Open Math. 2016; 14: 586–602 **DE GRUYTER** OPEN

Open Mathematics **Open Access**

Research Article

Xiaoyao Jia*, Juanjuan Gao, and Xiaoquan Ding

Random attractors for stochastic two-compartment Gray-Scott equations with a multiplicative noise

DOI 10.1515/math-2016-0052
Received February 4, 2016; accepted July 25, 2016.

Abstract: In this paper, we consider the existence of a pullback attractor for the random dynamical system generated by stochastic two-compartment Gray-Scott equation for a multiplicative noise with the homogeneous Neumann boundary condition on a bounded domain of space dimension $n \leq 3$. We first show that the stochastic Gray-Scott equation generates a random dynamical system by transforming this stochastic equation into a random one. We also show that the existence of a random attractor for the stochastic equation follows from the conjugation relation between systems. Then, we prove pullback asymptotical compactness of solutions through the uniform estimate on the solutions. Finally, we obtain the existence of a pullback attractor.

Keywords: Gray-Scott equation, Random attractor, Random dynamical system, Multiplicative noise

MSC: 35K45, 35B40

1 Introduction

In this paper, we consider the following coupled stochastic two-compartment Gray-Scott equation, which is a reaction-diffusion system with multiplicative noise:

$$\frac{\partial \tilde{u}}{\partial t} = d_1 \Delta \tilde{u} - (F + k)\tilde{u} + \tilde{u}^2 \tilde{v} + D_1(\tilde{w} - \tilde{u}) + \sigma \tilde{u} \circ \frac{dW_t}{dt}, \tag{1}$$

$$\frac{\partial \tilde{v}}{\partial t} = d_2 \Delta \tilde{v} + F(1 - \tilde{v}) - \tilde{u}^2 \tilde{v} + D_2(\tilde{y} - \tilde{v}) + \sigma \tilde{v} \circ \frac{dW_t}{dt}, \tag{2}$$

$$\frac{\partial \tilde{w}}{\partial t} = d_1 \Delta \tilde{w} - (F + k)\tilde{w} + \tilde{w}^2 \tilde{y} + D_1(\tilde{u} - \tilde{w}) + \sigma \tilde{w} \circ \frac{dW_t}{dt}, \tag{3}$$

$$\frac{\partial \tilde{y}}{\partial t} = d_2 \Delta \tilde{y} + F(1 - \tilde{y}) - \tilde{w}^2 \tilde{y} + D_2(\tilde{v} - \tilde{y}) + \sigma \tilde{y} \circ \frac{dW_t}{dt}, \tag{4}$$

for $t > 0$, on a bounded domain D. Here D is an open bounded set of $\mathbb{R}^n$ $(n \leq 3)$, and it has a locally Lipschitz coutinuous boundary ∂D. Suppose that the equations have the following homogeneous Neumann boundary condition:

$$\frac{\partial \tilde{u}}{\partial \nu}(t,x) = \frac{\partial \tilde{v}}{\partial \nu}(t,x) = \frac{\partial \tilde{w}}{\partial \nu}(t,x) = \frac{\partial \tilde{y}}{\partial \nu}(t,x) = 0, \quad t > 0, \ x \in \partial D, \tag{5}$$

***Corresponding Author: Xiaoyao Jia:** School of Mathematics and Statistics, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: jiaxiaoyao@haust.edu.cn
Juanjuan Gao: School of Mathematics and Statistics, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: gaojj@haust.edu.cn
Xiaoquan Ding: School of Mathematics and Statistics, Henan University of Science and Technology, No.263 Kai-Yuan Road, Luo-Long District, Luoyang, Henan Province, 471023, China, E-mail: xqding@haust.edu.cn

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative, and have the following initial condition

$$\tilde{u}(0,x) = \tilde{u}_0(x), \quad \tilde{v}(0,x) = \tilde{v}_0(x), \quad \tilde{w}(0,x) = \tilde{w}_0(x), \quad \tilde{y}(0,x) = \tilde{y}_0(x), \quad x \in D. \tag{6}$$

Here $d_1,\ d_2,\ F,\ k,\ D_1$ and D_2 are positive constants; σ is a positive parameter; Δ is the Laplacian operator with respect to $x \in D$; $\tilde{u}(x,t),\ \tilde{v}(x,t),\ \tilde{w}(x,t)$ and $\tilde{y}(x,t)$ are real functions of $x \in D$; and W_t is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

the Borel sigma-algebra $\mathcal{F}$ on Ω is generated by the compact open topology (see [1]), $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$; $\circ$ denotes the Stratonovich sense in the stochastic term. We identify $\omega(t)$ with $W_t(\omega)$, i.e. $W_t(\omega) = W(t,\omega) = \omega(t),\ t \in \mathbb{R}$.

The Gray-Scott equation is a kind of very important reaction-diffusion system, which arises from many chemical or biological systems [2–5]. This equation has been researched by many authors (see [2–10]). One of the most important problems in mathematical physics is the asymptotic behavior of dynamical system, which has been developed greatly in recent years. For the deterministic system, the global attractor is a very important tool to study the asymptotic behavior of dynamical system (see [9–16]). If $\sigma = 0$, system (1)-(4) reduces to the two-compartment Gray-Scott equation without random terms, which has been investigated by You [10], where we proved the existence of the global attractor for the coupled two-compartment Gray-Scott equations with homogeneous Neumann boundary condition on a bounded domain.

Stochastic differential equations of this type arise from many chemical or biological systems when random spatiotemporal force is taken into consideration. These random perturbations play important roles in macroscopic phenomena. To study the properties of stochastic dynamical systems, the concept of pullback random attractor is introduced [1, 17, 18]. The existence of random attractors for stochastic dynamical systems has been studied [6, 7, 19–21]. In this paper, we study the existence of random attractor for stochastic two-compartment Gray-Scott equation on bounded domain D of space dimension $n \le 3$.

The paper is organized as follows. In Section 2, we recall a theorem about the existence of random pullback attractor for random dynamical system, and transform the stochastic system (1)-(6) into a continuous random dynamical system (18)-(19) Ornstein-Uhlenbeck process. Moreover, we show that, for each ω, the random dynamical system has a unique global solution. In Section 3, we obtain some uniform estimates of solutions for system (18)-(19) as $t \to \infty$. These estimates are used to prove the existence of bounded absorbing sets and the asymptotic compactness of the solutions. In the last section, we obtain the existence of a pullback random attractor.

The following notations will be used throughout this paper. $||\cdot||$ and $(\cdot,\cdot)$ denote the norm and the inner product in $L^2(D)$ or $[L^2(D)]^4$ respectively. $||\cdot||_{L^p}$ and $||\cdot||_{H^1}$ are used to denote the norm in $L^p(D)$ and $H^1(D)$.

By the Poincaré's inequality, there is a constant $\gamma > 0$ such that

$$||\nabla\phi||^2 \ge \gamma||\phi||^2, \quad \text{for } \phi \in H_0^1(D) \ \text{ or } [H_0^1(D)]^4. \tag{7}$$

Note that $H_0^1(D) \hookrightarrow L^6(D)$ for $n \le 3$. There exists a constant $\eta > 0$ such that the following embedding inequality holds:

$$||f||_{H^1}^2 \ge \eta||f||_{L^6}^2, \quad \text{for} \quad f \in H_0^1(D) \ \text{ or } \ [H_0^1(D)]^4. \tag{8}$$

2 RDS Generated by Stochastic Gray-Scott equation

In this section, we first recall a theorem for the existence of random attractors. Please note that here we omit the basic knowledge about random dynamical systems (RDS) and random attractor. Reader can refer to [1, 11, 17–19] for these knowledge.

Suppose that $(X, ||\cdot||_X)$ is a separable Banach space with Borel $\sigma-$algebra $\mathcal{B}(X)$, and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system, and assume that ϕ is a continuous RDS on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. We recall a Proposition which will be used to prove the existence of random attractor ([11], [19]) for RDS.

