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Simple sufficient conditions for starlikeness and convexity for meromorphic functions

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Abstract: In this paper we investigate some extensions of sufficient conditions for meromorphic multivalent functions in the open unit disk to be meromorphic multivalent starlike and convex of order α . Our results unify and extend some starlikeness and convexity conditions for meromorphic multivalent functions obtained by Xu et al. [2], and some interesting special cases are given.

Keywords: Meromorphic functions, Multivalent functions, Meromorphic starlike functions, Meromorphic convex functions, Analytic branch

MSC: 30C45

1 Introduction and definitions

Let $\Sigma_{p,n}$ denote the class meromorphic multivalent functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k-p} z^{k-p} \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1)$$

which are analytic in the punctured unit disk $\mathcal{U}^* := \mathcal{U} \setminus \{0\}$, where $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.1.

(i) A function $f \in \Sigma_{p,n}$ is said to be meromorphic starlike functions of order α , if it satisfies the inequality

$$-\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U},$$

for some real α ($0 \leq \alpha < p$), and we denote this subclass by $\Sigma_{p,n}^*(\alpha)$.

(ii) A function $f \in \Sigma_{p,n}$ is said to be meromorphic convex functions of order α , if it satisfies the inequality

$$-\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathcal{U},$$

for some real α ($0 \leq \alpha < p$), and we denote this subclass by $\Sigma K_{p,n}(\alpha)$.

Let \mathcal{A}_n denote the class of analytic functions in \mathcal{U} of the form

$$f(z) = z + \sum_{k=n}^{\infty} a_{k+1} z^{k+1}, \quad z \in \mathcal{U} \quad (n \in \mathbb{N}).$$

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Definition 1.2.

(i) Let \mathcal{S}_n^* denote the class of n -starlike functions in \mathcal{U} , i.e. $f \in \mathcal{A}_n$ and satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathcal{U}.$$

(ii) Further, we denote by \mathcal{K}_n the class of n -convex functions in \mathcal{U} , i.e. $f \in \mathcal{A}_n$ and satisfies

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0, \quad z \in \mathcal{U}.$$

In the recent papers of Goyal et al. [1] and Xu et al. [2], the authors obtained some sufficient conditions for multivalent and meromorphic starlikeness and convexity, respectively. In this paper we will derive some extensions of these sufficient conditions for starlikeness and convexity of order α for meromorphic multivalent functions.

2 Main results

In order to find some simple sufficient conditions for the starlikeness and convexity of order α for a function $f \in \Sigma_{p,n}$, we will recall the following lemma due to P. T. Mocanu (see also [3]):

Lemma 2.1 ([4, Corollary 1.2]). *If $f \in \mathcal{A}_n$ and satisfies the inequality*

$$|f'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U},$$

then $f \in \mathcal{S}_n^$.*

Remark that, for the special case $n = 1$, this result was previously obtained in [5, Theorem 3].

Theorem 2.2. *If $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, satisfies the inequality*

$$\left| [z^p f(z)]^{\frac{1}{\alpha-p}} \left(\frac{zf'(z)}{f(z)} + \alpha \right) + p - \alpha \right| < \frac{(n+1)(p-\alpha)}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}, \quad (2)$$

for some real values of α ($0 \leq \alpha < p$), then $f \in \Sigma_{p,n}^(\alpha)$. (The power is the principal one).*

Proof. For $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, let us define a function h by

$$h(z) = z [z^p f(z)]^{\frac{1}{\alpha-p}}, \quad z \in \mathcal{U}. \quad (3)$$

Since $f \in \Sigma_{p,n}$ and $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, it follows that the power function

$$\varphi(z) = [z^p f(z)]^{\frac{1}{\alpha-p}}$$

has an analytic branch in \mathcal{U} with $\varphi(0) = 1$, so h is analytic in \mathcal{U} . If the function f is of the form (1), a simple computation shows that

$$h(z) = z + \frac{a_{n-p}}{\alpha-p} z^{n+1} + \dots,$$

$z \in \mathcal{U}$. Thus $h \in \mathcal{A}_n$.

