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# The fuzzy metric space based on fuzzy measure

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**Abstract:** In this paper, we study the relation between a fuzzy measure and a fuzzy metric which is induced by the fuzzy measure. We also discuss some basic properties of the constructed fuzzy metric space. In particular, we show that the nonatom of fuzzy measure space can be characterized in the constructed fuzzy metric space.

**Keywords:** Fuzzy measure, Fuzzy metric space,  $t$ -Norm, Nonatom, Convex

**MSC:** 54E70, 54A40, 60B05

## 1 Introduction

The theory of metric space is an important topic in topology. The methods of constructing a fuzzy metric have been extensively studied [1–4]. It is worth noting that George and Veeramani [5] introduced the concept of a fuzzy metric with the help of continuous  $t$ -norms. Despite being restrictive, this kind of fuzzy metric provides a more natural and intuitive way to connect with the metrizable topological spaces. This concept has been widely used in various papers devoted to fuzzy topology [5–11]. It also has been applied to color image filtering to improve image quality (see [9] and the references therein).

On the other hand, measure theory is one of the most important theories in mathematics and it has been extensively studied. The concept of fuzzy measure was first introduced by Sugeno [12]. It can be regarded as an extension of classical measure in which the additivity is replaced by a weaker condition, monotonicity. Klement et. al establish the axiomatic theory of fuzzy  $\sigma$ -algebras and develop a measure theory of fuzzy sets [13–15]. So far, there are many different classes of fuzzy measures, such as possibility measure [16, 17], decomposable measure [18–20], pseudo-additive measure [21, 22], and  $T$ -measure [23–26] etc. A systematic study of fuzzy measure theory can be found in [27–30].

Recently, the study of constructing a fuzzy metric using a fuzzy measure technique has been actively pursued. In particular, a fuzzy Prokhorov metric and ultrametric defined on the set of all probability measures in a compact fuzzy metric space have been developed in [31, 32]. Cao et. al [33] introduce fuzzy analogue of the Kantorovich metric among the set of possibility distributions. In [34, 35], the authors discuss the relations between the decomposable measure and the fuzzy metric. More specifically, it has been proven that, with a Hausdorff fuzzy pseudo-metric constructed on its power set, a stationary fuzzy ultrametric space can induce a  $\sigma$ - $\perp$ -superdecomposable measure. The authors of [35] further use a topological approach to extend the  $t$ -conorm-based decomposable measures by introducing a fuzzy pseudometric structure on an algebra of sets. In [36] we constructed a pseudo-metric (in the

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sense of Pap) on the measurable sets of a given  $\sigma$ - $\perp$ -decomposable measure, and then analyzed the connection between the induced pseudo-metric and the  $\sigma$ - $\perp$ -decomposable measure.

In this paper we focus on the following problems: how to construct a fuzzy metric by using a fuzzy measure developed in [14, 15] and what is the relation between these two? And what is the relations between these two? Specifically, by introducing the concept of an equivalence relation on fuzzy measurable sets, we construct a fuzzy metric on the associated quotient sets from a given fuzzy measure. Furthermore, we study some basic properties of the constructed fuzzy metric space such as completeness and continuity. To gain better insight into our proposed method of constructing a fuzzy metric, we study the properties of the constructed fuzzy metric which can precisely reflect those of fuzzy measure. As an illustration we obtain that the nonatom of fuzzy measure space can be characterized in the constructed fuzzy metric space.

The rest of the paper is organized as follows. In Section 2, some basic notions and results are given. Sections 3 and 4 are devoted to constructing a fuzzy metric and discussing its properties. In Section 5, we discuss the relationships between the constructed fuzzy metric and the fuzzy measure. Finally, some concluding remarks are given in Section 6.

## 2 Preliminaries

We start this section by recalling the concept of triangular norms from [20, 37]. They are an important tool in extending the classical metric space to fuzzy metric space.

**Definition 2.1** (Klement et al.). A function  $\top : [0, 1]^2 \rightarrow [0, 1]$  is called triangular norm ( $t$ -norm for short) if it satisfies the following conditions for all  $a, b, c, d \in [0, 1]$ :

- (T1)  $a \top 1 = a$ . (boundary condition)
- (T2)  $a \top b \leq c \top d$  whenever  $a \leq c$  and  $b \leq d$ . (monotonicity)
- (T3)  $a \top b = b \top a$ . (commutativity)
- (T4)  $a \top (b \top c) = (a \top b) \top c$ . (associativity)

A  $t$ -Norm  $\top$  is said to be *continuous* if it is a continuous function in  $[0, 1]^2$ . Typical examples of continuous  $t$ -Norms are the minimum  $T_M$ , the product  $T_P$  and the Łukasiewicz  $t$ -norm  $T_L$ , which are given by, respectively:

$$T_M(x, y) = \min(x, y), \quad T_P(x, y) = x \cdot y, \quad T_L(x, y) = \max(0, x + y - 1).$$

Because of the associative property, the  $t$ -Norm  $\top$  can be extended by induction to  $n$ -ary operation by setting

$$\bigtop_{i=1}^n x_i = \left( \bigtop_{i=1}^{n-1} x_i \right) \top x_n.$$

Due to monotonicity, for each sequence  $(x_i)_{i \in \mathbb{N}}$  of elements of  $[0, 1]$ , the following limit can be considered:

$$\bigtop_{i=1}^{\infty} x_i = \lim_{n \rightarrow \infty} \bigtop_{i=1}^n x_i.$$

Next we recall the concept of a fuzzy metric with the help of the continuous  $t$ -norm, which is a generalization of the concept of Menger probabilistic metric to the fuzzy setting.

