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Research Article

Ninghua Gao, Qingguo Li*, and Zhaowen Li

When do L-fuzzy ideals of a ring generate a distributive lattice?

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Abstract: The notion of L-fuzzy extended ideals is introduced in a Boolean ring, and their essential properties are investigated. We also build the relation between an L-fuzzy ideal and the class of its L-fuzzy extended ideals. By defining an operator "$\rightsquigarrow$" between two arbitrary L-fuzzy ideals in terms of L-fuzzy extended ideals, the result that "the family of all L-fuzzy ideals in a Boolean ring is a complete Heyting algebra" is immediately obtained. Furthermore, the lattice structures of L-fuzzy extended ideals of an L-fuzzy ideal, L-fuzzy extended ideals relative to an L-fuzzy subset, L-fuzzy stable ideals relative to an L-fuzzy subset and their connections are studied in this paper.

Keywords: Boolean ring, Complete Heyting algebra, L-fuzzy extended ideals, L-fuzzy ideals

MSC: 28A60, 06B05

1 Introduction

In 1971, Rosenfeld [1] applied the concept of fuzzy sets to abstract algebra and introduced the notion of fuzzy subgroups, and now the references related to various fuzzy algebraic substructures have been increasing rapidly. For example, Kuroki [2] investigated the properties of fuzzy ideals for a semigroup. Liu [3] introduced the fuzzy subring, etc. Standing upon these achievements, many researchers explored the lattice theoretical properties of these structures, such as, modularity of the lattice of the fuzzy normal subgroups was established in a systematic and step wise manner in [4–10]. By supposing the value "t" at the additive identity of a given ring, the fact that the set of fuzzy ideals with sup property forms a sublattice of the lattice of fuzzy ideals was proved by Ajimal and Thomas [11]. Furthermore, Majumdar and Sultana [12] also investigated the lattice of fuzzy ideals of a ring, and found that it is distributive, however, Zhang and Meng [13] pointed out that this result is erroneous. Subsequently, using a different proof from those of earlier papers, the author in [14] also stated that the lattice of all fuzzy ideals of a ring is modular. Modularity of the lattice of L-ideals of a ring was proved in [15], where L is a completely distributive lattice. Several authors have carried out further studies in this area (see Ref. [3, 16–25]).

Based on the above work, the question "When do L-fuzzy ideals of a ring generate a distributive lattice?" draws our attention. In fact, there exists a nontrivial class of rings so as to all of whose fuzzy ideals form a distributive lattice. The main purpose of this paper is to show that the family of all Boolean rings is such a class. For the Boolean ring, Gao and Cai [26] pointed out that it played a significant role in automata theory, which is the fundamental theory for the computer science technology.

The rest of this paper is organized as follows. In Section 2, we recall some fundamental notions and results to be used in the present paper. In Section 3, the definition of L-fuzzy extended ideals in a Boolean ring is introduced and

Ninghua Gao: College of Mathematics and Econometrics, Hunan University, Changsha, Hunan, China 410082
***Corresponding Author: Qingguo Li:** College of Mathematics and Econometrics, Hunan University, Changsha, Hunan, China 410082, E-mail: liqingguoli@aliyun.com
Zhaowen Li: College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning, Guangxi, China 530006

basic properties are examined. In Section 4, we discuss the lattice structure of L-fuzzy ideals ($LI(R)$) in a Boolean ring by means of L-fuzzy extended ideals, moreover, the lattice structures of its three subsets are investigated. Conclusions are given in Section 5.

2 Preliminaries

We begin by recalling some definitions and results.

In a complete lattice L, for any $S \subseteq L$, write $\bigvee S$ for the lest upper bound of S and $\bigwedge S$ the greatest lower bound of S.

Definition 2.1 ([27]). *A residuated lattice is a structure*

$$(L, \vee, \wedge, \otimes, \rightarrow, 0, 1),$$

which satisfies the following conditions:
(1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice with the least element 0 and the greatest element 1;*
(2) $(L, \otimes, 1)$ *is a commutative monoid with the identity 1;*
(3) $(\otimes, \rightarrow)$ *forms an adjoint pair, i.e.,* $a \otimes b \leq c \Longleftrightarrow a \leq b \rightarrow c$ *for any* $a, b \in L$.
It is easy to check that $a \rightarrow b = \bigvee\{c \in L : a \otimes c \leq b\}$ *for any* $a, b \in L$ *(see [28]). For example* $[0, 1]$ *is a residuated lattice, in which for any* $x, y \in [0, 1]$*:*

$$x \rightarrow y = \bigvee\{z \in L : z \wedge x \leq y\} = \begin{cases} 1, & x \leq y, \\ y, & otherwise. \end{cases}$$