Now, differentiating logarithmically the definition relation (3) we obtain that

$$\frac{h'(z)}{h(z)} = -\frac{1}{p-\alpha} \left(\frac{f'(z)}{f(z)} + \frac{\alpha}{z} \right), \quad z \in \mathcal{U}^*, \quad (4)$$

which gives

$$|h'(z) - 1| = \frac{1}{p-\alpha} \left| [z^p f(z)]^{\frac{1}{\alpha-p}} \left(\frac{zf'(z)}{f(z)} + \alpha \right) + p - \alpha \right|, \quad z \in \mathcal{U}.$$

From the above relation, by using the assumption (2) of the theorem we get

$$|h'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U},$$

hence, according to Lemma 2.1, we deduce that $h \in \mathcal{S}_n^*$.

Using again (4), we get

$$\frac{zh'(z)}{h(z)} = -\frac{1}{p-\alpha} \left(\frac{zf'(z)}{f(z)} + \alpha \right),$$

and according to the fact that $h \in \mathcal{S}_n^*$, the above relation implies

$$-\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathcal{U},$$

that is $f \in \Sigma_{p,n}^*(\alpha)$. □

Theorem 2.3. For $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, let's define the function h by

$$h(z) = z [z^p f(z)]^{\frac{1}{\alpha-p}}, \quad z \in \mathcal{U} \quad (0 \leq \alpha < p). \quad (5)$$

If h satisfies the inequality

$$|h''(z)| \leq \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}, \quad (6)$$

then $f \in \Sigma_{p,n}^*(\alpha)$. (The power is the principal one).

Proof. As in the proof of Theorem 2.2, we have $h \in \mathcal{A}_n$. Moreover, from the assumption (6) we deduce that

$$|h'(z) - 1| = \left| \int_0^z h''(t) dt \right| \leq \int_0^{|z|} |h''(\rho e^{i\theta})| d\rho \leq \frac{(n+1)|z|}{\sqrt{(n+1)^2 + 1}} < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}.$$

Therefore, the function h satisfies the condition of Lemma 2.1, and thus $h \in \mathcal{S}_n^*$. Now, using the same reasons as in the last part of the proof of Theorem 2.2, we finally obtain that $f \in \Sigma_{p,n}^*(\alpha)$. □

Next, we will give some sufficient conditions for a function $f \in \Sigma_{p,n}$ to be a convex function of order α .

Theorem 2.4. If $f \in \Sigma_{p,n}$, with $f'(z) \neq 0$ for all $z \in \mathcal{U}^*$, satisfies the inequality

$$\left| \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + \alpha \right) + p - \alpha \right| < \frac{(n+1)(p-\alpha)}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}, \quad (7)$$

for some real values of α ($0 \leq \alpha < p$), then $f \in \Sigma_{K_{p,n}}(\alpha)$. (The power is the principal one).

Proof. For $f \in \Sigma_{p,n}$ with $f'(z) \neq 0$ for all $z \in \mathcal{U}^*$, the power function

$$\varphi(z) = \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \quad (8)$$

has an analytic branch in \mathcal{U} with $\varphi(0) = 1$, and if the function f is of the form (1), then

$$\varphi(z) = 1 - \frac{n-p}{p(\alpha-p)} a_{n-p} z^n + \dots, \quad z \in \mathcal{U}.$$

It follows that the function h defined by

$$h(z) = \int_0^z \varphi(t) dt = z - \frac{n-p}{p(\alpha-p)(n+1)} a_{n-p} z^{n+1} + \dots, \quad z \in \mathcal{U}, \quad (9)$$

belongs to \mathcal{A}_n . Thus, we deduce that the function g defined by

$$g(z) = zh'(z) = z - \frac{n-p}{p(\alpha-p)} a_{n-p} z^{n+1} + \dots, \quad z \in \mathcal{U},$$

is in \mathcal{A}_n . From the above definition relation, we get

$$g'(z) = \frac{1}{\alpha-p} \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + \alpha \right).$$

From here and using the assumption (7), we obtain

$$|g'(z) - 1| < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}.$$

Therefore, from Lemma 2.1 it follows that $g(z) = zh'(z) \in \mathcal{S}_n^*$, which is equivalent to $h \in \mathcal{K}_n$. Noting that