**Definition 2.2** (George and Veeramani [5]). The 3-tuple  $(X, M, \top)$  is said to be a fuzzy metric space if  $X$  is an arbitrary nonempty set,  $\top$  is a continuous  $t$ -norm and  $M$  is a fuzzy set on  $X^2 \times (0, \infty)$  satisfying the following conditions, for all  $x, y, z \in X, t, s > 0$ :

- (GV1)  $M(x, y, t) > 0$ ,
- (GV2)  $M(x, y, t) = 1$  iff  $x = y$ ,
- (GV3)  $M(x, y, t) = M(y, x, t)$ ,
- (GV4)  $M(x, z, t + s) \geq M(x, y, t) \top M(y, z, s)$ ,
- (GV5)  $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$  is continuous.

If the condition (GV2) is replaced by the condition (GVp):  $M(x, x, t) = 1$ , then  $(X, M, \top)$  is said to be a fuzzy pseudometric space.

It was proved in [5] that in a fuzzy metric space  $X$ , the function  $M(x, y, \cdot)$  is nondecreasing for all  $x, y \in X$ . A sequence  $(x_i)_{i \in \mathbb{N}}$  in a fuzzy metric space  $(X, M, \top)$  is said to converge [6] to  $x$  if  $\lim_{i \rightarrow \infty} M(x_i, x, t) = 1$  for all  $t > 0$ ; a sequence  $(x_i)_{i \in \mathbb{N}}$  in a fuzzy metric space  $(X, M, \top)$  is said to be Cauchy [6] if  $\lim_{i, j \rightarrow \infty} M(x_i, x_j, t) = 1$  for all  $t > 0$ ;  $(X, M, \top)$  is said to be complete [8] if every Cauchy sequence is convergent. A mapping  $f$  from a fuzzy metric space  $(X, M, \top_1)$  to a fuzzy metric space  $(Y, N, \top_2)$  is called uniformly continuous [7] if for each  $\varepsilon \in (0, 1)$  and each  $t > 0$ , there exist  $r \in (0, 1)$  and  $s > 0$  such that  $N(f(x), f(y), t) > 1 - \varepsilon$  whenever  $M(x, y, s) > 1 - r$ .

In this follow-up, we recall several concepts from the measure theory of fuzzy sets (see e.g. [13–15]).

**Definition 2.3.** Let  $X$  be a nonempty set,  $I$  the unit interval  $[0, 1]$ . A subset  $\mathcal{A}$  of  $I^X$  is a fuzzy  $\sigma$ -algebra iff

- (A1)  $\mathbf{0}, \mathbf{1} \in \mathcal{A}$ ,
- (A2)  $A \in \mathcal{A}$  implies  $\mathbf{1} - A \in \mathcal{A}$ ,
- (A3) if  $\{A_i\}_{i=1}^\infty$  is a sequence in  $\mathcal{A}$ , then  $\bigvee_{i=1}^\infty A_i = \sup A_i \in \mathcal{A}$ .

**Definition 2.4.** A finite fuzzy measure (or  $F$ -measure) on a fuzzy  $\sigma$ -algebra  $\mathcal{A}$  is a function  $\mu : \mathcal{A} \rightarrow [0, \infty)$  satisfying:

- (M1)  $\mu(\mathbf{0}) = 0$ ,
- (M2) for  $A \in \mathcal{A}$ ,  $\mu(\mathbf{1} - A) = \mu(\mathbf{1}) - \mu(A)$ ,
- (M3) for  $A, B \in \mathcal{A}$ ,  $\mu(A \vee B) + \mu(A \wedge B) = \mu(A) + \mu(B)$ ,
- (M4) if  $\{A_i\}_{i=1}^\infty$  is a sequence in  $\mathcal{A}$  such that  $A_i \nearrow A$ ,  $A \in \mathcal{A}$ , then  $\mu(A) = \sup \mu(A_i)$ .

We call  $(X, \mathcal{A}, \mu)$  an  $F$ -measure space, elements of  $\mathcal{A}$  are referred as fuzzy measurable sets.

### 3 Constructing fuzzy metric based on $F$ -measure

In this follow-up,  $\top$  stands for the minimum  $t$ -norm  $T_M$ . The following result is the natural fuzzy metric structure on fuzzy measurable sets.

**Theorem 3.1.** Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space. If we define the fuzzy set  $M$  on  $\mathcal{A}^2 \times (0, \infty)$  by

$$M(A, B, t) = \frac{t}{t + \mu(A \vee B) - \mu(A \wedge B)},$$

where  $A, B \in \mathcal{A}$ . Then  $M$  is a fuzzy pseudometric on  $\mathcal{A}$ .

