

Taekyun Kim*, Dae San Kim, and Jong-Jin Seo

Fully degenerate poly-Bernoulli numbers and polynomials

DOI 10.1515/math-2016-0048

Received May 26, 2015; accepted June 29, 2016.

Abstract: In this paper, we introduce the new fully degenerate poly-Bernoulli numbers and polynomials and investigate some properties of these polynomials and numbers. From our properties, we derive some identities for the fully degenerate poly-Bernoulli numbers and polynomials.

Keywords: Fully degenerate poly-Bernoulli polynomial, Fully degenerate poly-Bernoulli number, Umbral calculus

MSC: 11B75, 11B83, 05A19, 05A40

1 Introduction

It is well known that the Bernoulli polynomials are defined by the generating function

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see [1–21]}). \quad (1)$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. From (1), we note that

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (2)$$

and

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad (n \in \mathbb{N}), \quad (\text{see [1, 19]}), \quad (3)$$

where $\delta_{n,k}$ is the Kronecker's symbol.

In [3], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \quad (4)$$

When $x = 0$, $\beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. From (1) and (4), we note that

$$\frac{t}{e^t - 1} e^{xt} = \lim_{\lambda \rightarrow 0} \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}(x) \frac{t^n}{n!}. \quad (5)$$

Thus, by (5), we get

$$B_n(x) = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}(x), \quad (\text{see [3, 15]}). \quad (6)$$

*Corresponding Author: **Taekyun Kim:** Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China and Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea, E-mail: tkkim@kw.ac.kr

Dae San Kim: Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea, E-mail: dskim@sogang.ac.kr

Jong-Jin Seo: Department of Applied Mathematics, Pukyong National University, Pusan, Republic of Korea, E-mail: seo2011@pknu.ac.kr

By (4), we get

$$\sum_{n=0}^{\infty} (\beta_{m,\lambda}(1) - \beta_{m,\lambda}) \frac{t^n}{n!} = t, \tag{7}$$

and

$$\beta_{n,\lambda}(x) = \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda} \lambda^{n-l} \left(\frac{x}{\lambda}\right)_{n-l}. \tag{8}$$

From (7), we have

$$\beta_{n,\lambda}(1) - \beta_{n,\lambda} = \delta_{1,n}, \quad (n \geq 0), \quad \beta_{0,\lambda} = 1. \tag{9}$$

Now, we consider the degenerate Bernoulli polynomials which are different from the degenerate Bernoulli polynomials of L. Carlitz as follows:

$$\frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{10}$$

When $x = 0$, $b_{n,\lambda} = b_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

Note. The degenerate Bernoulli polynomials are also called Daehee polynomials with λ -parameter (see [13]).

From (10), we note that

$$\frac{t}{e^t - 1} e^{xt} = \lim_{\lambda \rightarrow 0} \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} b_{n,\lambda}(x) \frac{t^n}{n!}. \tag{11}$$

By (1) and (11), we see that

$$B_n(x) = \lim_{\lambda \rightarrow 0} b_{n,\lambda}(x), \quad (n \geq 0).$$

The classical polylogarithm function $\text{Li}_k(x)$ is defined by

$$\text{Li}_k(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^k}, \quad (k \in \mathbb{Z}), \quad (\text{see [10, 11]}). \tag{12}$$

It is known that the poly-Bernoulli polynomials are defined by the generating function

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} B_n^{(k)}(x) \frac{t^n}{n!}, \quad (\text{see [9, 10, 12]}). \tag{13}$$

When $k = 1$, we have

$$\sum_{n=0}^{\infty} B_n^{(1)}(x) \frac{t^n}{n!} = \frac{t}{1 - e^{-t}} e^{xt} = \frac{t}{e^t - 1} e^{(x+1)t} = \sum_{n=0}^{\infty} B_n(x+1) \frac{t^n}{n!}. \tag{14}$$

By (14), we easily get

$$B_n^{(1)}(x) = B_n(x+1), \quad (n \geq 0).$$

Let $x = 0$. Then $B_n^{(k)} = B_n^{(k)}(0)$ are called the poly-Bernoulli numbers.

In this paper, we introduce the new fully degenerate poly-Bernoulli numbers and polynomials and investigate some properties of these polynomials and numbers. From our investigation, we derive some identities for the fully degenerate poly-Bernoulli numbers and polynomials.