Proposition 2.1. *Suppose $\mathcal{D}$ is the collection of random subsets of X, and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ is a random absorbing set for RDS ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor $\{A(\omega)\}_{\omega\in\Omega}$ which has the following form*

$$A(\omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \tag{9}$$

Next, we shall show that system (1)-(6) generates a random dynamical system. For our purpose, we first transform this stochastic system into a deterministic dynamical with random attractor. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space defined in Section 1. Define $(\theta_t)_{t\in\mathbb{R}}$ on Ω by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R},$$

then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a metric dynamical system.

Set $\tilde{g} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{y})^T$, then the system (1)-(6) can be rewritten as follows:

$$\frac{d\tilde{g}}{dt} = A\tilde{g} + \tilde{F}(\tilde{g}) + \sigma\tilde{g} \circ \frac{dW_t}{dt}, \ t > 0, \tag{10}$$

$$\tilde{g}(0, x) = \tilde{g}_0(x) = (\tilde{u}_0(x), \tilde{v}_0(x), \tilde{w}_0(x), \tilde{y}_0(x))^T, \ x \in D, \tag{11}$$

where

$$A = \begin{pmatrix} d_1\Delta & 0 & 0 & 0 \\ 0 & d_2\Delta & 0 & 0 \\ 0 & 0 & d_1\Delta & 0 \\ 0 & 0 & 0 & d_2\Delta \end{pmatrix}, \quad \tilde{F}(\tilde{g}) = \begin{pmatrix} -(F+k)\tilde{u} + \tilde{u}^2\tilde{v} + D_1(\tilde{w} - \tilde{u}) \\ F(1-\tilde{v}) - \tilde{u}^2\tilde{v} + D_2(\tilde{y} - \tilde{v}) \\ -(F+k)\tilde{w} + \tilde{w}^2\tilde{y} + D_1(\tilde{u} - \tilde{w}) \\ F(1-\tilde{y}) - \tilde{w}^2\tilde{y} + D_2(\tilde{v} - \tilde{y}) \end{pmatrix}.$$

To transform the stochastic system into a deterministic system with random parameter, we introduce the following one-dimensional Ornstein-Uhlenbeck process:

$$dz + zdt = dW_t. \tag{12}$$

From [13], we know that the stationary solution of Ornstein-Uhlenbeck process has the following form :

$$z(\theta_t\omega) \equiv -\int_{-\infty}^{0} e^s(\theta_t\omega)(s)ds, \quad t \in \mathbb{R}.$$

Moreover, the random variable $z(\theta_t\omega)$ is tempered, and $\mathbb{P}$−a.e. $\omega \in \Omega$, $t \mapsto z(\theta_t\omega)$ is continuous in t, and satisfies the properties (see [1, 11, 13]):

$$\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|} = 0; \qquad \lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds = 0. \tag{13}$$

Set $(u(t), v(t), w(t), y(t))^T = e^{-\sigma z(\theta_t\omega)}(\tilde{u}(t), \tilde{v}(t), \tilde{w}(t), \tilde{y}(t))^T$. Then, we obtain the equivalent system of (10) and (11) as:

$$\frac{\partial u}{\partial t} = d_1\Delta u - (F + k - \sigma z(\theta_t\omega))u + e^{2\sigma z(\theta_t\omega)}u^2 v + D_1(w - u), \tag{14}$$

$$\frac{\partial v}{\partial t} = d_2\Delta v + Fe^{-\sigma z(\theta_t\omega)} + (\sigma z(\theta_t\omega) - F)v - e^{2\sigma z(\theta_t\omega)}u^2 v + D_2(y - v), \tag{15}$$

$$\frac{\partial w}{\partial t} = d_1\Delta w - (F + k - \sigma z(\theta_t\omega))w + e^{2\sigma z(\theta_t\omega)}w^2 y + D_1(u - w), \tag{16}$$

$$\frac{\partial y}{\partial t} = d_2\Delta y + Fe^{-\sigma z(\theta_t\omega)} + (\sigma z(\theta_t\omega) - F)y - e^{2\sigma z(\theta_t\omega)}w^2 y + D_2(v - y), \tag{17}$$

that is, $g = (u, v, w, y)^T$ satisfies

$$\frac{dg}{dt} = Ag + F(g, \omega) + \sigma z(\theta_t\omega)g, \ t > 0, \tag{18}$$

$$g(0,x) = g_0(x) = e^{-\sigma z(\omega)}\tilde{g}_0(x) = (u_0(x), v_0(x), w_0(x), y_0(x))^T, \ x \in D, \tag{19}$$

with

$$F(g,\omega) = \begin{pmatrix} -(F+k)u + e^{2\sigma z(\theta_t\omega)}u^2v + D_1(w-u) \\ Fe^{-\sigma z(\theta_t\omega)} - Fv - e^{2\sigma z(\theta_t\omega)}u^2v + D_2(y-v) \\ -(F+k)w + e^{2\sigma z(\theta_t\omega)}w^2y + D_1(u-w) \\ Fe^{-\sigma z(\theta_t\omega)} - Fy - e^{2\sigma z(\theta_t\omega)}w^2y + D_2(v-y) \end{pmatrix}.$$

Notice that for $\mathbb{P}-a.e.\ \omega \in \Omega$, $F(g,\omega)$ is locally Lipschtiz continuous with respect to g. In [10], You proved that the deterministic system has a unique solution by the Galekin method. Similar to deterministic system, by the Galekin method, for $\mathbb{P}-a.e.\ \omega \in \Omega$, we can prove that for $g_0 \in [L^2(D)]^4$, (18)-(19) has a unique solution $g(\cdot,\omega,g_0) \in C([0,\infty),[L^2(D)]^4) \cap L^2((0,\infty),[H^1(D)]^4)$ with $g(0,\omega,g_0) = g_0$. Moreover, similarly to Lemma 3 of [10], we can prove that $g(t,\omega,g_0)$ is a unique, global, weak solution with respect to $g_0 \in [L^2(D)]^4$, for $t \in [0,\infty)$. This shows that (18) and (19) generate a continuous random dynamical system $(\varphi(t))_{t\geq 0}$ over $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ with

$$\varphi(t,\omega,g_0) = g(t,\omega,g_0), \qquad \forall (t,\omega,g_0) \in \mathbb{R}^+ \times \Omega \times [L^2(D)]^4. \tag{20}$$

Now assume that $\phi : \mathbb{R}^+ \times \Omega \times [L^2(D)]^4 \to [L^2(D)]^4$ is given by

$$\phi(t,\omega,\tilde{g}_0) = \tilde{g}(t,\omega,\tilde{g}_0) = g(t,\omega,e^{-\sigma z(\omega)}g_0)e^{\sigma z(\theta_t\omega)}. \tag{21}$$

Then ϕ is a continuous dynamical system associated to (10)-(11). Notice that two dynamical systems are conjugate to each other. Thus, in the following sections, we consider only the existence of a random attractor of φ.

3 Uniform estimates of solutions

To find the existence of the random attractor, we first need to obtain some uniform estimates of the solutions. Therefore in this section we first prove the uniform estimates about the solution of the two-compartment stochastic Gray-Scott equation on D, as $t \to +\infty$. We assume that $\mathcal{D}$ is a collection of all tempered random subsets of $[L^2(D)]^4$. First, we define some functions which will be used in this section. Set

$$\begin{aligned} &Y_1(t,x) = u(t,x) + v(t,x) + w(t,x) + y(t,x), \ \ Y_{1,0} = u_0 + v_0 + w_0 + y_0; \\ &Y_2(t,x) = u(t,x) + w(t,x), \ \ Y_{2,0} = u_0 + w_0; \\ &Y_3(t,x) = u(t,x) + v(t,x) - w(t,x) - y(t,x), \ \ Y_{3,0} = u_0 + v_0 - w_0 - y_0; \\ &Y_4(t,x) = v(t,x) - y(t,x); \\ &Y_5(t,x) = u(t,x) - w(t,x). \end{aligned}$$

The next lemma shows that φ has a random absorbing set in $\mathcal{D}$.