*Proof.* From the definition of  $M$ , it is obvious that for any  $A, B \in \mathcal{A}$ ,  $t > 0$ , we have (i)  $M(A, B, t) > 0$ ; (ii)  $M(A, A, t) = 1$  and  $M(A, B, t) = M(B, A, t)$ ; (iii)  $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$  is continuous. The only thing that we need to prove is the triangular inequality. For any  $A, B, C \in \mathcal{A}$  and  $t, s > 0$ , we have  $M(A, B, t) = \frac{t}{t + \mu(A \vee B) - \mu(A \wedge B)}$ ,  $M(B, C, s) = \frac{s}{s + \mu(B \vee C) - \mu(B \wedge C)}$  and  $M(A, C, t + s) = \frac{t + s}{t + s + \mu(A \vee C) - \mu(A \wedge C)}$ . We are going to verify  $M(A, C, t + s) \geq M(A, B, t) \top M(B, C, s)$ . For each  $A, B, C \in \mathcal{A}$ , we have

$$\begin{aligned} \mu(A \vee B) + \mu(B \vee C) + \mu(A \wedge C) &= \mu(A \vee B \vee C) + \mu((A \vee B) \wedge (B \vee C)) + \mu(A \wedge C) \\ &= \mu(A \vee B \vee C) + \mu(B \vee (A \wedge C)) + \mu(A \wedge C) \\ &\geq \mu(A \vee C) + \mu(B \wedge ((A \wedge B) \vee C)) + \mu(A \wedge B \wedge C) \\ &= \mu(A \vee C) + \mu(A \wedge B) + \mu(B \wedge C) \end{aligned}$$

and so

$$\mu(A \vee C) - \mu(A \wedge C) \leq \mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee C) - \mu(B \wedge C)$$

we can distinguish two cases:  $M(B, C, s) \geq M(A, B, t)$  or  $M(A, B, t) \geq M(B, C, s)$ .

Case (i). If  $M(B, C, s) \geq M(A, B, t)$ , or equivalently,

$$\frac{s}{s + \mu(B \vee C) - \mu(B \wedge C)} \geq \frac{t}{t + \mu(A \vee B) - \mu(A \wedge B)}$$

and hence

$$s(\mu(A \vee B) - \mu(A \wedge B)) \geq t(\mu(B \vee C) - \mu(B \wedge C)).$$

Consequently,

$$\begin{aligned} t(\mu(A \vee C) - \mu(A \wedge C)) &\leq t(\mu(A \vee B) - \mu(A \wedge B)) + t(\mu(B \vee C) - \mu(B \wedge C)) \\ &\leq t(\mu(A \vee B) - \mu(A \wedge B)) + s(\mu(A \vee B) - \mu(A \wedge B)). \end{aligned}$$

This implies that

$$\frac{t}{t + \mu(A \vee B) - \mu(A \wedge B)} \leq \frac{t + s}{t + s + \mu(A \vee C) - \mu(A \wedge C)},$$

and hence

$$M(A, C, t + s) \geq M(A, B, t) = M(A, B, t) \top M(B, C, s).$$

Case (ii). Similar to (i). Thus,  $M$  is a fuzzy pseudometric on  $\mathcal{A}$ .  $\square$

**Remark 3.2.** Based on Theorem 3.1,  $M$  is a fuzzy pseudometric on  $\mathcal{A}$  under any left-continuous  $t$ -norms since  $T_M$  is the biggest left-continuous  $t$ -norm.

Generally speaking, the fuzzy pseudometric space  $(\mathcal{A}, M, \top)$  is usually not a fuzzy metric space. But we can construct a fuzzy metric space from a fuzzy pseudometric metric space  $(\mathcal{A}, M, \top)$  and, at the same time, keep the general characteristics of the fuzzy pseudometric metric space.

**Lemma 3.3.** Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space. For each  $A, B \in \mathcal{A}$ , we define the relation “ $\sim$ ” on  $\mathcal{A} : A \sim B$  if and only if  $\mu(A \vee B) = \mu(A \wedge B)$ . Then  $\sim$  is an equivalence relation on  $\mathcal{A}$ .

*Proof.* The identity  $\mu(A \vee C) - \mu(A \wedge C) \leq \mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee C) - \mu(B \wedge C)$ , which was proved in Theorem 3.1, and the monotonicity of  $\mu$  imply that  $\sim$  is transitive, and it is clearly reflexive and symmetric.  $\square$

**Theorem 3.4.** Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space. Let  $\mathcal{A}/\mu$  be the set of all equivalence classes for the relation “ $\sim$ ”. The fuzzy pseudometric  $M$  has a natural extension to  $\mathcal{A}/\mu \times \mathcal{A}/\mu$  :

$$M([A], [B], t) = M(A, B, t),$$

where  $[A]$  ( $[B]$ ) denote the equivalence class of  $A$  ( $B$ ). Then  $M$  is a fuzzy metric on  $\mathcal{A}/\mu$ .