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha-p} \left(\frac{zf''(z)}{f'(z)} + p + 1 \right),$$

since $h \in \mathcal{K}_n$, we obtain that

$$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) = \operatorname{Re} \left[\frac{1}{\alpha-p} \left(1 + \alpha + \frac{zf''(z)}{f'(z)} \right) \right] > 0, \quad z \in \mathcal{U},$$

that is

$$-\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, \quad z \in \mathcal{U},$$

hence $f \in \Sigma_{p,n}(\alpha)$. □

Theorem 2.5. If $f \in \Sigma_{p,n}$, with $f'(z) \neq 0$ for all $z \in \mathcal{U}^*$, satisfies the inequality

$$\left| \frac{1}{z} \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right) \right| \leq \frac{(n+1)(p-\alpha)}{2\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}, \quad (10)$$

for some real values of α ($0 \leq \alpha < p$), then $f \in \Sigma_{p,n}(\alpha)$. (The power is the principal one).

Proof. For $f \in \Sigma_{p,n}$ with $f'(z) \neq 0$ for all $z \in \mathcal{U}^*$, the function φ defined by (8) is in \mathcal{A}_n , therefore the function h given by (9) is in \mathcal{A}_n .

Further, letting $g(z) = zh'(z)$, we obtain that

$$g(z) = z - \frac{n-p}{p(\alpha-p)} a_{n-p} z^{n+1} + \dots \in \mathcal{A}_n,$$

and

$$\begin{aligned} |g'(z) - 1| &= |h'(z) + zh''(z) - 1| \leq |h'(z) - 1| + |zh''(z)| = \left| \int_0^z h''(t) dt \right| + |zh''(z)| \\ &\leq \int_0^{|z|} |h''(\rho e^{i\theta})| d\rho + |zh''(z)|, \quad z \in \mathcal{U}. \end{aligned} \quad (11)$$

Since

$$h''(z) = \frac{1}{z(\alpha-p)} \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right),$$

using the assumption (10) we get

$$|h''(\rho e^{i\theta})| = \frac{1}{p-\alpha} \left| \frac{1}{z} \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{zf''(z)}{f'(z)} + p \right) \right| \Big|_{z=\rho e^{i\theta}} \leq \frac{n+1}{2\sqrt{(n+1)^2 + 1}},$$

and from (11), using again (10) we deduce that

$$|g'(z) - 1| \leq \frac{(n+1)|z|}{2\sqrt{(n+1)^2 + 1}} + \frac{(n+1)|z|}{2\sqrt{(n+1)^2 + 1}} < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}.$$

According to Lemma 2.1 we obtain that $g(z) = zh'(z) \in \mathcal{S}_n^*$, which is equivalent to $h \in \mathcal{K}_n$. Consequently, as in the last part of the proof of Theorem 2.4 it follows that $f \in \Sigma K_{p,n}(\alpha)$. \square

Remarks 2.6.

- (i) If we put $n = 1$ in Theorem 2.2 and Theorem 2.3, we get the results established by Xu et al. [2].
- (ii) For the special case $n = 1$, Theorem 2.4 and Theorem 2.5 represent the results of Xu et al. [2].

For $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, let's define the function F by

$$F(z) = \int_0^z [t^p f(t)]^\gamma dt, \quad z \in \mathcal{U} \quad (\gamma \in \mathbb{C}),$$

where the power is the principal one. Thus, $F(z) = z + \frac{\gamma}{n+1} a_{n-p} z^{n+1} + \dots \in \mathcal{A}_n$, and considering this integral operator we derive the next result:

Theorem 2.7. If $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, satisfies the inequality

$$\left| \frac{\gamma [z^p f(z)]^\gamma}{z} \left(\frac{zf'(z)}{f(z)} + p \right) \right| \leq \frac{n+1}{2\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}, \quad (12)$$

for $\gamma \leq -\frac{1}{p}$, then $f \in \Sigma S_{p,n}^* \left(p + \frac{1}{\gamma} \right)$. (The power is the principal one).