Lemma 3.1. *Random dynamical system φ has a random absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{D}$, that is, for any $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$, and for $\mathbb{P}-a.e.\ \omega \in \Omega$, there is $T_B(\omega) > 0$ such that $\varphi(t,\theta_{-t}\omega,B(\theta_{-t}\omega)) \subset K(\omega)$, for any $t \geq T_B(\omega)$.*

Proof. Taking the inner products of (15) and (17) with v and y respectively, and adding them up, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(||v||^2 + ||y||^2) + d_2(||\nabla v||^2 + ||\nabla y||^2) \\ = &\ F\int_D e^{-\sigma z(\theta_t\omega)}(v+y)dx + \int_D (\sigma z(\theta_t\omega) - F)(v^2+y^2)dx \\ &- \int_D e^{2\sigma z(\theta_t\omega)}(u^2v^2 + w^2y^2)dx - \int_D D_2(y-v)^2dx \end{aligned} \tag{22}$$

$$= \left(\sigma z(\theta_t\omega) - \frac{F}{2}\right)(||v||^2 + ||y||^2) - \frac{F}{2}\int_D (v - e^{-\sigma z(\theta_t\omega)})^2 + (y - e^{-\sigma z(\theta_t\omega)})^2 dx \tag{23}$$

$$+F|D|e^{-2\sigma z(\theta_t\omega)} - \int_D D_2(y-v)^2 dx$$

$$\leq \left(\sigma z(\theta_t\omega) - \frac{F}{2}\right)(||v||^2 + ||y||^2) + F|D|e^{-2\sigma z(\theta_t\omega)}. \tag{24}$$

Gronwall's inequality yields that

$$||v||^2 + ||y||^2 \leq e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}(||v_0||^2 + ||y_0||^2)$$
$$+2F|D|e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_\tau\omega)d\tau + Fs - 2\sigma z(\theta_s\omega)}ds$$
$$-2d_2 e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_\tau\omega)d\tau + Fs}(||\nabla v||^2 + ||\nabla y||^2)ds. \tag{25}$$

Replacing ω by $\theta_{-t}\omega$ in the above inequality, we obtain that, for all $t \geq 0$,

$$||v(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + ||y(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2$$
$$\leq e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}(||v_0(\theta_{-t}\omega)||^2 + ||y_0(\theta_{-t}\omega)||^2)$$
$$+2F|D|e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs - 2\sigma z(\theta_{s-t}\omega)}ds$$
$$-2d_2 e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs}(||\nabla v(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \tag{26}$$
$$+||\nabla y(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2)ds. \tag{27}$$

Using the properties of the Ornstein-Uhlenbeck process (13), for any $g_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, we obtain that

$$\lim_{t\to+\infty} e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}(||y_0(\theta_{-t}\omega)||^2 + ||v_0(\theta_{-t}\omega)||^2)$$
$$= \lim_{t\to+\infty} e^{\int_{-t}^0 2\sigma z(\theta_\tau\omega)d\tau - Ft}(||y_0(\theta_{-t}\omega)||^2 + ||v_0(\theta_{-t}\omega)||^2) = 0, \tag{28}$$

and

$$2F|D|e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs - 2\sigma z(\theta_{s-t}\omega)}ds$$
$$= 2F|D|\int_0^t e^{\int_s^t 2\sigma z(\theta_{\tau-t}\omega)d\tau + F(s-t) - 2\sigma z(\theta_{s-t}\omega)}ds$$
$$= 2F|D|\int_{-t}^0 e^{\int_s^0 2\sigma z(\theta_\tau\omega)d\tau + Fs - 2\sigma z(\theta_s\omega)}ds$$
$$\leq 2F|D|\int_{-\infty}^0 e^{\int_s^0 2\sigma z(\theta_\tau\omega)d\tau + Fs - 2\sigma z(\theta_s\omega)}ds < +\infty. \tag{29}$$

Thus, there exists a $T_B(\omega) > 0$, such that, for all $t \geq T_B(\omega)$,

$$||v(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + ||y(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \leq \rho_0^2(\omega), \tag{30}$$

with

$$\rho_0^2(\omega) = 1 + 2F|D| \int_{-\infty}^{0} e^{\int_s^0 2\sigma z(\theta_\tau \omega) d\tau + Fs - 2\sigma z(\theta_s \omega)} ds. \tag{31}$$

It is easy to check that $\rho_0^2(\omega)$ is a tempered random variable. To estimate u and w, we use (14), (15), (16) and (17) to get the equation for $Y_1(t,x)$. Now

$$\frac{\partial Y_1}{\partial t} = d_1 \Delta Y_1 - (F + k - \sigma z(\theta_t \omega))Y_1 + (d_2 - d_1)\Delta(v + y) + k(v + y) + 2Fe^{-\sigma z(\theta_t \omega)}. \tag{32}$$

Taking the inner product of (32) and Y_1, then apply the Hölder's inequality and Poincaré's inequality (7), we get:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||Y_1||^2 + d_1||\nabla Y_1||^2 + (F + k - \sigma z(\theta_t \omega))||Y_1||^2 \\
&= \int_D (d_2 - d_1)\Delta(v + y)Y_1 dx + \int_D k(v + y)Y_1 dx + \int_D 2Fe^{-\sigma z(\theta_t \omega)} Y_1 dx \\
&\leq |d_1 - d_2| \, ||\nabla(v + y)|| \, ||\nabla Y_1|| + k||v + y|| \, ||Y_1|| + 2F|D|^{\frac{1}{2}} e^{-\sigma z(\theta_t \omega)}||Y_1|| \\
&\leq \frac{d_1}{2}||\nabla Y_1||^2 + \frac{(d_1 - d_2)^2}{2d_1}||\nabla(v + y)||^2 \qquad (33)\\
&\quad + \frac{k}{2}||Y_1||^2 + \frac{k}{2\gamma}||\nabla(v + y)||^2 + \frac{F}{2}||Y_1||^2 + 2F|D|e^{-2\sigma z(\theta_t \omega)}. \qquad (34)
\end{aligned}$$

Therefore,

$$\frac{d}{dt}||Y_1||^2 + d_1||\nabla Y_1||^2 + (F - 2\sigma z(\theta_t \omega))||Y_1||^2 \leq \left(\frac{(d_1 - d_2)^2}{d_1} + \frac{k}{\gamma}\right)||\nabla(v + y)||^2 + 4F|D|e^{-2\sigma z(\theta_t \omega)}.$$

Applying Gronwall's inequality, we get that

$$\begin{aligned}
||Y_1||^2 &\leq e^{\int_0^t 2\sigma z(\theta_\tau \omega) d\tau - Ft}||Y_{1,0}||^2 \\
&\quad + \left(\frac{(d_1 - d_2)^2}{d_1} + \frac{k}{\gamma}\right) e^{\int_0^t 2\sigma z(\theta_\tau \omega) d\tau - Ft} \int_0^t e^{\int_0^s -2\sigma z(\theta_\tau \omega) d\tau + Fs}||\nabla(v + y)||^2 ds \\
&\quad + 4F|D|e^{\int_0^t 2\sigma z(\theta_\tau \omega) d\tau - Ft} \int_0^t e^{\int_0^s -2\sigma z(\theta_\tau \omega) d\tau + Fs - 2\sigma z(\theta_s \omega)} ds \\
&\quad - d_1 e^{\int_0^t 2\sigma z(\theta_\tau \omega) d\tau - Ft} \int_0^t e^{\int_0^s -2\sigma z(\theta_\tau \omega) d\tau + Fs}||\nabla Y_1||^2 ds. \qquad (35)
\end{aligned}$$