*Proof.* We first prove that  $M([A], [B], t)$  is well defined on  $\mathcal{A}/\mu$ . If  $A_1 \in [A]$  and  $B_1 \in [B]$ , we conclude that  $\mu(A_1 \vee B_1) - \mu(A_1 \wedge B_1) = \mu(A \vee B) - \mu(A \wedge B)$ . In fact,

$$\mu(A_1 \vee B_1) - \mu(A_1 \wedge B_1) \leq \mu(A_1 \vee A) - \mu(A_1 \wedge A) + \mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee B_1) - \mu(B \wedge B_1).$$

Since  $A_1 \in [A]$  and  $B_1 \in [B]$ , we have  $\mu(A_1 \vee A) - \mu(A_1 \wedge A) = \mu(B \vee B_1) - \mu(B \wedge B_1) = 0$ , then  $\mu(A_1 \vee B_1) - \mu(A_1 \wedge B_1) \leq \mu(A \vee B) - \mu(A \wedge B)$ . In a similar way that  $\mu(A \vee B) - \mu(A \wedge B) \leq \mu(A_1 \vee B_1) - \mu(A_1 \wedge B_1)$  holds. We obtain  $M(A_1, B_1, t) = M(A, B, t)$ , which shows that  $M([A], [B], t)$  does not depend on our choice of representatives in the equivalence classes and is well-defined on  $\mathcal{A}/\mu$ . Secondly, we need to prove that  $M$  is a fuzzy metric on  $\mathcal{A}/\mu$ . It is obvious that  $M([A], [B], t) = 1 \Leftrightarrow \mu(A \vee B) - \mu(A \wedge B) = 0 \Leftrightarrow [A] = [B]$ . Thus,  $M$  is a fuzzy metric on  $\mathcal{A}/\mu$ .  $\square$

The following Lemma shows that the collection of equivalence class  $\mathcal{A}/\mu$  forms a fuzzy  $\sigma$ -algebra.

**Lemma 3.5.** Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space. Then for  $A_i, B_i \in \mathcal{A}$ ,  $A_i \sim B_i$ ,  $i \in \mathbb{N}$ , we have

- (i)  $\mathbf{1} - A_i \sim \mathbf{1} - B_i$ ;
- (ii)  $A_1 \vee A_2 \sim B_1 \vee B_2$  and  $A_1 \wedge A_2 \sim B_1 \wedge B_2$ ;
- (iii)  $\bigwedge_{i=1}^{\infty} A_i \sim \bigwedge_{i=1}^{\infty} B_i$ .

*Proof.* (i) For  $A_i, B_i \in \mathcal{A}$ ,  $A_i \sim B_i$ , then  $\mu(A_i \vee B_i) - \mu(A_i \wedge B_i) = 0$ . Hence,  $\mathbf{1} - A_i \sim \mathbf{1} - B_i$  holds by  $\mu((\mathbf{1} - A_i) \vee (\mathbf{1} - B_i)) - \mu((\mathbf{1} - A_i) \wedge (\mathbf{1} - B_i)) = \mu(A_i \vee B_i) - \mu(A_i \wedge B_i) = 0$ .

(ii) We first show that for each  $P \in \mathcal{A}$ ,  $A_1 \vee P \sim B_1 \vee P$ . Since  $A_1 \sim B_1$ , we have

$$\begin{aligned} \mu(A_1 \vee P) &\leq \mu(A_1 \vee B_1 \vee P) \\ &= \mu(A_1 \vee B_1) + \mu(P) - \mu((A_1 \vee B_1) \wedge P) \\ &\leq \mu(B_1) + \mu(P) - \mu(B_1 \wedge P) \\ &= \mu(B_1 \vee P). \end{aligned}$$

Similarly,

$$\mu(B_1 \vee P) \leq \mu(A_1 \vee B_1 \vee P) \leq \mu(A_1 \vee P).$$

Consequently,

$$\mu(A_1 \vee P) = \mu(A_1 \vee B_1 \vee P) = \mu(B_1 \vee P).$$

Thus we obtain

$$\mu(A_1 \vee A_2) = \mu(B_1 \vee B_2) = \mu(A_1 \vee A_2 \vee B_1 \vee B_2).$$

This implies that  $A_1 \vee A_2 \sim B_1 \vee B_2$ . Similarly, we have  $A_1 \wedge A_2 \sim B_1 \wedge B_2$ .