Proof. If $f \in \Sigma_{p,n}$, with $f(z) \neq 0$ for all $z \in \mathcal{U}^*$, then

$$F''(z) = \frac{\gamma [z^p f(z)]^\gamma}{z} \left(\frac{zf'(z)}{f(z)} + p \right), \quad z \in \mathcal{U}. \quad (13)$$

Defining the function $g(z) = zF'(z)$, it follows that $g \in \mathcal{A}_n$, and

$$\begin{aligned} |g'(z) - 1| &= |F'(z) + zF''(z) - 1| \leq |F'(z) - 1| + |zF''(z)| \\ &= \left| \int_0^z F''(t) dt \right| + |zF''(z)| \leq \int_0^{|z|} |F''(\rho e^{i\theta})| d\rho + |zF''(z)|, \quad z \in \mathcal{U}. \end{aligned} \quad (14)$$

From (13), using the assumption (12) we get

$$|F''(\rho e^{i\theta})| = \left| \frac{\gamma [z^p f(z)]^\gamma}{z} \left(\frac{zf'(z)}{f(z)} + p \right) \right| \Big|_{z=\rho e^{i\theta}} \leq \frac{n+1}{2\sqrt{(n+1)^2 + 1}},$$

and the inequality (14) combined again with (12) implies that

$$|g'(z) - 1| \leq \frac{(n+1)|z|}{2\sqrt{(n+1)^2 + 1}} + \frac{(n+1)|z|}{2\sqrt{(n+1)^2 + 1}} < \frac{n+1}{\sqrt{(n+1)^2 + 1}}, \quad z \in \mathcal{U}.$$

Consequently, from Lemma 2.1 we obtain that $g(z) = zF'(z) \in \mathcal{S}_n^*$, which is equivalent to $F \in \mathcal{K}_n$. Using the fact that $F'(z) = [z^p f(z)]^\gamma$, it follows that

$$1 + \frac{zF''(z)}{F'(z)} = \gamma \left(p + \frac{zf'(z)}{f(z)} \right) + 1,$$

and since $F \in \mathcal{K}_n$ we conclude that $f \in \Sigma S_{p,n}^* \left(p + \frac{1}{\gamma} \right)$. \square

3 Special cases

Let's consider the function f defined by

$$f(z) = \frac{1}{z^p} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\alpha-p}, \quad z \in \mathcal{U}, \quad (15)$$

where $0 \leq \alpha < p$, the power is the principal one, and assuming that the parameter $\lambda \in \mathbb{C}$ is chosen such that

$$\frac{1}{\lambda} \neq 1 - \frac{\sin z}{z}, \quad z \in \mathcal{U}. \quad (16)$$

Using MAPLE™ software, from Figure 1a we may see that

$$\max_{|z| \leq 1} \left| 1 - \frac{\sin z}{z} \right| < 0.18, \quad (17)$$

therefore (16) holds whenever $|\lambda| \leq \frac{50}{9} = 5.555\dots$, and consequently, if $\lambda \in \mathbb{C}$ satisfies this inequality then $f \in \Sigma_{p,2}$.

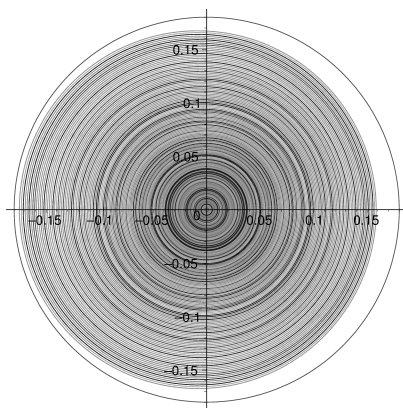
Using again MAPLE™ software, from Figure 1b we have that

$$\max_{|z| \leq 1} \left| \sin^2 \frac{z}{2} \right| < 0.28, \quad (18)$$

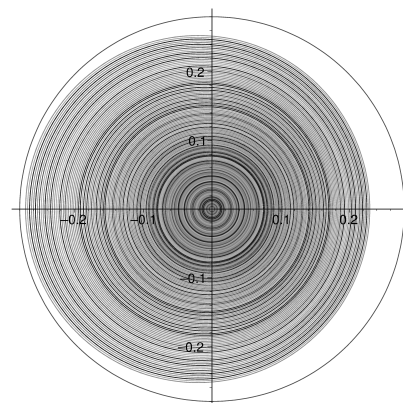
and a simple computation leads to

$$\left| [z^p f(z)]^{\frac{1}{\alpha-p}} \left(\frac{zf'(z)}{f(z)} + \alpha \right) + p - \alpha \right| = 2(p - \alpha)|\lambda| \left| \sin^2 \frac{z}{2} \right| < 2(p - \alpha)|\lambda| \cdot 0.28, \quad z \in \mathcal{U}.$$

