rightarrow C^{q-1}(\mathcal{W}, M_{\Delta, \alpha})$ by

$$(\tau_3 \gamma)_{\lambda_1, \dots, \lambda_{q-1}}(X_1, \dots, X_{q-1}) = (-1)^{q-1} \gamma_{\lambda_1, \dots, \lambda_{q-1}, \lambda}(X_1, \dots, X_{q-1}, L)|_{\lambda=0},$$

for $X_1, \dots, X_{q-1} \in \{L, M\}$. Then

$$\begin{aligned} ((d\tau_3 + \tau_3 d)\gamma)_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) &= L_\lambda \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q)|_{\lambda=0} + \left(\sum_{i=1}^q \lambda_i \right) \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \\ &= (\partial + \alpha + \sum_{i=1}^q \lambda_i) \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \\ &\equiv \alpha \gamma_{\lambda_1, \dots, \lambda_q}(X_1, \dots, X_q) \pmod{\partial \tilde{C}^q(\mathcal{W}, M_{\Delta, \alpha})}. \end{aligned} \quad (69)$$

If γ is a reduced q -cocycle, it follows from (69) that $\gamma \equiv d(\alpha^{-1} \tau_3 \gamma)$ is a reduced coboundary, since $\alpha \neq 0$. Thus $H^q(\mathcal{W}, M_{\Delta, \alpha}) = 0$ for all $q \geq 0$.

This completes the proof of Theorem 5.3. \square

Remark 5.4. Denote by $\text{Lie}(\mathcal{W})_-$ the annihilation Lie algebra of \mathcal{W} . Note that $\text{Lie}(\mathcal{W})_-$ is isomorphic to the subalgebra spanned by $\{L_n, M_n \mid -1 \leq n \in \mathbb{Z}\}$ of the centerless W -algebra $W(2, 2)$. By [6, Corollary 6.1], $\tilde{H}^q(\mathcal{W}, \mathbb{C}) \cong H^q(\text{Lie}(\mathcal{W})_-, \mathbb{C})$. Thus we have determined the cohomology of $\text{Lie}(\mathcal{W})_-$ with trivial coefficients.

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