2 Fully degenerate poly-Bernoulli polynomials

For $k \in \mathbb{Z}$, we define the fully degenerate poly-Bernoulli polynomials which are given by the generating function

$$\frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{15}$$

When $x = 0$, $\beta_{n,\lambda}^{(k)} = \beta_{n,\lambda}^{(k)}(0)$ are called the fully degenerate poly-Bernoulli numbers.

From (13) and (15), we have

$$\frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} e^{xt} = \lim_{\lambda \rightarrow 0} \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!}. \tag{16}$$

Thus, we get

$$\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(k)}(x) = B_n^{(k)}(x), \quad (n \geq 0). \tag{17}$$

By (15), we get

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}^{(k)} \left(\frac{x}{\lambda} \right) \lambda^{n-l} \right) \frac{t^n}{n!}. \tag{18}$$

Thus, from (18), we have

$$\beta_{n,\lambda}^{(k)}(x + y) = \sum_{l=0}^n \binom{n}{l} \left(\frac{y}{\lambda} \right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)}(x), \quad (n \geq 0), \tag{19}$$

and

$$\beta_{n,\lambda}^{(k)}(x) = \sum_{l=0}^n \binom{n}{l} \left(\frac{x}{\lambda} \right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)}.$$

Therefore, by (17) and (19), we obtain the following theorem.

Theorem 2.1. For $k \in \mathbb{Z}$, $n \geq 0$, we have

$$\beta_{n,\lambda}^{(k)}(x + y) = \sum_{l=0}^n \binom{n}{l} \left(\frac{y}{\lambda} \right)_{n-l} \lambda^{n-l} \beta_{l,\lambda}^{(k)}(x), \quad (n \geq 0), \tag{20}$$

and

$$\lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(k)}(x) = B_n^{(k)}(x),$$

where $(x)_n = x(x - 1) \cdots (x - n + 1) = \sum_{l=0}^n S_1(n, l) x^l$.

From (15), we can derive the following equation:

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x+1}{\lambda}}. \tag{21}$$

Thus, by (21), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ \beta_{n,\lambda}^{(k)} - \beta_{n,\lambda}^{(k)}(-1) \right\} \frac{t^n}{n!} \\ &= \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \end{aligned} \tag{22}$$

$$\begin{aligned}
 &= \sum_{m=1}^{\infty} \frac{\left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)^m}{m^k} \\
 &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)^k} \left(e^{-\frac{1}{\lambda} \log(1 + \lambda t)} - 1\right)^{m+1} \\
 &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)^k} (m+1)! \sum_{l=m+1}^{\infty} S_2(l, m+1) (-1)^l \lambda^{-l} \frac{(\log(1 + \lambda t))^l}{l!} \\
 &= \sum_{l=1}^{\infty} \sum_{m=0}^{l-1} \frac{(-1)^{m+1}}{(m+1)^k} (m+1)! S_2(l, m+1) (-1)^l \lambda^{-l} \sum_{n=l}^{\infty} S_1(n, l) \lambda^n \frac{t^n}{n!} \\
 &= \sum_{n=1}^{\infty} \left(\sum_{l=1}^n \sum_{m=0}^{l-1} \frac{m! (m+1) (-1)^{l-m-1} \lambda^{n-l} S_2(l, m+1) S_1(n, l)}{(m+1)^k} \right) \frac{t^n}{n!},
 \end{aligned}$$

where $S_2(n, l)$ and $S_1(n, l)$ are the Stirling numbers of the second kind and of the first kind, respectively.

Therefore, by (22), we obtain the following theorem.

Theorem 2.2. For $k \in \mathbb{Z}$, $n \geq 1$, we have

$$\beta_{n,\lambda}^{(k)} - \beta_{n,\lambda}^{(k)} (-1) = \sum_{l=1}^n \sum_{m=0}^{l-1} \frac{m! (-1)^{l-m-1} \lambda^{n-l} S_2(l, m+1) S_1(n, l)}{(m+1)^{k-1}}.$$

From (12), we can easily derive the following equation:

$$\text{Li}'_k(t) = \frac{d}{dt} \text{Li}_k(t) = \frac{1}{t} \text{Li}_{k-1}(t). \tag{23}$$