(iii) For  $i \in \mathbb{N}$ , we have  $\mu(A_i \vee B_i) = \mu(A_i \wedge B_i)$ , then  $\mu(A_i) = \mu(B_i) = \mu(A_i \wedge B_i)$ . Hence

$$\mu\left(\bigwedge_{i=1}^{\infty} A_i\right) = \lim_{k \rightarrow \infty} \mu\left(\bigwedge_{i=1}^k A_i\right) = \lim_{k \rightarrow \infty} \mu\left(\bigwedge_{i=1}^k B_i\right) = \mu\left(\bigwedge_{i=1}^{\infty} B_i\right).$$

Also,

$$\begin{aligned} \mu\left(\bigwedge_{i=1}^{\infty} A_i\right) &= \lim_{k \rightarrow \infty} \mu\left(\bigwedge_{i=1}^k A_i\right) \\ &= \lim_{k \rightarrow \infty} \mu\left(\bigwedge_{i=1}^k (A_i \wedge B_i)\right) \\ &= \mu\left(\bigwedge_{i=1}^{\infty} (A_i \wedge B_i)\right) \\ &= \mu\left(\left(\bigwedge_{i=1}^{\infty} A_i\right) \wedge \left(\bigwedge_{i=1}^{\infty} B_i\right)\right). \end{aligned}$$

Thus,  $\bigwedge_{i=1}^{\infty} A_i \sim \bigwedge_{i=1}^{\infty} B_i$ . □

**Lemma 3.6.** Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space. If, for  $A_i, B_i \in \mathcal{A}$ ,  $A_i \sim B_i, i \in \mathbb{N}$ , and  $A_i \nearrow A, B_i \nearrow B$ , then  $A \sim B$ .

*Proof.* The proof is straightforward. □

According to Lemma 3.5 and Lemma 3.6, by means of representatives of classes, we can introduce the operations of union, intersection and complementation on  $\mathcal{A}/\mu$ :  $\bigvee_{i=1}^{\infty} [A_i] = [\bigvee_{i=1}^{\infty} A_i]$ ;  $\bigwedge_{i=1}^{\infty} [A_i] = [\bigwedge_{i=1}^{\infty} A_i]$ ;  $\mathbf{1} - [A_i] = [\mathbf{1} - A_i]$ , where  $[A_i] \in \mathcal{A}/\mu$  denote the equivalence class of  $A_i$  in  $\mathcal{A}$ . Hence,  $\mathcal{A}/\mu$  is a fuzzy  $\sigma$ -algebra. We therefore properly define  $\mu$  on  $\mathcal{A}/\mu$  by setting

$$\mu([A]) = \mu(A), \text{ for all } A \in \mathcal{A}.$$

The pair  $(\mathcal{A}/\mu, \mu)$  is said to be an  $F$ -measure algebra.

For convenience and simplicity, we denote members  $[A]$  of  $\mathcal{A}/\mu$  by  $A$ , and functions  $\mu: \mathcal{A}/\mu \rightarrow [0, \infty)$  by  $\mu: \mathcal{A} \rightarrow [0, \infty)$ .

## 4 Properties of the fuzzy metric space $(\mathcal{A}, M, \top)$

In this section, we study some properties of the fuzzy metric space  $(\mathcal{A}, M, \top)$  based on  $F$ -measure  $\mu$ .

**Theorem 4.1.** *Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space and  $M$  be the fuzzy metric on  $\mathcal{A}$  defined in Theorem 3.4. Then the maps (i)  $(A, B) \mapsto A \vee B$  and (ii)  $(A, B) \mapsto A \wedge B$  are uniformly continuous from  $\mathcal{A} \times \mathcal{A}$  to  $\mathcal{A}$ .*

*Proof.* (i) For any  $A_1, B_1, A_2, B_2 \in \mathcal{A}$ ,  $t, s > 0$ , we first prove the relation  $M(A_1, A_2, t) \top M(B_1, B_2, s) \leq M(A_1 \vee B_1, A_2 \vee B_2, t + s)$ . For each  $A_1, B_1, A_2, B_2 \in \mathcal{A}$ , we have

$$\begin{aligned} & \mu((A_1 \vee B_1) \vee (A_2 \vee B_2)) - \mu((A_1 \vee B_1) \wedge (A_2 \vee B_2)) \\ & \leq \mu((A_1 \vee A_2) \vee (B_1 \vee B_2)) - \mu((A_1 \wedge A_2) \vee (B_1 \wedge B_2)) \\ & = \mu(A_1 \vee A_2) + \mu(B_1 \vee B_2) - \mu((A_1 \vee A_2) \wedge (B_1 \vee B_2)) \\ & \quad - \mu(A_1 \wedge A_2) - \mu(B_1 \wedge B_2) + \mu((A_1 \wedge A_2) \wedge (B_1 \wedge B_2)) \\ & \leq \mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2) + \mu(B_1 \vee B_2) - \mu(B_1 \wedge B_2). \end{aligned}$$

we can distinguish two cases:  $M(A_2, B_2, s) \geq M(A_1, B_1, t)$  or  $M(A_1, B_1, t) \geq M(A_2, B_2, s)$ .