By replacing ω by $\theta_{-t}\omega$ in the above inequality, it follows that

$$\begin{aligned}
&||Y_1(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \leq e^{\int_0^t 2\sigma z(\theta_{\tau - t} \omega) d\tau - Ft}||Y_{1,0}(\theta_{-t}\omega)||^2 \\
&+ \left(\frac{(d_1 - d_2)^2}{d_1} + \frac{k}{\gamma}\right) e^{\int_0^t 2\sigma z(\theta_{\tau - t} \omega) d\tau - Ft} \times \\
&\qquad \int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau - t} \omega) d\tau + Fs}||\nabla(v + y)(s, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 ds \\
&+ 4F|D|e^{\int_0^t 2\sigma z(\theta_{\tau - t} \omega) d\tau - Ft} \int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau - t} \omega) d\tau + Fs - 2\sigma z(\theta_{s-t} \omega)} ds \\
&- d_1 e^{\int_0^t 2\sigma z(\theta_{\tau - t} \omega) d\tau - Ft} \int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau - t} \omega) d\tau + Fs}||\nabla Y_1(s, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 ds. \qquad (36)
\end{aligned}$$

Similar to (28), we have, for any $g_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,

$$\lim_{t\to+\infty} e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}||Y_{1,0}(\theta_{-t}\omega)||^2 = \lim_{t\to+\infty} e^{\int_{-t}^0 2\sigma z(\theta_\tau\omega)d\tau - Ft}||Y_{1,0}(\theta_{-t}\omega)||^2 = 0. \tag{37}$$

By (27)-(30), we obtain that, for all $t \geq T_B(\omega)$,

$$2d_2 e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs}||\nabla v(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2$$
$$+||\nabla y(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 ds \leq \rho_0^2(\omega). \tag{38}$$

Let $c_1 = \frac{1}{2}(\frac{(d_1-d_2)^2}{d_1d_2} + \frac{k}{\gamma d_2})$, then the above inequality yields that for all $t \geq T_B(\omega)$,

$$\left(\frac{(d_1-d_2)^2}{d_1} + \frac{k}{\gamma}\right) e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs}||\nabla v(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2$$
$$+||\nabla y(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 ds \leq c_1\rho_0^2(\omega). \tag{39}$$

Therefore (29), (36) and (39) imply that

$$||Y_1(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \leq (c_1+2)\rho_0^2(\omega). \tag{40}$$

Hence, we obtain from (30) and (40) that

$$\begin{aligned}||Y_2(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 &= ||Y_1(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega)) - (v+y)(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2\\ &\leq 2\left(||Y_1(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + ||(v+y)(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2\right)\\ &\leq 2||Y_1(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + 4||v(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2\\ &\quad +4||y(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2\\ &\leq (2c_1+8)\rho_0^2(\omega).\end{aligned} \tag{41}$$

It is easy to check that $Y_3(t,x)$ satisfies the following equation

$$\begin{aligned}\frac{\partial Y_3}{\partial t} &= d_1\Delta Y_3 - (F+k-\sigma z(\theta_t\omega))Y_3 + 2D_1(w-u) + 2D_2(y-v) + k(v-y) + (d_2-d_1)\Delta(v-y)\\ &= d_1\Delta Y_3 - (F+k+2D_1-\sigma z(\theta_t\omega))Y_3 + (k+2D_1-2D_2)(v-y) + (d_2-d_1)\Delta(v-y).\end{aligned} \tag{42}$$

Taking the inner product of the last equation with Y_3, we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}||Y_3||^2 + d_1||\nabla Y_3||^2 + (F+k+2D_1-\sigma z(\theta_t\omega))||Y_3||^2\\ &= \int_D (k+2D_1-2D_2)(v-y)Y_3dx - \int_D (d_2-d_1)\nabla(v-y)\nabla Y_3dx\\ &\leq |k+2D_1-2D_2|\,||v-y||\,||Y_3|| + |d_1-d_2|\,||\nabla(v-y)||\,||\nabla Y_3||\\ &\leq \left(\frac{(d_1-d_2)^2}{2d_1} + \frac{(k+2D_1-2D_2)^2}{2\gamma(F+k+2D_1)}\right)||\nabla(v-y)||^2 + \frac{d_1}{2}||\nabla Y_3||^2 + \frac{F+k+2D_1}{2}||Y_3||^2.\end{aligned} \tag{43}$$

In the last step we used Poincaré's inequality (7). Thus,

$$\frac{d}{dt}||Y_3||^2 + d_1||\nabla Y_3||^2 + (F-2\sigma z(\theta_t\omega))||Y_3||^2 \leq c_2(||\nabla v||^2 + ||\nabla y||^2)$$

with $c_2 = \frac{2(d_1-d_2)^2}{d_1} + \frac{2(k+2D_1-2D_2)^2}{\gamma(F+k+2D_1)}$. It follows from Gronwall's inequality that

$$||Y_3||^2 \leq e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}||Y_{3,0}||^2 - d_1 e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_\tau\omega)d\tau + Fs}||\nabla Y_3||^2 ds$$

$$+c_2\, e^{\int_0^t 2\sigma z(\theta_\tau\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_\tau\omega)d\tau+Fs}(||\nabla v||^2+||\nabla y||^2)ds. \tag{44}$$

Replacing ω by $\theta_{-t}\omega$, we find that, for all $t \ge 0$,

$$\begin{aligned}
&||Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&\le e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau-Ft}||Y_{3,0}(\theta_{-t}\omega)||^2\\
&-d_1 e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau-Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau+Fs}||\nabla Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2 ds\\
&+c_2\, e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau-Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau+Fs}\Big(||\nabla v(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&\qquad +||\nabla y(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\Big)ds.
\end{aligned} \tag{45}$$

Similar to (28) and (39), we obtain that

$$\lim_{t\to+\infty} e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau-Ft}||Y_{3,0}(\theta_{-t}\omega)||^2 = 0 \tag{46}$$

and

$$\begin{aligned}
&c_2 e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau-Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau+Fs}\Big(||\nabla v(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&\qquad +||\nabla y(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega+||\nabla y(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\Big)ds \le \frac{c_2}{2d_2}\rho_0^2(\omega).
\end{aligned} \tag{47}$$

Set $c_3 = \frac{c_2}{2d_2}+1$, then by (45)-(47), one has that, for all $t \ge T_B(\omega)$,

$$||Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2 \le c_3\rho_0^2(\omega). \tag{48}$$

It follows from (30) and (48) that, for all $t \ge T_B(\omega)$,

$$\begin{aligned}
&||Y_5(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&= ||Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega)) - Y_4(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&\le 2\Big(||Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2+||Y_4(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\Big)\\
&\le 2||Y_3(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2+4\Big(||v(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2+||y(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\Big)\\
&= c_4\rho_0^2(\omega)
\end{aligned} \tag{49}$$

with $c_4 = 2c_3+4$. Consequently, we obtain that, for all $t \ge T_B(\omega)$,

$$\begin{aligned}
&||u(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2+||w(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&= ||\frac{1}{2}\left[Y_2(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))+Y_5(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))\right]||^2\\
&\qquad +||\frac{1}{2}\left[Y_2(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))-Y_5(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))\right]||^2\\
&\le ||Y_2(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2+||Y_5(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2\\
&\le c_5\rho_0^2(\omega)
\end{aligned} \tag{50}$$

with $c_5 = 2c_1+8+c_4$. We finally obtain that, for all $t \ge T_B(\omega)$,

$$||g(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||^2 \le (c_5+1)\rho_0^2(\omega). \tag{51}$$