A residuated lattice L is called complete residuated if $(L, \vee, \wedge, 0, 1)$ *is a complete lattice.*

Throughout this paper, we denote by L a complete residuated lattice and R a Boolean ring unless otherwise stated. A ring R is called a Boolean ring if every element is idempotent (i.e., $aa = a$ for all $a \in R$), such a ring is necessarily commutative and addition is modulo 2 (i.e., $a + a = 0$ for all $a \in R$, see [29]).

Properties of (complete) residuated lattices can be found in many papers, e.g. [27, 30–33]. We only give some which are used in the further text.

Proposition 2.2 ([27, 32, 33]). *In any complete residuated lattice* $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$*, the following properties hold for any* $a, b, a_i, b_i, c \in L$ $(i \in \tau)$*:*
(1) $\otimes$ *is isotone in both arguments,* $\rightarrow$ *antitone in the 1st argument and isotone in the 2nd argument;*
(2) $a \leq b \rightarrow (a \otimes b)$*;*
(3) $a \otimes b \leq a \wedge b$*;*
(4) $a \otimes \bigvee_{i\in\tau} b_i = \bigvee_{i\in\tau} (a \otimes b_i),\ a \rightarrow \bigwedge_{i\in\tau} b_i = \bigwedge_{i\in\tau} (a \rightarrow b_i),\ \bigvee_{i\in\tau} a_i \rightarrow b = \bigwedge_{i\in\tau} (a_i \rightarrow b)$*;*
(5) $(a \otimes b) \rightarrow c = b \rightarrow (a \rightarrow c) = a \rightarrow (b \rightarrow c)$*;*
(6) $a \leq b \Longleftrightarrow a \rightarrow b = 1,\ 1 \rightarrow a = a$*;*
(7) $a \otimes (b \rightarrow c) \leq (a \rightarrow b) \rightarrow c$*, specially,* $a \otimes (a \rightarrow b) \leq b$*;*
(8) $\bigvee_{i\in\tau} (a_i \rightarrow b_i) \leq (\bigwedge_{i\in\tau} a_i) \rightarrow (\bigvee_{i\in\tau} b_i)$.

It is well known that a complete Heyting algebra (or frame) is a complete lattice L satisfying the following infinite distributive law:

$$a \wedge (\bigvee B) = \bigvee\{a \wedge b \mid b \in B\},\ \ \forall a \in L,\ B \subseteq L.$$

Thus, a complete residuated lattice is a complete Heyting algebra (i.e., frame) if $\otimes = \wedge$.

An L-fuzzy subset of X is a function from X into L. The set of all L-fuzzy subsets of X is denoted by L^X. For any $f, g \in L^X$, f is called contained in g if $f(x) \leq g(x)$ for every $x \in X$, which is denoted by $f \subseteq g$. In particular, when L is [0,1], the L-fuzzy subsets of X are called fuzzy subsets of X.

Specially, for any $A \subseteq X$, the characteristic function χ_A is defined as follows:

$$\chi_A(y) = \begin{cases} 1, & y \in A, \\ 0, & y \notin A. \end{cases}$$

We denote by χ_x instead of $\chi_{\{x\}}$. Furthermore, $\bar{0}, \bar{1} \in L^X$ are defined as follows:

$$\bar{0} : X \longrightarrow L \ \text{ by } \ x \longmapsto \bar{0}(x) := 0,$$

$$\bar{1} : X \longrightarrow L \ \text{ by } \ x \longmapsto \bar{1}(x) := 1.$$

In [34], the author introduced the concept of L-fuzzy ideals of a commutative ring and gave some propositions (Propositions 2.4-2.6), where L is a completely distributive lattice. Here, we give the similar definition and results when L is a complete residuated lattice, which is a more general structure of truth values than a completely distributive lattice, and we omit the proof.