Fig. 1



(a) Functions $G_1(z) = 1 - \frac{\sin z}{z}$ and $G_2(z) = 0.18z$



(b) Functions $H_1(z) = \sin^2 \frac{z}{2}$ and $H_2(z) = 0.28z$

Thus, according to Theorem 2.2 we obtain the following special case:

Example 3.1. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{75}{14\sqrt{10}} = 1.6941\dots$, then

$$f(z) = \frac{1}{z^p} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\alpha-p} \in \Sigma_{p,2}^*(\alpha),$$

for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.

Remark 3.2. For the function f given by (15), the function h defined by (5) is of the form

$$h(z) = (1 - \lambda)z + \lambda \sin z, \quad z \in \mathcal{U}.$$

Therefore, $|h''(z)| = |\lambda| |\sin z|$, and using MAPLE™ software, the Figure 2a yields that

$$\max_{|z| \leq 1} |\sin z| < 1.2.$$

Now, according to Theorem 2.3 we obtain the following special case:

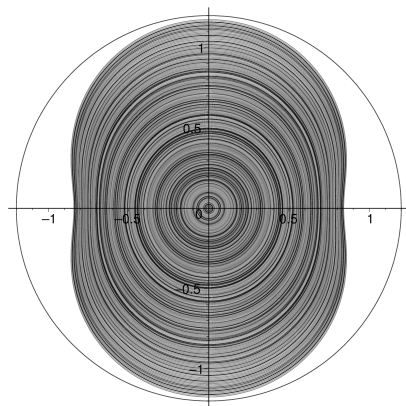
If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{5}{2\sqrt{10}} = 0.79057\dots$, then

$$f(z) = \frac{1}{z^p} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\alpha-p} \in \Sigma_{p,2}^*(\alpha),$$

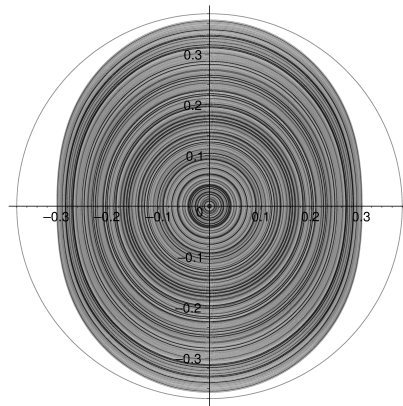
for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.

If we compare the result given by Example 3.1 with the above one, for this special choice of the function f the Example 3.1 gives a better result.

Fig. 2



(a) Functions $J_1(z) = \sin z$ and $J_2(z) = 1.2z$



(b) Functions $K_1(z) = \frac{z \cos z - \sin z}{z^2}$ and $K_2(z) = 0.38z$

As we already proved, if $|\lambda| \leq \frac{50}{9} = 5.555\dots$ then the relation (16) holds. Therefore, there exists a function $f \in \Sigma_{p,2}$ such that

$$f'(z) = -\frac{p}{z^{p+1}} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\alpha-p}, \quad z \in \mathcal{U},$$

where $0 \leq \alpha < p$, and the power is the principal one, assuming that $\lambda \in \mathbb{C}$ is chosen such that $|\lambda| \leq \frac{50}{9} = 5.555\dots$

A simple computation combined with (18) shows that

$$\left| \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{z f''(z)}{f'(z)} + \alpha \right) + p - \alpha \right| = 2(p - \alpha) |\lambda| \left| \sin^2 \frac{z}{2} \right| < 2(p - \alpha) |\lambda| \cdot 0.28, \quad z \in \mathcal{U},$$

and from Theorem 2.4 we obtain the following special case:

Example 3.3. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{75}{14\sqrt{10}} = 1.6941\dots$, then $f \in \Sigma_{p,2}$ with

$$f'(z) = -\frac{p}{z^{p+1}} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\alpha-p}, \quad z \in \mathcal{U}, \quad (19)$$

for some real values of α ($0 \leq \alpha < p$), is in $\Sigma_{K_p,n}(\alpha)$. (The power is the principal one).