It is easy to check that $\rho_0^2(\omega)$ is tempered. This ends the proof. □

Lemma 3.2. *There exists a random variable $\rho_1(\omega)$, such that, for any $B(\omega) \in \mathcal{D}$, and $g_0(\omega) \in B(\omega)$, for $\mathbb{P}-a.e.\ \omega \in \Omega$, there is a $T_B(\omega) > 0$, such that, for any $t \geq T_B(\omega)$, the following estimate holds*

$$||v(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||_{L^6}^6 + ||y(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||_{L^6}^6 \leq \rho_1(\omega). \tag{52}$$

Proof. Because the unique global weak solution of system (18)-(19) satisfies $g(t,\omega,g_0(\omega)) \in C((0,\infty), [L^2(D)]^4) \cap L^2((0,\infty),[H^1(D)]^4)$, then, for any initial value $g_0 \in [L^2(D)]^4$, there exists a small time $t_0 \in (0,1)$ such that,

$$g(t_0,\omega,g_0(\omega)) \in [H^1(D)]^4 \hookrightarrow [L^6(D)]^4. \tag{53}$$

This means that the weak solution $g(t,\omega,g_0(\omega))$ becomes a strong solution on $[t_0,+\infty)$ and satisfies $g(t,\omega,g_0(\omega)) \in C([t_0,\infty),[H^1(D)]^4) \cap L^2([t_0,\infty),[H^2(D)]^4) \subset C([t_0,\infty),[L^6(D)]^4)$. Thus, without loss of generality, we can assume that $g_0 \in [L^6(D)]^4$. Taking the inner products of (15) and (17) with v^5 and y^5 respectively, and adding them up, we obtain

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}(||v||_{L^6}^6 + ||y||_{L^6}^6) + 5d_2(||v^2\nabla v||^2 + ||y^2\nabla y||^2)\\
&= F\int_D e^{-\sigma z(\theta_t\omega)}(v^5+y^5) - (y^6+v^6)dx + \int_D \sigma z(\theta_t(\omega))(v^6+y^6)dx\\
&\quad - \int_D e^{2\sigma z(\theta_t\omega)}(u^2v^6 + w^2y^6)dx + D_2\int_D (y-v)(v^5-y^5)dx.
\end{aligned} \tag{54}$$

We now estimate all terms on the right hand side of (54). For the fourth term, by Young's inequality, we get that

$$\begin{aligned}
&\int_D (y-v)(v^5-y^5)dx = \int_D -v^6 - y^6 + yv^5 + vy^5 dx\\
&\leq \int_D -v^6 - y^6 + (\frac{1}{6}y^6 + \frac{5}{6}v^6) + (\frac{1}{6}v^6 + \frac{5}{6}y^6)dx = 0.
\end{aligned} \tag{55}$$

For the first term, by Young's inequality,

$$\begin{aligned}
&F\int_D e^{-\sigma z(\theta_t\omega)}(v^5+y^5) - (y^6+v^6)dx\\
&\leq F\int_D \frac{5}{6}(v^6+y^6) + \frac{1}{3}e^{-6\sigma z(\theta_t\omega)} - (y^6+v^6)dx\\
&\leq -\frac{F}{6}(||v||_{L^6}^6 + ||y||_{L^6}^6) + \frac{F}{3}|D|e^{-6\sigma z(\theta_t\omega)}.
\end{aligned} \tag{56}$$

By (54)-(56), we arrive at the following estimate, for all $t \geq 0$

$$\frac{d}{dt}(||v||_{L^6}^6 + ||y||_{L^6}^6) \leq (6\sigma z(\theta_t\omega) - F)\left(||v||_{L^6}^6 + ||y||_{L^6}^6\right) + 2F|D|e^{-6\sigma z(\theta_t\omega)}. \tag{57}$$

By Gronwall's inequality, we obtain that, for all $t \geq 0$,

$$\begin{aligned}
&||v||_{L^6}^6 + ||y||_{L^6}^6 \leq e^{\int_0^t 6\sigma z(\theta_\tau\omega)d\tau - Ft}(||v_0||_{L^6}^6 + ||y_0||_{L^6}^6)\\
&+2F|D|e^{\int_0^t 6\sigma z(\theta_\tau\omega)d\tau - Ft}\int_0^t e^{\int_0^s -6\sigma z(\theta_\tau\omega)d\tau + Fs - 6\sigma z(\theta_s\omega)}ds.
\end{aligned} \tag{58}$$

Replacing ω by $\theta_{-t}\omega$ in the above inequality, we have that, for all $t \geq 0$,

$$||v(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||_{L^6}^6 + ||y(t,\theta_{-t}\omega,g_0(\theta_{-t}\omega))||_{L^6}^6$$

$$\leq e^{\int_0^t 6\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}(||v_0(\theta_{-t}\omega)||_{L^6}^6 + ||y_0(\theta_{-t}\omega)||_{L^6}^6)$$
$$+2F|D|e^{\int_0^t 6\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -6\sigma z(\theta_{\tau-t}\omega)d\tau + Fs - 6\sigma z(\theta_{s-t}\omega)}ds$$
$$= e^{\int_0^t 6\sigma z(\theta_\tau\omega)d\tau - Ft}(||v_0(\theta_{-t}\omega)||_{L^6}^6 + ||y_0(\theta_{-t}\omega)||_{L^6}^6)$$
$$+2F|D|\int_{-t}^0 e^{\int_s^0 -6\sigma z(\theta_\tau\omega)d\tau + Fs - 6\sigma z(\theta_s\omega)}ds. \tag{59}$$

For $g_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$, the properties of the Ornstein-Uhlenbeck process implies that, for all $t \geq T_B(\omega)$

$$\lim_{t\to+\infty} e^{\int_{-t}^0 6\sigma z(\theta_\tau\omega)d\tau - Ft}(||v_0(\theta_{-t}\omega)||_{L^6}^6 + ||y_0(\theta_{-t}\omega)||_{L^6}^6) = 0, \tag{60}$$

and

$$2F|D|\int_{-t}^0 e^{\int_s^0 -6\sigma z(\theta_\tau\omega)d\tau + Fs - 6\sigma z(\theta_s\omega)}ds$$
$$\leq 2F|D|\int_{-\infty}^0 e^{\int_s^0 -6\sigma z(\theta_\tau\omega)d\tau + Fs - 6\sigma z(\theta_s\omega)}ds < \infty. \tag{61}$$

Set

$$\rho_1(\omega) \equiv 1 + 2F|D|\int_{-\infty}^0 e^{\int_s^0 -6\sigma z(\theta_\tau\omega)d\tau + Fs - 6\sigma z(\theta_s\omega)}ds. \tag{62}$$

Then there exists a $T_B(\omega) > 0$, independent of σ, such that for all $t > T_B(\omega)$,

$$||v(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||_{L^6}^6 + ||y(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||_{L^6}^6 \leq \rho_1(\omega). \tag{63}$$

This ends the proof. □

Lemma 3.3. *There exists a random variable $\rho_2(\omega) > 0$ such that, for any $B(\omega) \in \mathcal{D}$, $g_0(\omega) \in B(\omega)$, for $\mathbb{P}-a.e.\ \omega \in \Omega$, there exists a $T_B(\omega) > 0$ such that, for all $t \geq T_B(\omega)$, the following estimate holds,*

$$\int_t^{t+1} ||\nabla g(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 ds \leq \rho_2(\omega). \tag{64}$$

Proof. Using (27), (28) and (29), we obtain:

$$2d_2 e^{\int_0^t 2\sigma z(\theta_{\tau-t}\omega)d\tau - Ft}\int_0^t e^{\int_0^s -2\sigma z(\theta_{\tau-t}\omega)d\tau + Fs}\Big(||\nabla v(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2$$
$$+||\nabla y(s,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2\Big)ds \leq \rho_0^2(\omega). \tag{65}$$