Definition 2.3. *Let $\mu \in L^R$. Then μ is called an L-fuzzy ideal of R if and only if for any $x, y \in R$, it satisfies the following conditions:*
(I1) $\mu(x) \wedge \mu(y) \leq \mu(x-y)$;
(I2) $\mu(x) \wedge \mu(y) \leq \mu(xy)$;
(I3) $\mu(x) \leq \mu(xy)$.
Denote $LI(R) = \{\mu \mid \mu$ is an L-fuzzy ideal of $R\}$, obviously $\bar{0}, \bar{1} \in LI(R)$.

Proposition 2.4. *For any family $\{\mu_i\}_{i\in\tau} \subseteq LI(R)$, the intersection $\bigcap_{i\in\tau} \mu_i$ is an L-fuzzy ideal of R.*

Proposition 2.5. *Let $\nu \in L^R$. Then the L-fuzzy ideal generated by ν is defined to be the least L-fuzzy ideal of R which contains ν. It is denoted by $\langle \nu \rangle$, that is*

$$\langle \nu \rangle = \bigcap_{\nu \subseteq \mu_i} \{\mu_i \mid \mu_i \in LI(R)\}.$$

Theorem 2.6. *The set of all L-fuzzy ideals $LI(R)$ is a complete lattice under the ordering of L-fuzzy subset inclusion, where for any $\{\mu_i\}_{i\in\tau} \subseteq LI(R)$, the infimum and the supremum are defined as:*

$$\bigwedge_{i\in\tau} \mu_i = \bigcap_{i\in\tau} \mu_i, \quad \bigvee_{i\in\tau} \mu_i = \left\langle \bigcup_{i\in\tau} \mu_i \right\rangle.$$

3 L-fuzzy extended ideals

Let μ be an L-fuzzy ideal of R and $\nu \in L^R$. We define the L-fuzzy extended ideal of μ associated with ν as follows:

$$(\forall x \in R) \ \ \varepsilon_\mu(\nu)(x) = \bigwedge_{b\in R} \nu(b) \to \mu(xb).$$

Specially

$$(\forall x, y \in R) \ \ \varepsilon_\mu(\chi_y)(x) = \bigwedge_{b\in R} \chi_y(b) \to \mu(xb) = \mu(xy).$$

For any L-fuzzy ideal μ, L-fuzzy subset ν of R, we put $\varepsilon_\mu = \{\varepsilon_\mu(\nu) \mid \nu \in L^R\}$ and $\varepsilon^\nu = \{\varepsilon_\mu(\nu) \mid \mu \in LI(R)\}$, respectively.

Proposition 3.1. *Let μ be an L-fuzzy ideal and $\nu \in L^R$. Then we have*
(1) *$\varepsilon_\mu(\nu)$ is an L-fuzzy ideal of R;*
(2) *$\mu \subseteq \varepsilon_\mu(\nu)$.*

Proof. (1) For any $x, y \in R$, μ is an L-fuzzy ideal of R, which implies (i)

$$\begin{aligned}\varepsilon_\mu(\nu)(x-y) &= \bigwedge_{b\in R} \nu(b) \to \mu((x-y)b)\\ &= \bigwedge_{b\in R} \nu(b) \to \mu(xb-yb)\\ &\geq \bigwedge_{b\in R} \nu(b) \to (\mu(xb)\wedge\mu(yb))\\ &= \varepsilon_\mu(\nu)(x)\wedge\varepsilon_\mu(\nu)(y),\end{aligned}$$

(ii)

$$\begin{aligned}\varepsilon_\mu(\nu)(xy) &= \bigwedge_{b\in R} \nu(b) \to \mu(xyb)\\ &= \bigwedge_{b\in R} \nu(b) \to \mu(xybb)\\ &\geq \bigwedge_{b\in R} \nu(b) \to (\mu(xb)\wedge\mu(yb))\\ &= \varepsilon_\mu(\nu)(x)\wedge\varepsilon_\mu(\nu)(y),\end{aligned}$$

(iii)

$$\varepsilon_\mu(\nu)(xy) = \bigwedge_{b\in R} \nu(b) \to \mu(xyb) \geq \bigwedge_{b\in R} \nu(b) \to \mu(xb) = \varepsilon_\mu(\nu)(x),$$

thus, $\varepsilon_\mu(\nu)$ is an L-fuzzy ideal of R.