Remark 3.4. Using MAPLE™ software, from Figure 2b we have that

$$\max_{|z| \leq 1} \left| \frac{z \cos z - \sin z}{z^2} \right| < 0.38. \quad (20)$$

From a simple computation combined with (20) we get

$$\left| \frac{1}{z} \left(\frac{z^{p+1} f'(z)}{-p} \right)^{\frac{1}{\alpha-p}} \left(1 + \frac{z f''(z)}{f'(z)} + p \right) \right| = (p - \alpha) |\lambda| \left| \frac{z \cos z - \sin z}{z^2} \right| < (p - \alpha) |\lambda| \cdot 0.38, \quad z \in \mathcal{U},$$

and using Theorem 2.5 we obtain the next special case:

If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{75}{19\sqrt{10}} = 1.2483\dots$, then $f \in \Sigma_{p,2}$ that satisfies (19) has, moreover, the property that $f \in \Sigma K_{p,n}(\alpha)$, for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.

Comparing the result given by Example 3.3 with the above one, for this special choice of the function f the Example 3.3 gives a better result.

As we proved at the beginning of this section, the function $f \in \Sigma_{p,2}$, where

$$f(z) = \frac{1}{z^p} \left(1 - \lambda + \lambda \frac{\sin z}{z} \right)^{\frac{1}{\gamma}}, \quad z \in \mathcal{U}, \quad (21)$$

with $\lambda \in \mathbb{C}$, $|\lambda| \leq \frac{50}{9} = 5.555\dots$, and the power is the principal one. Using the inequality (20), we deduce that

$$\left| \frac{\gamma [z^p f(z)]^\gamma}{z} \left(\frac{z f'(z)}{f(z)} + p \right) \right| = |\lambda| \left| \frac{z \cos z - \sin z}{z^2} \right| < |\lambda| \cdot 0.38, \quad z \in \mathcal{U},$$

and from Theorem 2.7 we obtain the next special case:

Example 3.5. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{75}{19\sqrt{10}} = 1.2483\dots$, then the function f given by (21) is in $\Sigma S_{p,2}^* \left(p + \frac{1}{\gamma} \right)$, for $\gamma \leq -\frac{1}{p}$. (The power is the principal one).

Using MAPLE™ software, we could check that the next inequalities hold (see Figures 3a, 3b, 4a, and 4b):

$$\max_{|z| \leq 1} \left| 1 + \frac{z}{2} - \frac{e^z - 1}{z} \right| < 0.22, \quad (22)$$

$$\max_{|z| \leq 1} |e^z - 1 - z| < 0.73, \quad (23)$$

$$\max_{|z| \leq 1} |e^z - 1| < 1.73, \quad (24)$$

$$\max_{|z| \leq 1} \left| \frac{ze^z - e^z + 1}{z^2} - \frac{1}{2} \right| < 0.51. \quad (25)$$

From (22) and (23), using Theorem 2.2 we may easily obtain the following special case:

Example 3.6. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{300}{73\sqrt{10}} = 1.2996\dots$, then

$$f(z) = \frac{1}{z^p} \left[1 + \lambda \left(\frac{e^z - 1}{z} - 1 - \frac{z}{2} \right) \right]^{\alpha-p} \in \Sigma S_{p,2}^*(\alpha),$$

for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.