Thus, by (23), the generating function of the fully degenerate poly-Bernoulli numbers is also written in terms of the following iterated integral:

$$\begin{aligned}
 &\frac{(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \int_0^t \frac{1}{\left((1 + \lambda t)^{\frac{1}{\lambda}} - 1\right) (1 + \lambda t)} \\
 &\times \int_0^t \frac{1}{\left((1 + \lambda t)^{\frac{1}{\lambda}} - 1\right) (1 + \lambda t)} \cdots \int_0^t \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{\left((1 + \lambda t)^{\frac{1}{\lambda}} - 1\right) (1 + \lambda t)} \underbrace{dt \cdots dt}_{k-1 \text{ times}} \\
 &= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!}.
 \end{aligned} \tag{24}$$

For $k = 2$, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(2)} \frac{t^n}{n!} &= \frac{(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \int_0^t \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{-\frac{1}{\lambda}} dt \\
 &= \frac{(1 + \lambda t)^{\frac{1}{\lambda}}}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \left(\sum_{m=0}^{\infty} b_{m,\lambda} (-\lambda) \frac{1}{m!} \int_0^t t^m dt \right) \\
 &= \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{1}{\lambda}} \right) \left(\sum_{m=0}^{\infty} \frac{b_{m,\lambda} (-\lambda)}{(m+1) m!} t^m \right) \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda} (1) \frac{b_{n-l,\lambda} (-\lambda)}{n-l+1} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$\beta_{n,\lambda}^{(2)} = \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}(1) \frac{b_{n-l,\lambda}(-\lambda)}{n-l+1}.$$

Note that

$$B_n^{(2)} = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(2)} = \sum_{l=0}^n \binom{n}{l} B_l(1) \frac{B_{n-l}}{n-l+1}.$$

From (15), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} &= \frac{\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \tag{26} \\ &= \sum_{m=0}^{\infty} \frac{1}{(m+1)^k} \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right)^m \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} \left(e^{-\frac{1}{\lambda} \log(1 + \lambda t)} - 1 \right)^m \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^k} m! \sum_{l=m}^{\infty} S_2(l, m) \left(-\frac{1}{\lambda} \right)^l \frac{(\log(1 + \lambda t))^l}{l!} \\ &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^l \frac{(-1)^{m+l} m!}{(m+1)^k} S_2(l, m) \lambda^{-l} \right) \frac{1}{l!} (\log(1 + \lambda t))^l \\ &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^l \frac{(-1)^{m+l} m!}{(m+1)^k} S_2(l, m) \lambda^{-l} \right) \sum_{n=l}^{\infty} S_1(n, l) \lambda^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^n \sum_{m=0}^l \frac{(-1)^{m+l} m!}{(m+1)^k} S_2(l, m) S_1(n, l) \lambda^{n-l} \right\} \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (26), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$\beta_{n,\lambda}^{(k)} = \sum_{l=0}^n \sum_{m=0}^l \frac{(-1)^{m+l} m!}{(m+1)^k} S_2(l, m) S_1(n, l) \lambda^{n-l}.$$

Note that

$$B_n^{(k)} = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(k)} = \sum_{m=0}^n \frac{(-1)^{m+n} m!}{(m+1)^k} S_2(n, m).$$

From (23), we have

$$\begin{aligned} \frac{d}{dt} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) &= \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{-\frac{1}{\lambda}-1} \text{Li}_{k-1} \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \tag{27} \\ &= (1 + \lambda t)^{-\frac{1}{\lambda}-1} \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k-1)} \frac{t^n}{n!}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{d}{dt} \left(\text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \right) \tag{28} \\ &= \frac{d}{dt} \left(\left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= (1 + \lambda t)^{-\frac{1}{\lambda}-1} \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}} \right) + (1 - (1 + \lambda t)^{-\frac{1}{\lambda}}) \frac{d}{dt} \left(\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} \right) \\
&= (1 + \lambda t)^{-\frac{1}{\lambda}-1} \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + (1 - (1 + \lambda t)^{-\frac{1}{\lambda}}) \sum_{n=1}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^{n-1}}{(n-1)!}.
\end{aligned}$$