(2) For any $x \in R$,

$$\varepsilon_\mu(\nu)(x) = \bigwedge_{b\in R} \nu(b) \to \mu(xb) \geq \bigwedge_{b\in R} 1 \to \mu(xb) = \bigwedge_{b\in R} \mu(xb) = \mu(x)$$

by (I3), i.e., $\mu \subseteq \varepsilon_\mu(\nu)$. □

Example 3.2. *Let $R = \{0, p, q, r\}$ with the following Cayley tables:*

+	0	p	q	r
0	0	p	q	r
p	p	0	r	q
q	q	r	0	p
r	r	q	p	0

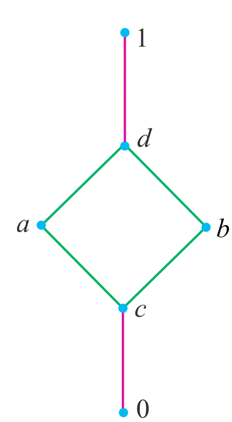
and

·	0	p	q	r
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
r	0	p	q	r

.

One can easily verify that it is a Boolean ring. Let $L = \{0, a, b, c, d, 1\}$ be a complete lattice depicted in Figure 1.

Fig. 1. The lattice L



The precomplement operator ¬ is given in Table 1.

Table 1

x	0	a	b	c	d	1
$\neg x$	1	b	a	d	c	0

The generalized triangular norm ⊗ and the implication operator → in L are defined as follows: for any $x, y \in L$,

$$x \otimes y = \begin{cases} 0, & x \leq \neg y, \\ x \wedge y, & x \nleq \neg y. \end{cases}$$

$$x \to y = \begin{cases} 1, & x \leq y, \\ \neg x \vee y, & x \nleq y. \end{cases}$$

Then $(L, \vee, \wedge, \otimes, \to, 0, 1)$ is a complete residuated lattice.

We check that an L-fuzzy subset

$$f = \left\{\frac{t_1}{0}, \frac{t_2}{p}, \frac{t_3}{q}, \frac{t_4}{r}\right\} \quad (t_i \in L, i = 1, 2, 3, 4)$$

of R is an L-fuzzy ideal of R if and only if $t_1 \geq t_2, t_3, t_4$ and $t_4 = t_2 \wedge t_3$. Now, considering the L-fuzzy ideal

$$f_0 = \left\{\frac{d}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\}$$

and the L-fuzzy subset

$$\nu = \left\{\frac{0}{0}, \frac{d}{p}, \frac{a}{q}, \frac{b}{r}\right\},$$

we can calculate

$$\varepsilon_{f_0}(\nu) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\}.$$

Definition 3.3. *An L-fuzzy ideal μ is called stable relative to the L-fuzzy subset ν if $\mu = \varepsilon_\mu(\nu)$.*

For any L-fuzzy subset ν, denote $S(\nu) = \{\mu \in LI(R) \mid \varepsilon_\mu(\nu) = \mu\}$.

Example 3.4. *Let R and L be the Boolean ring and complete residuated lattice defined as in Example 3.1, respectively. For the L-fuzzy ideal*

$$f_1 = \left\{\frac{1}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\}$$

and the L-fuzzy subset ν in Example 3.1, we have

$$\varepsilon_{f_1}(\nu) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\} = f_1.$$

i.e., f_1 is an L-fuzzy ideal stable relative to ν.

Next we present basic properties of L-fuzzy extended ideals in a Boolean ring.

Proposition 3.5. *Let μ, μ_1, μ_2 be L-fuzzy ideals and $\nu, \nu_1, \nu_2, \omega$ L-fuzzy subsets of R. We have*

(1) $\varepsilon_\mu(\bar{0}) = \varepsilon_{\bar{1}}(\nu) = \bar{1}$. *Moreover, if R has identity e, then $\varepsilon_\mu(\chi_e) = \mu$;*

(2) *if $\nu_1 \subseteq \nu_2$ then $\varepsilon_\mu(\nu_2) \subseteq \varepsilon_\mu(\nu_1)$;*

(3) *if $\mu_1 \subseteq \mu_2$ then $\varepsilon_{\mu_1}(\nu) \subseteq \varepsilon_{\mu_2}(\nu)$;*