Case one. If  $M(B_1, B_2, s) \geq M(A_1, A_2, t)$ , or equivalently,

$$\frac{s}{s + \mu(B_1 \vee B_2) - \mu(B_1 \wedge B_2)} \geq \frac{t}{t + \mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)}$$

and hence

$$s(\mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)) \geq t(\mu(B_1 \vee B_2) - \mu(B_1 \wedge B_2))$$

In consequence,

$$\begin{aligned} & t(\mu((A_1 \vee B_1) \vee (A_2 \vee B_2)) - \mu((A_1 \vee B_1) \wedge (A_2 \vee B_2))) \\ & \leq t(\mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)) + t(\mu(B_1 \vee B_2) - \mu(B_1 \wedge B_2)) \\ & \leq t(\mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)) + s(\mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)) \end{aligned}$$

This implies that

$$\frac{t}{t + \mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2)} \leq \frac{t + s}{t + s + \mu((A_1 \vee B_1) \vee (A_2 \vee B_2)) - \mu((A_1 \vee B_1) \wedge (A_2 \vee B_2))}$$

and hence

$$M(A_1 \vee B_1, A_2 \vee B_2, t + s) \geq M(A_1, A_2, t) = M(A_1, A_2, t) \top M(B_1, B_2, s).$$

Case two. Similar to case one.

Thus,  $M(A_1 \vee B_1, A_2 \vee B_2, t + s) \geq M(A_1, A_2, t) \top M(B_1, B_2, s)$ .

If we fix  $\varepsilon \in (0, 1)$ , there exists  $\delta \in (0, 1)$  such that  $(1 - \delta) \top (1 - \delta) > 1 - \varepsilon$ , by the continuity of  $\top$ . Thus, for each  $\varepsilon \in (0, 1)$  there exist  $\delta \in (0, 1)$  such that  $M(A_1 \vee B_1, A_2 \vee B_2, t + s) > 1 - \varepsilon$  whenever  $M(A_1, A_2, t) > 1 - \delta$  and  $M(B_1, B_2, s) > 1 - \delta$ . We conclude that  $(A, B) \mapsto A \vee B$  is uniformly continuous.

(ii) Proceeding as in the proof of (i), we prove the relation  $M(A_1 \wedge B_1, A_2 \wedge B_2, t + s) \geq M(A_1, A_2, t) \top M(B_1, B_2, s)$ , for any  $A_1, B_1, A_2, B_2 \in \mathcal{A}$ ,  $t, s > 0$ . Moreover, we conclude that  $(A, B) \mapsto A \wedge B$  is uniformly continuous by using similar technique in (i).  $\square$

Now, we shall consider the completeness of the fuzzy metric space  $(\mathcal{A}, M, \top)$ .

**Lemma 4.2.** *Let  $(X, \mathcal{A}, \mu)$  be an  $F$ -measure space and  $A_i$  ( $i = 1, 2, \dots$ ) be elements of  $\mathcal{A}$ . Then the fuzzy metric  $M$  as defined in Theorem 3.4 satisfies the following properties:*

- (i)  $M(A_1, A_2, t) \leq M(A_1, A_1 \vee A_2, t)$  for each  $t > 0$ ;
- (ii)  $M(A_k, A_{k+1}, t_k) \leq M(\bigvee_{i=1}^k A_i, \bigvee_{i=1}^{k+1} A_i, \sum_{i=1}^k t_i)$ , for all  $k \in \mathbb{N}$ ,  $t_i > 0$ .

*Proof.* (i) For any  $A_1, A_2 \in \mathcal{A}$ , we get

$$\begin{aligned} & \mu(A_1 \vee (A_1 \vee A_2)) - \mu(A_1 \wedge (A_1 \vee A_2)) \\ &= \mu(A_1 \vee A_2) - \mu(A_1) \\ &\leq \mu(A_1 \vee A_2) - \mu(A_1 \wedge A_2), \end{aligned}$$

and so  $M(A_1, A_2, t) \leq M(A_1, A_1 \vee A_2, t)$ , for each  $t > 0$ .

(ii) By (i) and Theorem 4.1 (i), for all  $k \in \mathbb{N}, t_i > 0$ , we have

$$\begin{aligned} M\left(\bigvee_{i=1}^k A_i, \bigvee_{i=1}^{k+1} A_i, \sum_{i=1}^k t_i\right) &= M\left(\left(\bigvee_{i=1}^{k-1} A_i\right) \vee A_k, \left(\bigvee_{i=1}^{k-1} A_i\right) \vee (A_k \vee A_{k+1}), \sum_{i=1}^k t_i\right) \\ &\geq M\left(\bigvee_{i=1}^{k-1} A_i, \bigvee_{i=1}^{k-1} A_i, \sum_{i=1}^{k-1} t_i\right) \top M(A_k, A_k \vee A_{k+1}, t_k) \\ &\geq 1 \top M(A_k, A_{k+1}, t_k) \\ &= M(A_k, A_{k+1}, t_k). \end{aligned} \quad \square$$

**Lemma 4.3.** *Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in fuzzy metric space  $(X, M, \top)$ . If there is a subsequence  $(x_{k(n)})_{n \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  that converges to  $x$  in  $X$ , then the Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ .*