Setting $t = t+1$, we have:

$$2d_2 e^{\int_0^{t+1} 2\sigma z(\theta_{\tau-t-1}\omega)d\tau - F(t+1)}\int_0^{t+1} e^{\int_0^s -2\sigma z(\theta_{\tau-t-1}\omega)d\tau + Fs}\Big(||\nabla v(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2$$
$$+||\nabla y(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2\Big)ds$$
$$\geq 2d_2\int_t^{t+1} e^{\int_s^{t+1} 2\sigma z(\theta_{\tau-t-1}\omega)d\tau + F(s-t-1)}\Big(||\nabla v(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2$$

$$+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\Big)\,ds$$

$$\geq 2d_2 e^{-2\sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|-F}\int_t^{t+1}||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds.$$

Let $T_B(\omega)$ be the constant defined in Lemma 3.1. It follows that, for all $t\geq T_B(\omega)$,

$$\int_t^{t+1}||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\leq c_6\rho_0^2(\omega), \tag{66}$$

with $c_6=\frac{1}{2d_2}e^{2\sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|+F}$. Similarly, by (36)-(40) and (45)-(48), we get:

$$\int_t^{t+1}||\nabla Y_1(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\leq c_7\rho_0^2(\omega),$$

$$\int_t^{t+1}||\nabla Y_3(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\leq c_8\rho_0^2(\omega),$$

where $c_7=\frac{c_1+2}{d_1}e^{2\sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|+F}$, $c_8=\frac{c_3}{d_1}e^{2\sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|+F}$. Set $c_9=2c_7+4c_6$, and $c_{10}=2c_8+4c_6$. It follows that, for all $t\geq T_B(\omega)$,

$$\begin{aligned}
&\int_t^{t+1}||\nabla Y_2(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\\
&=\int_t^{t+1}||\nabla Y_1(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))-\nabla(v+y)(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\\
&\leq\int_t^{t+1}2||\nabla Y_1(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+4(||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\\
&\quad+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2)ds\\
&\leq c_9\rho_0^2(\omega),
\end{aligned}\tag{67}$$

and

$$\begin{aligned}
&\int_t^{t+1}||\nabla Y_5(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\\
&=\int_t^{t+1}||\nabla Y_3(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))-\nabla Y_4(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\\
&\leq 2\int_t^{t+1}||\nabla Y_3(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla Y_4(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2 ds\\
&\leq 2\int_t^{t+1}||\nabla Y_3(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+2(||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\\
&\quad+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2)ds\\
&\leq c_{10}\rho_0^2(\omega).
\end{aligned}\tag{68}$$

Therefore, by (67) and (68), we get

$$\begin{aligned}
&\int_t^{t+1} ||\nabla u(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 + ||\nabla w(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 ds \\
&= \frac{1}{4}\int_t^{t+1} ||\nabla Y_2(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega)) + \nabla Y_5(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 \\
&\qquad + ||\nabla Y_2(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega)) - \nabla Y_5(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 ds \\
&\le \int_t^{t+1} ||\nabla Y_2(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 + ||\nabla Y_5(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 ds \\
&\le (c_9 + c_{10})\rho_0^2(\omega).
\end{aligned} \tag{69}$$

Finally, by (66) and (69), we obtain that, for all $t \ge T_B(\omega)$,

$$\int_t^{t+1} ||\nabla g(s,\theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 \le (c_6 + c_9 + c_{10})\rho_0^2(\omega) \equiv \rho_2(\omega). \tag{70}$$

This ends the proof. □

Lemma 3.4. *There exists a random variable $\rho_3(\omega)$, such that, for any $B(\omega) \in \mathcal{D}$, and $g_0(\omega) \in B(\omega)$, for $\mathbb{P}-a.e.\ \omega \in \Omega$, there is a $T_B(\omega) > 0$, such that for all $t \ge T_B(\omega)$, the following estimate holds ,*

$$||\nabla u(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + ||\nabla w(t,\theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \le \rho_3(\omega). \tag{71}$$

Proof. Taking the inner products of (14) and (16) with $-\Delta u$ and $-\Delta w$ respectively, and then summing them up, we obtain that,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(||\nabla u||^2 + ||\nabla w||^2) + d_1(||\Delta u||^2 + ||\Delta w||^2) + (F + k - \sigma z(\theta_t\omega))(||\nabla u||^2 + ||\nabla w||^2) \\
&= -\int_D e^{2\sigma z(\theta_t\omega)}(u^2 v\Delta u + w^2 y\Delta w)dx + D_1\int_D (w-u)(\Delta w - \Delta u)dx.
\end{aligned} \tag{72}$$

Now, we estimate each term on the right hand side of (72). For the second term,

$$\int_D (w-u)(\Delta w - \Delta u)dx = -\int_D (\nabla w - \nabla u)^2 dx \le 0. \tag{73}$$

For the first term, by Hölder's inequality and (8), we have that

$$\begin{aligned}
&-\int_D e^{2\sigma z(\theta_t\omega)}(u^2 v\Delta u + w^2 y\Delta w)dx \\
&\le d_1(||\Delta u||^2 + ||\Delta w||^2) + \frac{1}{4d_1}e^{4\sigma z(\theta_t\omega)}\int_D u^4 v^2 + w^4 y^2 dx \\
&\le d_1(||\Delta u||^2 + ||\Delta w||^2) + \frac{1}{4d_1}e^{4\sigma z(\theta_t\omega)}(||u||_{L^6}^4 ||v||_{L^6}^2 + ||w||_{L^6}^4 ||y||_{L^6}^2) \\
&\le d_1(||\Delta u||^2 + ||\Delta w||^2) + \frac{1}{2d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\left((||u||^4 + ||\nabla u||^4)||v||_{L^6}^2 + (||w||^4 + ||\nabla w||^4)||y||_{L^6}^2\right).
\end{aligned} \tag{74}$$

It follows from (72)-(74) that

$$\frac{d}{dt}(||\nabla u||^2 + ||\nabla w||^2) + 2(F + k - \sigma z(\theta_t\omega))(||\nabla u||^2 + ||\nabla w||^2)$$

$$
\begin{aligned}
&\le \frac{1}{d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\Big[(||u||^4+||\nabla u||^4)||v||^2_{L^6}+(||w||^4+||\nabla w||^4)||y||^2_{L^6}\Big]\\
&\le \frac{1}{d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\left(||u||^2+||w||^2\right)^2\left(||v||^2_{L^6}+||y||^2_{L^6}\right)\\
&\quad+\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\left(||v||^2_{L^6}+||y||^2_{L^6}\right)\left(||\nabla u||^2+||\nabla w||^2\right)^2. \qquad (75)
\end{aligned}
$$

Hence, for all $t \ge 0$

$$
\begin{aligned}
&\frac{d}{dt}(||\nabla u||^2+||\nabla w||^2)\\
&\le \left[\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\left(||v||^2_{L^6}+||y||^2_{L^6}\right)\left(||\nabla u||^2+||\nabla w||^2\right)-2(F+k-\sigma z(\theta_t\omega))\right]\left(||\nabla u||^2+||\nabla w||^2\right)\\
&\qquad+\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_t\omega)}\left(||u||^2+||w||^2\right)^2\left(||v||^2_{L^6}+||y||^2_{L^6}\right). \qquad (76)
\end{aligned}
$$