By (27) and (28), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k-1)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + (1 + \lambda t) \left((1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right) \sum_{n=1}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + \left((1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right) \sum_{m=0}^{\infty} \beta_{m+1,\lambda}^{(k)} \frac{t^m}{m!} \\
&\quad + \lambda \left((1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right) \sum_{m=0}^{\infty} \beta_{m,\lambda}^{(k)} m \frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + \left(\sum_{l=1}^{\infty} \left(\frac{1}{\lambda} \right)_l \lambda^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \beta_{m+1,\lambda}^{(k)} \frac{t^m}{m!} \right) \\
&\quad + \lambda \left(\sum_{l=1}^{\infty} \left(\frac{1}{\lambda} \right)_l \lambda^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \beta_{m,\lambda}^{(k)} m \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} \frac{n!}{(n-m)!m!} \beta_{m+1,\lambda}^{(k)} \right) \frac{t^n}{n!} \\
&\quad + \lambda \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} \frac{m \cdot n!}{(n-m)!m!} \beta_{m,\lambda}^{(k)} \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!} + \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} \beta_{m+1,\lambda}^{(k)} \right) \frac{t^n}{n!} \\
&\quad + \lambda \sum_{n=1}^{\infty} \left(\sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} m \beta_{m,\lambda}^{(k)} \right) \frac{t^n}{n!},
\end{aligned} \tag{29}$$

where

$$\left(\frac{1}{\lambda} \right)_n = \left(\frac{1}{\lambda} \right) \left(\frac{1}{\lambda} - 1 \right) \cdots \left(\frac{1}{\lambda} - n + 1 \right) = \sum_{l=0}^n S_1(n, l) \lambda^{-l}, \quad (n \geq 0).$$

Thus, by (29), we have

$$\begin{aligned}
\beta_{n,\lambda}^{(k-1)} &= \beta_{n,\lambda}^{(k)} + \sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} \beta_{m+1,\lambda}^{(k)} \\
&\quad + \lambda \sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} m \beta_{m,\lambda}^{(k)} \\
&= (n+1) \beta_{n,\lambda}^{(k)} + \sum_{m=1}^{n-1} \binom{n}{m-1} \left(\frac{1}{\lambda} \right)_{n-m+1} \lambda^{n-m+1} \beta_{m,\lambda}^{(k)} \\
&\quad + \lambda \sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda} \right)_{n-m} \lambda^{n-m} m \beta_{m,\lambda}^{(k)}, \quad (n \geq 1).
\end{aligned} \tag{30}$$

Therefore, by (30), we obtain the following theorem.

Theorem 2.5. For $n \geq 1$, we have

$$\beta_{n,\lambda}^{(k)} = \frac{1}{n+1} \left\{ \beta_{n,\lambda}^{(k-1)} - \sum_{m=1}^{n-1} \binom{n}{m-1} \beta_{m,\lambda}^{(k)} \left(\frac{1}{\lambda}\right)_{n-m+1} \lambda^{n-m+1} - \lambda \sum_{m=0}^{n-1} \binom{n}{m} \left(\frac{1}{\lambda}\right)_{n-m} \lambda^{n-m} m \beta_{m,\lambda}^{(k)} \right\}.$$

Note that

$$B_n^{(k)} = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(k)} = \frac{1}{n+1} \left\{ B_n^{(k-1)} - \sum_{m=1}^{n-1} \binom{n}{m-1} B_m^{(k)} \right\}.$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)^n (n+1)^{-k} &= \sum_{n=1}^{\infty} \frac{\left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)^n}{n^k} \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \quad (31) \\ &= \frac{1}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} \text{Li}_k \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right) \\ &= \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)} \frac{t^n}{n!}. \end{aligned}$$