*Proof.* Let  $(x_{k(n)})_{n \in \mathbb{N}}$  be a subsequence of  $(x_n)_{n \in \mathbb{N}}$ . Then, given  $r$  with  $0 < r < 1$  and  $t > 0$ , there is an  $n_0 \in \mathbb{N}$  such that for each  $n \geq n_0$ ,  $M(x, x_{k(n)}, \frac{t}{2}) > 1 - s$ , where  $s > 0$  satisfies  $(1 - s) \top (1 - s) > 1 - r$ . Since  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence, there is  $n_1 \geq k(n_0)$  such that  $M(x_n, x_m, \frac{t}{2}) > 1 - s$  for each  $n, m \geq n_1$ . Therefore, for each  $n \geq n_1$ , we have

$$M(x, x_n, t) \geq M(x, x_{k(n)}, \frac{t}{2}) \top M(x_{k(n)}, x_n, \frac{t}{2}) \geq (1 - s) \top (1 - s) > 1 - r.$$

We conclude that the Cauchy sequence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ . □

**Theorem 4.4.** *The fuzzy metric space  $(\mathcal{A}, M, \top)$  based on  $F$ -measure  $\mu$  is complete.*

*Proof.* Let  $(A_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{A}$ . We need to show that there exists a set  $A \in \mathcal{A}$  such that  $\lim_{n \rightarrow \infty} M(A_n, A, t) = 1$  for each  $t > 0$ . By Lemma 4.3, it suffices to show that there is a convergent subsequence in  $(A_n)_{n \in \mathbb{N}}$ . Hence, passing to subsequence, there exists  $n_0 \in \mathbb{N}$  such that  $M(A_n, A_{n+1}, t_n) > 1 - 2^{-n}$  for all  $n \geq n_0, t_n > 0$ .

For  $n \in \mathbb{N}$ , let  $B_p = \bigvee_{i=n}^{n+p} A_i$ . Then  $(B_p)$  is a monotonic increasing sequence and  $\lim_{p \rightarrow \infty} B_p = \bigvee_{i=n}^{\infty} A_i \triangleq D_n$ , so,  $\mu(D_n) = \lim_{p \rightarrow \infty} \mu(B_p)$ . By Lemma 4.2, we have

$$\begin{aligned} & M(A_n, B_p, pt_n + (p - 1)t_{n+1} + \dots + t_{n+p-1}) \\ &\geq M(A_n, B_1, t_n) \top M(B_1, B_2, t_n + t_{n+1}) \top \dots \top M(B_{p-1}, B_p, t_n + t_{n+1} + \dots + t_{n+p-1}) \\ &\geq M(A_n, A_{n+1}, t_n) \top M(A_{n+1}, A_{n+2}, t_{n+1}) \top \dots \top M(A_{n+p-1}, A_{n+p}, t_{n+p-1}) \\ &> (1 - 2^{-n}) \top (1 - 2^{-n-1}) \top \dots \top (1 - 2^{-n-p+1}). \end{aligned}$$

Hence,  $\lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} M(A_n, D_n, pt_n + (p - 1)t_{n+1} + \dots + t_{n+p-1}) = 1$ . Moreover, since the sequence  $(D_n)_{n \in \mathbb{N}}$

is monotonic decreasing, we can set  $A = \bigcap_{n=1}^{\infty} D_n, t = \lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} (pt_n + (p - 1)t_{n+1} + \dots + t_{n+p-1}), A \in \mathcal{A}$  in the light of  $\mathcal{A}$  is a fuzzy  $\sigma$ -algebra. Thus,  $\lim_{n \rightarrow \infty} M(A_n, A, t) = 1$  and complete the proof. □

## 5 The correspondence between fuzzy metric space and $F$ -measure space

In the following, we give the characteristics of the nonatom of the  $F$ -measure algebra  $(\mathcal{A}, \mu)$ .

**Definition 5.1.** If, for two distinct elements  $A, B \subset X$ , there exists  $t, s > 0$  and an element  $P \subset X$ , different from both  $A$  and  $B$  such that  $M(A, B, t + s) = M(A, P, t) \top M(P, B, s)$ , then fuzzy metric space  $(X, M, \top)$  is called convex.

**Definition 5.2.** Let  $(\mathcal{A}, \mu)$  be an  $F$ -measure algebra. Then the measure  $\mu$  is called nonatom if for  $A, B \in \mathcal{A}$ ,  $A \leq B$  and  $\mu(A) < \mu(B)$ , there exists  $P \in \mathcal{A}$ ,  $A \leq P \leq B$  such that  $\mu(A) < \mu(P) < \mu(B)$ .

**Theorem 5.3.** The  $F$ -measure algebra  $(\mathcal{A}, \mu)$  is nonatom if and only if fuzzy metric space  $(\mathcal{A}, M, \top)$  is convex.

*Proof. (Sufficiency)* Suppose that  $\mathcal{A}$  is convex. Let  $A, C \in \mathcal{A}$  with  $A \leq C$  and  $\mu(A) < \mu(C)$ . Since  $A \neq C$ , there exists  $t, s > 0$  and  $B \in \mathcal{A}$ , which is different from both  $A$  and  $C$  such that  $M(A, C, t + s) = M(A, B, t) \top M(B, C, s)$ . We can distinguish two cases:  $M(A, B, t) \geq M(B, C, s)$  or  $M(B, C, s) \geq M(A, B, t)$ . Since  $M$  is nondecreasing with respect to  $t$ , it is easy to verify that in the two cases the inequalities

$$\mu(A \vee C) - \mu(A \wedge C) \geq \mu(A \vee B) - \mu(A \wedge B), \quad \mu(A \vee C) - \mu(A \wedge C) \geq \mu(B \vee C) - \mu(B \wedge C) (*)$$

hold. Set  $P = A \vee (B \wedge C) = (A \vee B) \wedge C$ , then  $A \leq P \leq C$ .