Replacing t by s, and replacing ω by $\theta_{-t-1}\omega$ in the last inequality, for every $s > 0$, we obtain that

$$
\begin{aligned}
&\frac{d}{ds}(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2)\\
&\le \Big[\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_{s-t-1}\omega)}\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right)\\
&\quad\times\left(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)-2(F+k-\sigma z(\theta_{s-t-1}\omega))\Big]\\
&\quad\times\left(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)\\
&\quad+\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_{s-t-1}\omega)}\left(||u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\\
&\quad\times\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right), \qquad (77)
\end{aligned}
$$

which can be rewritten as

$$\frac{d\rho}{ds}\le\beta(s)\rho(s)+\alpha(s), \qquad (78)$$

with

$$
\begin{aligned}
\rho(s)&=||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2;\\
\beta(s)&=\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_{s-t-1}\omega)}\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right)\\
&\quad\times\left(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)-2(F+k-\sigma z(\theta_{s-t-1}\omega));\\
\alpha(s)&=\frac{1}{d_1\eta^2}e^{4\sigma z(\theta_{s-t-1}\omega)}\left(||u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\\
&\quad\times\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right). \qquad (79)
\end{aligned}
$$

We use uniform Gronwall inequality to estimate

$$||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2.$$

We need to compute $\int_t^{t+1}\beta(s)ds$ and $\int_t^{t+1}\alpha(s)ds$ first. By (50) and (52), we have that, for $t > T_B(\omega)$

$$
\begin{aligned}
&||u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega)||^2\\
&=||u(s,\theta_{-s}(\theta_{s-t-1}\omega),g_0(\theta_{-s}(\theta_{s-t-1}\omega)))||^2+||w(s,\theta_{-s}(\theta_{s-t-1}\omega),g_0(\theta_{-s}(\theta_{s-t-1}\omega))||^2\\
&\le c_5\rho_0^2(\theta_{s-t-1}\omega), \qquad (80)
\end{aligned}
$$

and

$$||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega)||^2_{L^6}$$

$$= ||u(s, \theta_{-s}(\theta_{s-t-1}\omega), g_0(\theta_{-s}(\theta_{s-t-1}\omega)))||_{L^6}^2 + ||w(s, \theta_{-s}(\theta_{s-t-1}\omega), g_0(\theta_{-s}(\theta_{s-t-1}\omega)))||_{L^6}^2$$
$$\leq c_{11}\rho_1^{1/3}(\theta_{s-t-1}\omega). \tag{81}$$

It follows from (80) and (81) that

$$\int_t^{t+1} \alpha(s)ds \leq \frac{c_5^2 c_{11}}{d_1\eta^2} \int_t^{t+1} e^{4\sigma|z(\theta_{s-t-1}\omega)|}\rho_0^4(\theta_{s-t-1}\omega)\rho_1^{1/3}(\theta_{s-t-1}\omega)ds$$
$$\leq \frac{c_5^2 c_{11}}{d_1\eta^2} \max_{-1\leq\tau\leq 0}\left[e^{4\sigma|z(\theta_\tau\omega)|}\rho_0^2(\theta_\tau\omega)\rho_1^{1/3}(\theta_\tau\omega)\right] \equiv M_1(\omega). \tag{82}$$

Next, we use Lemma 4.3 to estimate $\int_t^{t+1}\beta(s)ds$.

$$\int_t^{t+1} \beta(s)ds \leq \frac{c_5 c_{11}}{d_1\eta^2} \max_{-1\leq\tau\leq 0}\left[e^{4\sigma|z(\theta_\tau\omega)|}\rho_0^2(\theta_\tau\omega)\rho_1^{1/3}(\theta_\tau\omega)\right] \int_t^{t+1} (||\nabla u(s, \theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 +$$
$$||\nabla w(s, \theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 ds + 2(F + k + \sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|)$$
$$\leq \frac{c_5 c_{11}}{d_1\eta^2} \max_{-1\leq\tau\leq 0}\left[e^{4\sigma|z(\theta_\tau\omega)|}\rho_0^2(\theta_\tau\omega)\rho_1^{1/3}(\theta_\tau\omega)\right]\rho_2(\theta_\tau\omega)$$
$$+2(F + k + \sigma \max_{-1\leq\tau\leq 0}|z(\theta_\tau\omega)|) \equiv M_2(\omega). \tag{83}$$

Therefore, applying uniform Gronwall inequality, we can get that

$$||\nabla u(t+1, \theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2 + ||\nabla w(t+1, \theta_{-t-1}\omega, g_0(\theta_{-t-1}\omega))||^2$$
$$\leq (M_1(\omega) + \rho_2(\omega))e^{M_2(\omega)} \equiv \rho_3(\omega). \tag{84}$$

This ends the proof. □

Lemma 3.5. *There exists a random variable $\rho_4(\omega)$, such that, for any $B(\omega) \in \mathcal{D}$, and $g_0(\omega) \in B(\omega)$, for $\mathbb{P}-a.e.\ \omega \in \Omega$, there is a $T_B(\omega) > 0$, such that, for all $t \geq T_B(\omega)$, the following estimate holds*

$$||\nabla v(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 + ||\nabla y(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))||^2 \leq \rho_4(\omega). \tag{85}$$

Proof. Taking the inner products of (15) and (17) with $-\Delta v$ and $-\Delta y$ respectively, and summing these two equations up, one has that

$$\frac{1}{2}\frac{d}{dt}(||\nabla v||^2 + ||\nabla y||^2) + d_2(||\Delta v||^2 + ||\Delta y||^2)$$
$$= -\int_D Fe^{-\sigma z(\theta_t\omega)}(\Delta v + \Delta y)dx + (\sigma z(\theta_t\omega) - F)(||\nabla v||^2 + ||\nabla y||^2)$$
$$+e^{2\sigma z(\theta_t\omega)}\int_D u^2 v\Delta v + w^2 y\Delta y dx - D_2\int_D (y - v)\Delta(v - y)dx. \tag{86}$$

Now we estimate each term on the right hand side of (86). For the first and fourth term, by Green's formula,

$$\int_D Fe^{-\sigma z(\theta_t\omega)}(\Delta v + \Delta y)dx = 0,$$
$$-D_2\int_D (y - v)\Delta(v - y)dx = -\int_D ||\nabla(v - y)||^2 dx \leq 0.$$

As for the third term,

$$e^{2\sigma z(\theta_t\omega)}\int_D u^2 v\Delta v + w^2 y\Delta y dx \leq d_2(||\Delta v||^2 + ||\Delta y||^2) + \frac{e^{4\sigma z(\theta_t\omega)}}{4d_2}\int_D u^4 v^2 + w^4 y^2 dx. \tag{87}$$

It follows from Young's inequality that

$$\begin{aligned}
&\frac{d}{dt}(||\nabla v||^2+||\nabla y||^2)+2(F-\sigma z(\theta_t\omega))(||\nabla v||^2+||\nabla y||^2)\\
&\le \frac{e^{4\sigma z(\theta_t\omega)}}{2d_2}(||u||^4_{L^6}||v||^2_{L^6}+||w||^4_{L^6}||y||^2_{L^6})\\
&\le \frac{e^{4\sigma z(\theta_t\omega)}}{2d_2\eta^2}(||u||^4_{H^1}||v||^2_{L^6}+||w||^4_{H^1}||y||^2_{L^6})\\
&\le \frac{e^{4\sigma z(\theta_t\omega)}}{d_2\eta^2}\left[(||u||^4+||\nabla u||^4)||v||^2_{L^6}+(||w||^4+||\nabla w||^4)||y||^2_{L^6}\right]\\
&\le \frac{e^{4\sigma z(\theta_t\omega)}}{d_2\eta^2}\left[(||u||^2+||w||^2)^2+(||\nabla u||^2+||\nabla w||^2)^2\right](||v||^2_{L^6}+||y||^2_{L^6}).
\end{aligned}\tag{88}$$

Replacing t by s and replacing ω by $\theta_{-t-1}\omega$ in the last inequality, we obtain:

$$\begin{aligned}
&\frac{d}{dt}\left(||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)\\
&\quad+2\left(F-\sigma z(\theta_{s-t-1}\omega)\right)\left(||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)\\
&\le \frac{e^{4\sigma z(\theta_{s-t-1}\omega)}}{d_2\eta^2}\Big[\left(||u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\\
&\quad+\left(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\Big]\\
&\quad\times\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right).
\end{aligned}$$