(4) $\nu \subseteq \varepsilon_\mu(\varepsilon_\mu(\nu))$;
(5) $\varepsilon_{\varepsilon_\mu(\nu)}(\omega) = \varepsilon_{\varepsilon_\mu(\omega)}(\nu)$;
(6) $\varepsilon_\mu(\nu) = \varepsilon_\mu(\varepsilon_\mu(\varepsilon_\mu(\nu)))$;
(7) *if* $\nu \subseteq \mu$, *then* $\varepsilon_\mu(\nu) = \bar{1}$. *Furthermore, if R has identity e, then* $\varepsilon_\mu(\nu) = \bar{1}$ *if and only if* $\nu \subseteq \mu$;
(8) *if L is a complete Heyting algebra and* $\mu_1 \subseteq \mu_2$, *then* $\varepsilon_{\mu_1}(\mu_2) \cap \mu_2 = \mu_1$;
(9) *(a)* $\varepsilon_\mu(\bigcup_{i\in\tau} \nu_i) = \bigcap_{i\in\tau} \varepsilon_\mu(\nu_i)$, *(b)* $\bigcap_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \varepsilon_{\bigcap_{i\in\tau}\mu_i}(\nu)$;
(10) *if L is a complete Heyting algebra, then* $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) = \varepsilon_\mu(\nu)$;
(11) *(relative extension property) let* $\nu_1 \subseteq \nu_2$, *then if* μ *is an L-fuzzy stable ideal relative to* ν_1, μ *is an L-fuzzy stable ideal relative to* ν_2, *too.*

Proof. (1) It can be easily obtained by the definition of L-fuzzy extended ideals.

(2) Assume that $\nu_1 \subseteq \nu_2$, then for any $x \in R$,

$$\begin{aligned}\varepsilon_\mu(\nu_2)(x) &= \bigwedge_{b\in R} \nu_2(b) \to \mu(xb)\\ &\le \bigwedge_{b\in R} \nu_1(b) \to \mu(xb)\\ &= \varepsilon_\mu(\nu_1)(x),\end{aligned}$$

i.e., $\varepsilon_\mu(\nu_2) \subseteq \varepsilon_\mu(\nu_1)$.

(3) Let $\mu_1 \subseteq \mu_2$. For any $x \in R$,

$$\begin{aligned}\varepsilon_{\mu_1}(\nu)(x) &= \bigwedge_{b\in R} \nu(b) \to \mu_1(xb)\\ &\le \bigwedge_{b\in R} \nu(b) \to \mu_2(xb)\\ &= \varepsilon_{\mu_2}(\nu)(x),\end{aligned}$$

i.e., $\varepsilon_{\mu_1}(\nu) \subseteq \varepsilon_{\mu_2}(\nu)$.

(4) For any $x \in R$,

$$\begin{aligned}\varepsilon_\mu\left(\varepsilon_\mu\left(\nu\right)\right)(x) &= \bigwedge_{b\in R} \varepsilon_\mu(\nu)(b) \to \mu(xb)\\ &= \bigwedge_{b\in R}\left(\bigwedge_{c\in R} \nu(c) \to \mu(bc)\right) \to \mu(xb)\\ &\ge \bigwedge_{b\in R} (\nu(x) \to \mu(bx)) \to \mu(xb)\\ &\ge \bigwedge_{b\in R} \nu(x) \otimes (\mu(bx) \to \mu(xb))\\ &= \nu(x),\end{aligned}$$

hence, $\nu \subseteq \varepsilon_\mu(\varepsilon_\mu(\nu))$.

(5) For any $x \in R$, we get

$$\begin{aligned}\varepsilon_{\varepsilon_\mu(\nu)}(\omega)(x) &= \bigwedge_{b\in R} \omega(b) \to \varepsilon_\mu(\nu)(xb)\\ &= \bigwedge_{b\in R}\left(\omega(b) \to \bigwedge_{c\in R} (\nu(c) \to \mu(xbc))\right)\\ &= \bigwedge_{b\in R}\bigwedge_{c\in R} (\omega(b) \to (\nu(c) \to \mu(xbc)))\\ &= \bigwedge_{b\in R}\bigwedge_{c\in R} (\nu(c) \to (\omega(b) \to \mu(xbc)))\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{c\in R}\left(\nu(c)\rightarrow \bigwedge_{b\in R}(\omega(b)\rightarrow \mu(xbc))\right)\\
&= \bigwedge_{c\in R}\nu(c)\rightarrow \varepsilon_{\mu}(\omega)(xc)\\
&= \varepsilon_{\varepsilon_{\mu}(\omega)}(\nu)(x),
\end{aligned}
$$

therefore, $\varepsilon_{\varepsilon_{\mu}(\nu)}(\omega)=\varepsilon_{\varepsilon_{\mu}(\omega)}(\nu)$.

(6) It follows from (2) and (4).