By (31), we get

$$\begin{aligned} \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(-k)} \frac{x^n}{n!} \right) \frac{y^k}{k!} &= \sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} \left(1 - (1 + \lambda t)^{-\frac{1}{\lambda}}\right)^m (m+1)^k \right) \frac{y^k}{k!} \quad (32) \\ &= \sum_{m=0}^{\infty} \left(1 - (1 + \lambda x)^{-\frac{1}{\lambda}}\right)^m \sum_{k=0}^{\infty} (m+1)^k \frac{y^k}{k!} \\ &= \sum_{m=0}^{\infty} \left(1 - (1 + \lambda x)^{-\frac{1}{\lambda}}\right)^m e^{(m+1)y} \\ &= \sum_{j=0}^{\infty} (-1)^j \left(e^{-\frac{1}{\lambda} \log(1 + \lambda x)} - 1 \right)^j e^{(j+1)y} \\ &= \sum_{j=0}^{\infty} (-1)^j j! \sum_{m=j}^{\infty} S_2(m, j) (-1)^m \lambda^{-m} \frac{(\log(1 + \lambda x))^m}{m!} e^{(j+1)y} \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^m (-1)^{j+m} j! S_2(m, j) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n, m) \lambda^n \frac{x^n}{n!} e^{(j+1)y} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{j=0}^m (-1)^{j+m} j! S_2(m, j) \lambda^{n-m} S_1(n, m) e^{(j+1)y} \right) \frac{x^n}{n!} \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \sum_{j=0}^m (-1)^{j+m} j! \lambda^{n-m} S_2(m, j) S_1(n, m) (j+1)^k \right) \frac{x^n}{n!} \frac{y^k}{k!}. \end{aligned}$$

Therefore, by (32), we obtain the following theorem.

Theorem 2.6. For $k \in \mathbb{Z}$ and $n \geq 0$, we have

$$\beta_{n,\lambda}^{(-k)} = \sum_{m=0}^n \sum_{j=0}^m (-1)^{j+m} j! \lambda^{n-m} (j+1)^k S_2(m, j) S_1(n, m).$$

Note that

$$B_n^{(-k)} = \lim_{\lambda \rightarrow 0} \beta_{n,\lambda}^{(-k)} = \sum_{j=0}^n (-1)^{j+n} j! (j+1)^k S_2(n, j).$$

From Theorem 2.1, we have

$$\begin{aligned} \frac{d}{dx} \beta_{n,\lambda}^{(k)}(x) &= \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}^{(k)} \frac{d}{dx} \left(\prod_{i=0}^{n-l-1} (x-i\lambda) \right) \\ &= \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}^{(k)} \sum_{j=0}^{n-l-1} \frac{1}{(x-j\lambda)} \prod_{i=0}^{n-l-1} (x-i\lambda). \end{aligned}$$

The generalized falling factorial $(x | \lambda)_n$ is given by

$$(x | \lambda)_n = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \quad (n \geq 0). \tag{33}$$

As is well known, the Bernoulli numbers of the second kind are defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}, \quad (\text{see [20]}). \tag{34}$$

We observe that

$$\int_0^1 (1+\lambda t)^{\frac{x}{\lambda}} dx = \sum_{n=0}^{\infty} \lambda^n \int_0^1 \binom{x}{\lambda}_n dx \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_0^1 (x | \lambda)_n dx \frac{t^n}{n!}. \tag{35}$$

On the other hand,

$$\begin{aligned} \int_0^1 (1+\lambda t)^{\frac{x}{\lambda}} dx &= \frac{\lambda}{\log(1+\lambda t)} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right) \\ &= \frac{\lambda t}{\log(1+\lambda t)} \left(\frac{(1+\lambda t)^{\frac{1}{\lambda}} - 1}{t} \right) \\ &= \left(\sum_{m=0}^{\infty} b_m \lambda^m \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} \frac{(1|\lambda)_{l+1}}{l+1} \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \frac{(1|\lambda)_{l+1}}{l+1} \lambda^{n-l} b_{n-l} \binom{n}{l} \right) \frac{t^n}{n!}. \end{aligned} \tag{36}$$

From (35) and (36), we have

$$\int_0^1 (x | \lambda)_n dx = \sum_{l=0}^n \binom{n}{l} \lambda^{n-l} b_{n-l} \frac{(1|\lambda)_{l+1}}{l+1}, \quad (n \geq 0). \tag{37}$$

By Theorem 2.1, we get

$$\begin{aligned} \int_0^1 \beta_{n,\lambda}^{(k)}(x) dx &= \sum_{l=0}^n \binom{n}{l} \beta_{l,\lambda}^{(k)} \int_0^1 \binom{x}{\lambda}_{n-l} \lambda^{n-l} dx \\ &= \sum_{l=0}^n \binom{n}{l} \beta_{n-l,\lambda}^{(k)} \int_0^1 (x | \lambda)_l dx \\ &= \sum_{l=0}^n \left(\sum_{m=0}^l \binom{l}{m} \lambda^{l-m} b_{l-m} \frac{(1|\lambda)_{m+1}}{m+1} \right) \binom{n}{l} \beta_{n-l,\lambda}^{(k)}. \end{aligned}$$