It can be rewritten as

$$\frac{d\rho}{ds}\le \beta(s)\rho(s)+\alpha(s),\tag{89}$$

with

$$\begin{aligned}
\rho(s)&=||\nabla v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2;\\
\beta(s)&=-2(F-\sigma z(\theta_{s-t-1}\omega));\\
\alpha(s)&=\frac{e^{4\sigma z(\theta_{s-t-1}\omega)}}{d_2\eta^2}\Big[\left(||u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\\
&\quad+\left(||\nabla u(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla w(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\right)^2\Big]\\
&\quad\times\left(||v(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}+||y(s,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2_{L^6}\right).
\end{aligned}\tag{90}$$

Similarly to Lemma 4.4, we need to estimate $\int_t^{t+1}\alpha(s)ds$, $\int_t^{t+1}\beta(s)ds$ and $\int_t^{t+1}\rho(s)ds$.

$$\int_t^{t+1}\beta(s)ds\le 2(F+\sigma\max_{-1\le\tau\le 0}|z(\theta_\tau\omega)|)\equiv M_3(\omega).\tag{91}$$

By (80), (81) and Lemma 4.4, we have that, for $t>T_B(\omega)$,

$$\alpha(s)\le\frac{c_{11}}{d_2\eta^2}e^{4\sigma|z(\theta_{s-t-1}\omega)|}\left[c_5^2\rho_0^4(\theta_{s-t-1}\omega)+\rho_3^2(\theta_{s-t-1}\omega)\right]\rho_1^{1/3}(\theta_{s-t-1}\omega).\tag{92}$$

It follows that

$$\int_t^{t+1}\alpha(s)ds\le\frac{c_{11}}{d_2\eta^2}\max_{-1\le\tau\le 0}\left[e^{4\sigma|z(\theta_\tau\omega)|}(c_5^2\rho_0^4(\theta_\tau\omega)+\rho_3^2(\theta_\tau\omega))\rho_1^{1/3}(\theta_\tau\omega)\right]\equiv M_4(\omega).\tag{93}$$

By using uniform Gronwall inequality and (66), (93) and (91), we get that

$$\begin{aligned}
&||\nabla v(t+1,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2+||\nabla y(t+1,\theta_{-t-1}\omega,g_0(\theta_{-t-1}\omega))||^2\\
&\le(M_4(\omega)+\rho_2(\omega))\,e^{M_3(\omega)}\equiv\rho_4(\omega).
\end{aligned}\tag{94}$$

This ends the proof. □

4 Existence of random attractors

In this section we use Proposition 2.1 to prove the existence of a pullback attractor.

Theorem 4.1. *The random dynamical system ϕ has a unique $\mathcal{D}-$pullback random attractor in $[L^2(D)]^4$.*

Proof. In Lemma 3.1, we find that the system has a bounded absorbing set. By Lemma 3.4 and Lemma 3.5, we know that for any $g_0(\theta_{-t}\omega) \in B_0(\theta_{-t}\omega)$, the weak solution $g(t, \theta_{-t}\omega, g_0(\theta_{-t}\omega))$ is bounded in $[H^1(D)]^4$. Since the embedding $[H^1(D)]^4 \hookrightarrow [L^2(D)]^4$ is a compact mapping, This shows that the random dynamical system φ is asymptotically compact. Hence, by Proposition 2.1, we obtain the existence of a unique $\mathcal{D}$-pullback random attractor $\{A_1(\omega)\}_{\omega\in\Omega}$ for φ in $[L^2(D)]^4$.

Since ϕ and φ are conjugated by the random homeomorphism $T(\omega, \xi) = e^{\sigma z(\theta_t\omega)}\xi(\omega)$, then by Proposition 1.8.3 in [17], ϕ has a unique $\mathcal{D}$-random attractor $\{A_2(\omega)\}_{\omega\in\Omega}$ in $[L^2(D)]^4$ which is given by

$$A_2(\omega) = \{e^{\sigma z(\theta_t\omega)}\xi(\omega) : \xi(\omega) \in A_1(\omega)\}. \tag{95}$$

This completes the proof. □

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgement: This work is partially supported by the National Natural Science Foundation of China under Grants 11301153, 11271110 and 11201123, the Key Programs for Science and Technology of the Education Department of Henan Province under Grand 12A110007, and the Scientific Research Funds of Henan University of Science and Technology.

References

[1] Arnold L., Random Dynamical Systems, Springer-Verlag, New York, 1998

[2] Hale J.K., Troy W.C., Exact homoclinic and heteroclinic solutions of the Gray-Scott model for autocalysis, SIAM J. Appl. Math, 2000, 61, 102-130

[3] Pang P., Wang M., Non-constant positive stady states of a predator-prey system with non-monotonic functional response and diffusion, Proc London Math Soc, 2004, 1(1), 135-157

[4] Pang R., Wang M., Positive steady-state solutions of the Noyes-Field model for Belousov-Zhabotinskii reaction, Nonlinear Anal., 2004, 56(3), 451-464

[5] Wang M., Non-constant positive steady-states of the Sel'kov model, J. Differ. Equat., 2003, 190, 600-620

[6] Gu A., Xiang H., Upper semicontinuity of random attractors for stochastic three-component reversible Gray-Scott system, App. Math. Comp., 2013, 225(12), 387-400

[7] Gu A., Random Attractors of Stochastic Three-Component Reversible Gray-Scott System on Unbounded Domains, Abst. Appl. Anal., 2012, 7, 1065-1076

[8] Mahara H., Suzuki K.., Jahan RA., Yamaguchi T., Coexisting stable patterns in a reaction-diffusion system with reversible gray-scott dynamics, Phys. Rev. E, 2008, 78(6), 837-849

[9] You Y., Dynamics of three-component reversible Gray-Scott model. Disc. Cont. Dynam. Syst., 2010, 14(4), 1671-1688

[10] You Y., Dynamics of two-compartment Gray-Scott equations. Nonlin. Anal., 2011, 74(5), 1969-1986

[11] Bates P.W., Lisei H., Lu K., Attractors for stochastic lattics dynamical systems, Stoch. Dyn., 2011, 6, 1-21

[12] Caraballo T., Langa J.A., Robinson J.C., Upper semicontinuity of attractors for small random perturbations of dynamical systems, Comm.Part.Diff.Equa., 2002, 23(9), 1557-1581

[13] Fan X., Attractors for a damped stochastic wave equation of sina-Gordon type with sublinear multiplicative noise, Stoc. Anal. Appl. 2006, 24(4), 767-793

[14] Hale J.K., Lin X., Raugel G., Upper semicontinuity of attractors for approximations of semigroups and PDE's, Math. Comp., 1988, 50(181), 89-123

[15] Hale J.K., Raugel G., Lower semicontinuity of the attractor for a singularly perturbed hyperbolic equation, J. Diff. Equa., 1989, 2(1), 16-97

[16] You Y., Global dynamics of an autocatalytic reaction-diffusion system with functional response. J. App. Anal. Comp., 2011, 1(1), 121-142
[17] Chueshov I., Monotone Random Systems Theory and Applications, Springer Berlin Heidelberg, 2002.
[18] Crauel H., Debussche A., Flandoli F., Random attractors, J. Dyn. Diff. Equs. 1997, 9(2), 307-341.
[19] Flandoli F., Schmalfuβ B., Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative noise, Stoch. Rep., 2007, 59(1), 21-45
[20] Wang Z., Zhou S., Random attractor for stochastic reaction-diffusion equation with multiplicative noise on unbounded domains, J. Math. Anal. Appl., 2011, 384(1), 160-172
[21] Wang B., Upper semicontinuity of random attractors for non-compact random dynamical systems. J. Diff. Equa., 2009, 139, 1206-1219