(7) For any $x\in R$, by (I3), we have

$$
\begin{aligned}
\varepsilon_{\mu}(\nu)(x) &= \bigwedge_{b\in R}\nu(b)\rightarrow \mu(xb)\\
&\geq \bigwedge_{b\in R}\nu(b)\rightarrow \mu(b)\\
&= 1.
\end{aligned}
$$

If R has identity e, assuming $\varepsilon_{\mu}(\nu)=\bar{1}$, i.e., for any $x\in R$, $\nu(b)\leq\mu(xb)$ for all $b\in R$, pick $x=e$, we have $\nu\subseteq\mu$.

(8) Let $\mu_1\subseteq\mu_2$. We only prove $\varepsilon_{\mu_1}(\mu_2)\cap\mu_2\subseteq\mu_1$. L is a complete Heyting algebra, which implies that for any $x\in R$,

$$
\begin{aligned}
(\varepsilon_{\mu_1}(\mu_2)\cap\mu_2)(x) &= \varepsilon_{\mu_1}(\mu_2)(x)\wedge\mu_2(x)\\
&= \left(\bigwedge_{b\in R}\mu_2(b)\rightarrow\mu_1(xb)\right)\wedge\mu_2(x)\\
&\leq (\mu_2(x)\rightarrow\mu_1(xx))\wedge\mu_2(x)\\
&= (\mu_2(x)\rightarrow\mu_1(x))\wedge\mu_2(x)\\
&= \mu_1(x).
\end{aligned}
$$

(9) For any $x\in R$, (a)

$$
\begin{aligned}
\varepsilon_{\mu}(\bigcup_{i\in\tau}\nu_i)(x) &= \bigwedge_{b\in R}\left(\bigvee_{i\in\tau}\nu_i(b)\rightarrow\mu(xb)\right)\\
&= \bigwedge_{i\in\tau}\left(\bigwedge_{b\in R}\nu_i(b)\rightarrow\mu(xb)\right)\\
&= \bigcap_{i\in\tau}\varepsilon_{\mu}(\nu_i),
\end{aligned}
$$

i.e., $\varepsilon_{\mu}(\bigcup_{i\in\tau}\nu_i)=\bigcap_{i\in\tau}\varepsilon_{\mu}(\nu_i)$.

(b)

$$
\begin{aligned}
\left(\bigcap_{i\in\tau}\varepsilon_{\mu_i}(\nu)\right)(x) &= \bigwedge_{i\in\tau}\varepsilon_{\mu_i}(\nu)(x)\\
&= \bigwedge_{i\in\tau}\left(\bigwedge_{b\in R}\nu(b)\rightarrow\mu_i(xb)\right)\\
&= \bigwedge_{b\in R}\left(\nu(b)\rightarrow\bigwedge_{i\in\tau}\mu_i(xb)\right)
\end{aligned}
$$

$$= \bigwedge_{b\in R} (\nu(b) \to (\cap_{i\in\tau}\mu_i)(xb))$$
$$= \varepsilon_{\bigcap\limits_{i\in\tau} \mu_i}(\nu),$$

i.e., $\bigcap\limits_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \varepsilon_{\bigcap\limits_{i\in\tau} \mu_i}(\nu)$.

(10) It is sufficient to prove $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) \subseteq \varepsilon_\mu(\nu)$, for any $x \in R$,

$$\begin{aligned}
\left(\varepsilon_{\varepsilon_\mu(\nu)}(\nu)\right)(x) &= \bigwedge_{b\in R} (\nu(b) \to \varepsilon_\mu(\nu)(xb)) \\
&= \bigwedge_{b\in R} \left(\nu(b) \to \left(\bigwedge_{c\in R} \nu(c) \to \mu(xbc)\right)\right) \\
&\leq \bigwedge_{b\in R} (\nu(b) \to (\nu(b) \to \mu(xbb))) \\
&= \bigwedge_{b\in R} (\nu(b) \to (\nu(b) \to \mu(xb))) \\
&= \bigwedge_{b\in R} ((\nu(b) \wedge \nu(b)) \to \mu(xb)) \\
&= \bigwedge_{b\in R} (\nu(b) \to \mu(xb)) \\
&= \varepsilon_\mu(\nu),
\end{aligned}$$

i.e., $\varepsilon_{\varepsilon_\mu(\nu)}(\nu) = \varepsilon_\mu(\nu)$.