3 Further remarks

Let \mathbb{C} be complex number field and let \mathcal{F} be the set of all formal power series in the variable t over \mathbb{C} with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{38}$$

Let \mathbb{P} be the algebra of polynomials in a single variable x over \mathbb{C} and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . The action of linear functional $L \in \mathbb{P}^*$ on a polynomial $p(x)$ is denoted by $\langle L | p(x) \rangle$, and linearly extended as

$$\langle cL + c'L' | p(x) \rangle = c \langle L | p(x) \rangle + c' \langle L' | p(x) \rangle,$$

where $c, c' \in \mathbb{C}$.

For $f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$, we define a linear functional on \mathbb{P} by setting

$$\langle f(t) | x^n \rangle = a_n \quad \text{for all } n \geq 0. \tag{39}$$

Thus, by (39), we get

$$\langle t^k | x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [4, 16, 20]}). \tag{40}$$

For $f_L(t) = \sum_{k=0}^{\infty} \langle L | x^k \rangle \frac{t^k}{k!}$, by (40), we get $\langle f_L(t) | x^n \rangle = \langle L | x^n \rangle$. In addition, the mapping $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of the formal power series in t and the vector space of all linear functionals on \mathbb{P} and so an element $f(t)$ of \mathcal{F} can be regarded as both a formal power series and a linear functional. We refer to \mathcal{F} umbral algebra. The umbral calculus is the study of umbral algebra (see [5, 15, 20]). The order $o(f(t))$ of the non-zero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish.

If $o(f(t)) = 1$ (respectively, $o(f(t)) = 0$), then $f(t)$ is called a delta (respectively, an invertible) series (see [20]). For $o(f(t)) = 1$ and $o(g(t)) = 0$, there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, (n, k \geq 0)$.

The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, and we write $s_n(x) \sim (g(t), f(t))$ (see [20]).

Let $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then, by (40), we get

$$f(t) = \sum_{k=0}^{\infty} \langle f(t) | x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k | p(x) \rangle \frac{x^k}{k!}. \tag{41}$$

From (41), we have

$$p^{(k)}(0) = \langle t^k | p(x) \rangle = \langle 1 | p^{(k)}(x) \rangle, \tag{42}$$

where $p^k(x) = \frac{d^k}{dx^k} p(x)$, (see [11, 14, 20]).

By (42), we easily get

$$t^k p(x) = p^k(x), \quad e^{yt} p(x) = p(x+y), \quad \text{and } \langle e^{yt} | p(x) \rangle = p(y). \tag{43}$$

From (43), we have

$$\frac{e^{yt} - 1}{t} p(x) = \int_x^{x+y} p(u) du, \quad \langle e^{yt} - 1 | p(x) \rangle = p(y) - p(0).$$

Let $f(t)$ be the linear functional such that

$$\langle f(t) | p(x) \rangle = \int_0^y p(u) du, \tag{44}$$

for all polynomials $p(x)$. Then it can be determined by (41) to be

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t) | x^k \rangle}{k!} t^k = \sum_{k=0}^{\infty} \frac{y^{k+1}}{(k+1)!} t^k = \frac{1}{t} (e^{yt} - 1). \tag{45}$$

Thus, for $p(x) \in \mathbb{P}$, we have

$$\left\langle \frac{e^{yt} - 1}{t} \middle| p(x) \right\rangle = \int_0^y p(u) du. \tag{46}$$

It is known that

$$s_n(x) \sim (g(t), f(t)) \iff \frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{k=0}^{\infty} s_k(x) \frac{t^k}{k!}, \quad (x \in \mathbb{C}) \tag{47}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ such that $f(\bar{f}(t)) = \bar{f}(f(t)) = t$ (see [11, 20]).