Suppose that  $\mu(A \wedge B) = \mu(B \wedge C)$ . Then

$$\begin{aligned} \mu(B \vee C) - \mu(B \wedge C) &= \mu(B \vee C) - \mu(A \wedge B) \\ &\geq \mu(C) - \mu(A) \\ &= \mu(A \vee C) - \mu(A \wedge C) \end{aligned}$$

which contradicts the condition (\*). Hence, we have  $\mu(B \wedge C) > \mu(A \wedge B)$ . Consequently,  $\mu(P) - \mu(A) = \mu(B \wedge C) - \mu(A \wedge B) > 0$ , i.e.,  $\mu(P) > \mu(A)$ .

On the other hand, suppose that  $\mu(A \vee B) = \mu(B \vee C)$ . Then

$$\begin{aligned} \mu(A \vee B) - \mu(A \wedge B) &= \mu(B \vee C) - \mu(A \wedge B) \\ &\geq \mu(C) - \mu(A) \\ &= \mu(A \vee C) - \mu(A \wedge C) \end{aligned}$$

which contradicts the condition (\*). Hence, we obtain  $\mu(B \vee C) > \mu(A \vee B)$ . Consequently,  $\mu(C) - \mu(P) = \mu(B \vee C) - \mu(A \vee B) > 0$ , i.e.,  $\mu(C) > \mu(P)$ . Thus the  $F$ -measure  $\mu$  is nonatom.

**(Necessity)** Suppose that  $\mu$  is nonatom,  $A$  and  $C \in \mathcal{A}$  with  $A \neq C$ .

Case (i). If  $A \wedge C \neq A$  and  $A \wedge C \neq C$ , put  $B = A \wedge C$ . Then  $\mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee C) - \mu(B \wedge C) = \mu(A) - \mu(A \wedge C) + \mu(C) - \mu(A \wedge C) = \mu(A \vee C) - \mu(A \wedge C)$ .

Case (ii). Suppose  $A \wedge C = A$ . Then  $\mu(A \wedge C) = \mu(A)$  and so  $\mu(A \vee C) = \mu(C)$ . We deduce that  $\mu(A) < \mu(C)$  in the light of  $(\mathcal{A}, \mu)$  is  $F$ -measure algebra. By the assumption, there exists  $B \in \mathcal{A}$ ,  $A \leq B \leq C$  such that  $\mu(A) < \mu(B) < \mu(C)$ . Hence  $\mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee C) - \mu(B \wedge C) = \mu(B) - \mu(A) + \mu(C) - \mu(B) = \mu(A \vee C) - \mu(A \wedge C)$ . So, both in the case (i) and (ii) we deduce that

$$\mu(A \vee B) - \mu(A \wedge B) + \mu(B \vee C) - \mu(B \wedge C) = \mu(A \vee C) - \mu(A \wedge C). \quad (**)$$

Next we need to prove that there exists  $t, s > 0$ , such that  $M(A, C, t + s) = M(A, B, t) \top M(B, C, s)$ . If there exists  $t, s > 0$ , such that

$$\frac{t}{t + \mu(A \vee B) - \mu(A \wedge B)} = \frac{t + s}{t + s + \mu(A \vee C) - \mu(A \wedge C)},$$

so

$$\mu(A \vee B) - \mu(A \wedge B) = \frac{t(\mu(A \vee C) - \mu(A \wedge C))}{t + s},$$

according to the equality (\*\*), we get

$$\mu(B \vee C) - \mu(B \wedge C) = \frac{s(\mu(A \vee C) - \mu(A \wedge C))}{t + s},$$

and hence

$$\frac{s}{s + \mu(B \vee C) - \mu(B \wedge C)} = \frac{t + s}{t + s + \mu(A \vee C) - \mu(A \wedge C)}.$$

In consequence,  $M(A, C, t + s) = M(A, B, t) \top M(B, C, s)$ . Also, if there exists  $t, s > 0$ , such that

$$\frac{s}{s + \mu(B \vee C) - \mu(B \wedge C)} = \frac{t + s}{t + s + \mu(A \vee C) - \mu(A \wedge C)},$$

we can get  $M(A, C, t + s) = M(A, B, t) \top M(B, C, s)$  by the same method as employed in the above. This completes the proof of theorem.  $\square$

## 6 Conclusion

In this paper, by constructing a fuzzy metric on the fuzzy measurable sets, we studied the relations between these two. In particular, several satisfactory properties of the constructed fuzzy metric have been obtained. In addition, we investigated the characterization of the nonatom of the fuzzy measure and the corresponding properties of constructed fuzzy metric space. The main results and methods presented in this paper generalize some well known results in [38, 39].

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