Fig. 3

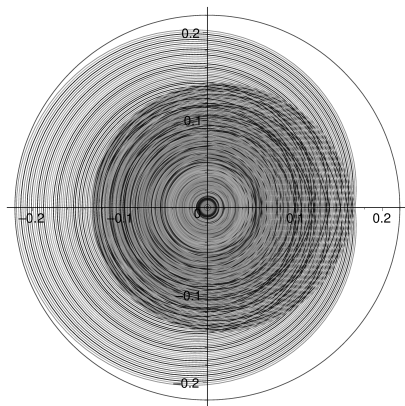
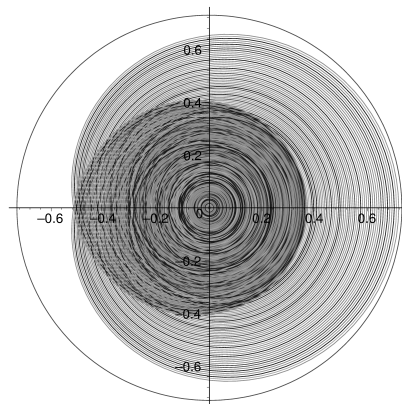
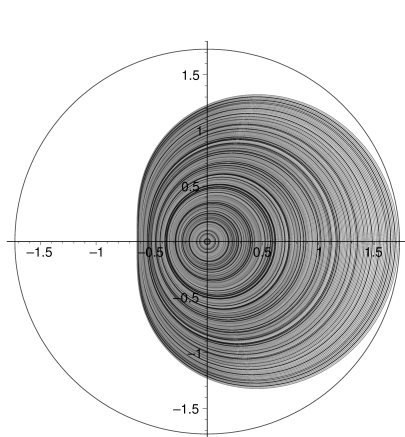
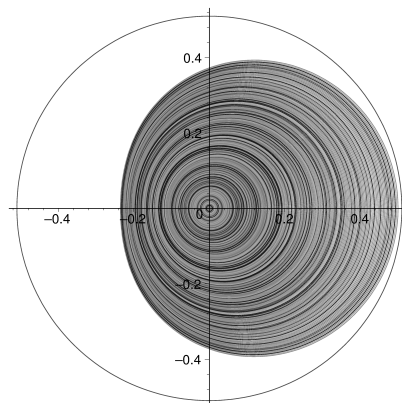
(a) Functions $L_1(z) = 1 + \frac{z}{2} - \frac{e^z - 1}{z}$ and $L_2(z) = 0.22z$ (b) Functions $M_1(z) = e^z - 1 - z$ and $M_2(z) = 0.73z$

Fig. 4

(a) Functions $N_1(z) = e^z - 1$ and $N_2(z) = 1.73z$ (b) Functions $P_1(z) = \frac{ze^z - e^z + 1}{z^2} - \frac{1}{2}$ and $P_2(z) = 0.51z$

Remark 3.7. From (22) and (24), according to Theorem 2.3 we could similarly obtain the next special case:

If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{300}{173\sqrt{10}} = 0.54837\dots$, then

$$f(z) = \frac{1}{z^p} \left[1 + \lambda \left(\frac{e^z - 1}{z} - 1 - \frac{z}{2} \right) \right]^{\alpha-p} \in \Sigma_{p,2}^*(\alpha),$$

for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.

Thus, for this special choice of the function f the Example 3.6 gives a better result.

From (22) and (24), using Theorem 2.4 we easily get the next special case:

Example 3.8. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{300}{73\sqrt{10}} = 1.2996\dots$, then $f \in \Sigma_{p,2}$ with

$$f'(z) = -\frac{p}{z^{p+1}} \left[1 + \lambda \left(\frac{e^z - 1}{z} - 1 - \frac{z}{2} \right) \right]^{\alpha-p}, \quad z \in \mathcal{U}, \quad (26)$$

for some real values of α ($0 \leq \alpha < p$), is in $\Sigma_{K_p,n}(\alpha)$. (The power is the principal one).

Remark 3.9. From (22) and (25), according to Theorem 2.5 we could similarly obtain the next special case:

If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{150}{51\sqrt{10}} = 0.93008\dots$, then $f \in \Sigma_{p,2}$ that satisfies (26) has, moreover, the property that $f \in \Sigma K_{p,n}(\alpha)$, for some real values of α ($0 \leq \alpha < p$), where the power is the principal one.
Consequently, for this special choice of the function f the Example 3.8 gives a better result.

Finally, from the inequalities (22) and (25), using Theorem 2.7 we obtain the next special case:

Example 3.10. If $\lambda \in \mathbb{C}$ and $|\lambda| \leq \frac{150}{51\sqrt{10}} = 0.93008\dots$, then

$$f(z) = \frac{1}{z^p} \left[1 + \lambda \left(\frac{e^z - 1}{z} - 1 - \frac{z}{2} \right) \right]^{\frac{1}{\gamma}} \in \Sigma S_{p,2}^* \left(p + \frac{1}{\gamma} \right),$$

for $\gamma \leq -\frac{1}{p}$. (The power is the principal one).

We will omit the detailed proofs of the last three examples, since these are similar with the previous ones.

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