From (15), we note that

$$\beta_{n,\lambda}^{(k)}(x) \sim \left(\frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})}, \frac{1}{\lambda} (e^{\lambda t} - 1) \right). \tag{48}$$

That is,

$$\sum_{n=0}^{\infty} \beta_{n,\lambda}^{(k)}(x) \frac{t^n}{n!} = \frac{\text{Li}_k(1 - (1 + \lambda t)^{-\frac{1}{\lambda}})}{1 - (1 + \lambda t)^{-\frac{1}{\lambda}}} (1 + \lambda t)^{\frac{x}{\lambda}}.$$

Thus, by (48),

$$\frac{1}{\lambda} (e^{\lambda t} - 1) \beta_{n,\lambda}^{(k)}(x) = n \beta_{n-1,\lambda}^{(k)}(x). \tag{49}$$

On the other hand,

$$(e^{\lambda t} - 1) \beta_{n,\lambda}^{(k)}(x) = \beta_{n,\lambda}^{(k)}(x + \lambda) - \beta_{n,\lambda}^{(k)}(x). \tag{50}$$

Therefore, by (49) and (50), we obtain the following theorem.

Theorem 3.1. For $n \in \mathbb{N}$, we have

$$\lambda \beta_{n-1,\lambda}^{(k)}(x) = \frac{1}{n} \left\{ \beta_{n,\lambda}^{(k)}(x + \lambda) - \beta_{n,\lambda}^{(k)}(x) \right\}.$$

By (46), we get

$$\frac{e^{yt} - 1}{t} \beta_{n,\lambda}^{(k)}(x) = \int_x^{x+y} \beta_{n,\lambda}^{(k)}(u) du. \tag{51}$$

From (51), we have

$$\left\langle \frac{e^{yt} - 1}{t} \middle| \beta_{n,\lambda}^{(k)}(x) \right\rangle = \int_0^y \beta_{n,\lambda}^{(k)}(u) du. \tag{52}$$

Thus, by (52), we get

$$\left\langle \frac{e^t - 1}{t} \middle| \beta_{n,\lambda}^{(k)}(x) \right\rangle = \int_0^1 \beta_{n,\lambda}^{(k)}(u) du = \sum_{l=0}^n \sum_{m=0}^l \binom{l}{m} \binom{n}{l} \lambda^{l-m} b_{l-m} \beta_{n-l,\lambda}^{(k)} \frac{(1|\lambda)_{m+1}}{m+1}. \tag{53}$$

Therefore, by (53), we obtain the following theorem.,

Theorem 3.2. For $n \geq 0$, we have

$$\left\langle \frac{e^t - 1}{t} \middle| \beta_{n,\lambda}^{(k)}(x) \right\rangle = \sum_{l=0}^n \sum_{m=0}^l \binom{l}{m} \binom{n}{l} \lambda^{l-m} b_{l-m} \beta_{n-l,\lambda}^{(k)} \frac{(1|\lambda)_{m+1}}{m+1}.$$

Note that

$$\left\langle \frac{e^t - 1}{t} \middle| B_n^{(k)}(x) \right\rangle = \lim_{\lambda \rightarrow 0} \left\langle \frac{e^t - 1}{t} \middle| \beta_{n,\lambda}^{(k)}(x) \right\rangle = \lim_{\lambda \rightarrow 0} \int_0^1 \beta_{n,\lambda}^{(k)}(u) du = \sum_{l=0}^n \binom{n}{l} B_{n-l}^{(k)} \frac{1}{l+1}$$

Let

$$\mathbb{P}_n = \{ p(x) \in \mathbb{C}[x] \mid \deg p(x) \leq n \}, \quad (n \geq 0).$$

For $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x)$, we have

$$\left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle = \sum_{l=0}^n a_l \left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| \beta_{l,\lambda}^{(k)}(x) \right\rangle \tag{54}$$

From (48), we note that

$$\left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| \beta_{l,\lambda}^{(k)}(x) \right\rangle = l! \delta_{l,m}. \tag{55}$$

By (54) and (55), we get

$$a_m = \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle, \quad (m \geq 0). \tag{56}$$

Therefore, by (56), we obtain the following theorem.

Theorem 3.3. For $p(x) \in \mathbb{P}_n$, we have

$$p(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x),$$

where

$$a_m = \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| p(x) \right\rangle.$$

For example, let us take $p(x) = B_n^{(k)}(x)$ ($n \geq 0$). Then, by Theorem 3.3, we have

$$B_n^{(k)}(x) = \sum_{m=0}^n a_m \beta_{m,\lambda}^{(k)}(x), \tag{57}$$

where

$$\begin{aligned} a_m &= \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| B_n^{(k)}(x) \right\rangle \\ &= \frac{1}{m!} \left\langle \frac{1 - e^{-t}}{\text{Li}_k(1 - e^{-t})} \left(\frac{1}{\lambda} (e^{\lambda t} - 1) \right)^m \middle| \frac{\text{Li}_k(1 - e^{-t})}{1 - e^{-t}} x^n \right\rangle \\ &= \frac{\lambda^{-m}}{m!} \left\langle (e^{\lambda t} - 1)^m \middle| x^n \right\rangle = \lambda^{-m} \sum_{l=m}^{\infty} S_2(l, m) \frac{\lambda^l}{l!} \langle t^l \middle| x^n \rangle \\ &= \lambda^{n-m} S_2(n, m). \end{aligned} \tag{58}$$

From (57) and (58), we have

$$B_n^{(k)}(x) = \sum_{m=0}^n \lambda^{n-m} S_2(n, m) \beta_{m,\lambda}^{(k)}(x). \tag{59}$$

References

- [1] Araci S., Acikgoz M., Kilicman A., Extended p -adic q -invariant integrals on \mathbb{Z}_p associated with applications of umbral calculus, *Adv. Difference Equ.*, 2013, 2013:96, 14 pp.
- [2] Bayad A., Simsek Y., Srivastava H. M., Some array type polynomials associated with special numbers and polynomials, *Appl. Math. Comput.*, 2014, 244, 149–157.
- [3] Carlitz L., Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Math.*, 1979, 15, 51–88.
- [4] Dere R., Simsek Y., Applications of umbral algebra to some special polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)*, 2012, 22, 433–438.
- [5] Dere R., Simsek Y., Hermite base Bernoulli type polynomials on the umbral algebra, *Russ. J. Math. Phys.* 2015, 22, 1–5.
- [6] Ding D., Yang J., Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)*, 2010, 20, 7–21.
- [7] Gaboury S., Tremblay R., and B.-J. Fugere, Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials, *Proc. Jangjeon Math. Soc.*, 2014, 17, 97–104.
- [8] He Y., Zhang W., A convolution formula for Bernoulli polynomials, *Ars Combin.*, 2013, 108, 97–104.
- [9] Kaneko M., Masanobu poly-Bernoulli numbers, *J Théor. Nombres Bordeaux*, 1997, 9, 221–228.
- [10] Kim D., Kim T., A note on poly-Bernoulli and higher-order poly-Bernoulli polynomials, *Russ. J. Math. Phys.* 2015, 22, 26–33.
- [11] Kim D. S., Kim T., Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials, *Adv. Stud. Contemp. Math. (Kyungshang)*, 2013, 23, 621–636.
- [12] Kim D. S., Kim T., Higher-order Frobenius-Euler and poly-Bernoulli mixed-type polynomials, *Adv. Difference Equ.*, 2013, 2013:251, 13 pp.
- [13] Kim D. S., Kim T., Daehee polynomials with q -parameter, *Adv. Studies Theor. Phys.*, 2014, 8, 561–569.
- [14] Kim D. S., Kim T., Mansour T., and Dolgy D. V., On poly-Bernoulli polynomials of the second kind with umbral calculus viewpoint, *Adv. Difference Equ.*, 2015, 2015:27, 13 pp.
- [15] Kim T., Barnes' type multiple degenerate Bernoulli and Euler polynomials, *Appl. Math. Comput.* 2015, 258, 556–564.
- [16] Kim T. and Mansour T., Umbral calculus associated with Frobenius-type Eulerian polynomials, *Russ. J. Math. Phys.*, 2014, 21, 484–493.
- [17] Luo Q.-M., Some recursion formula and relations for Bernoulli numbers and Euler numbers of higher order, *Adv. Stud. Contemp. Math. (Kyungshang)*, 2005, 10, 63–70.
- [18] Ozden H., Simsek Y., Srivastava H. M., A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials, *Comput. Math. Appl.*, 2010, 60, 2779–2787.
- [19] Qi F., An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind, *J. Number Theory*, 2014, 144, 244–255.
- [20] Roman S., *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984.
- [21] Sen E., The kind on Apostol-Euler polynomials of higher order arising from Euler basis, *Adv. Stud. Contemp. Math. (Kyungshang)*, 2013, 23, 337–345.