(11) This is a direct result of (2) in Proposition 3.1 and (2) of this proposition. □

The following example indicates that some equations corresponding to (9) in Proposition 3.2 are not always true, which contributes to studying the lattice structure in Section 4.

Example 3.6. *Let R and L be the Boolean ring and complete residuated lattice defined as in Example 3.1, respectively.*

(1) *For the L-fuzzy ideal f_0, L-fuzzy subsets ν in Example 3.1 and the L-fuzzy ideal*

$$f_2 = \left\{\frac{a}{0}, \frac{c}{p}, \frac{a}{q}, \frac{c}{r}\right\},$$

we obtain

$$f_0 \cup f_2 = \left\{\frac{d}{0}, \frac{a}{p}, \frac{d}{q}, \frac{c}{r}\right\}, \ \langle f_0 \cup f_2\rangle = \left\{\frac{d}{0}, \frac{a}{p}, \frac{d}{q}, \frac{a}{r}\right\}, \ \varepsilon_{f_2}(\nu) = \left\{\frac{a}{0}, \frac{c}{p}, \frac{a}{q}, \frac{c}{r}\right\},$$

$$\varepsilon_{\langle f_0\cup f_2\rangle}(\nu) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{1}{q}, \frac{a}{r}\right\} \neq \langle \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_2}(\nu)\rangle = \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_2}(\nu) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{d}{q}, \frac{c}{r}\right\},$$

i.e., $\langle \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_2}(\nu)\rangle \neq \varepsilon_{\langle f_0\cup f_2\rangle}(\nu)$.

(2) *Considering the L-fuzzy ideal f_0, L-fuzzy subset ν in Example 3.1 and the L-fuzzy subset*

$$\omega = \left\{\frac{1}{0}, \frac{b}{p}, \frac{b}{q}, \frac{a}{r}\right\},$$

we get

$$\nu \cap \omega = \left\{\frac{0}{0}, \frac{b}{p}, \frac{c}{q}, \frac{c}{r}\right\}, \ \varepsilon_{f_0}(\omega) = \left\{\frac{d}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\},$$

$$\varepsilon_{f_0}(\nu \cap \omega) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{1}{q}, \frac{a}{r}\right\} \neq \langle \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_0}(\omega)\rangle = \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_0}(\omega) = \left\{\frac{1}{0}, \frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right\},$$

i.e., $\langle \varepsilon_{f_0}(\nu) \cup \varepsilon_{f_0}(\omega)\rangle \neq \varepsilon_{f_0}(\nu \cap \omega)$.

For any L-fuzzy ideal μ of R, we present the following characterization theorem via L-fuzzy extended ideals.

Theorem 3.7. *Let μ be an L-fuzzy ideal of R. Then $\mu = \bigcap \varepsilon_\mu$.*

Proof. We need to prove $\mu = \bigcap\{\varepsilon_\mu(\nu) \mid \nu \in L^R\}$. Obviously, $\mu \subseteq \bigcap_{\nu \in L^R} \varepsilon_\mu(\nu)$ by (2) in Proposition 3.1 . On the other hand, for any $x \in R$, we have

$$\begin{aligned}\left(\bigcap_{\nu \in L^R} \varepsilon_\mu(\nu)\right)(x) &= \bigwedge_{\nu \in L^R} \varepsilon_\mu(\nu)(x) \\ &\leq \varepsilon_\mu(\chi_x)(x) \\ &= \mu(xx) \\ &= \mu(x),\end{aligned}$$

thus, $\mu = \bigcap \varepsilon_\mu$. □

4 Lattice structures

As mentioned in Theorem 2.1, the class of L-fuzzy ideals of a commutative ring is a complete lattice. In this section, three other subsets of this class are also investigated in a Boolean ring R. Moreover, in terms of L-fuzzy extended ideals, we show that all L-fuzzy ideals of R form a complete Heyting algebra, thus a distributive lattice. The following information are reviewed in order to discuss the lattice structures.

In a poset P, for any $S \subseteq P$, we denote $S^u = \{y \mid x \leq y \ \ (\forall x \in S)\}$.

Proposition 4.1 ([35])**.** *Let P be a poset such that $\bigwedge S$ exists in P for every non-empty subset S of P. Then $\bigvee Q$ exists for every non-empty subset Q of P, indeed, $\bigvee Q = \bigwedge Q^u$.*

Theorem 4.2 ([35])**.** *Let P be a non-empty ordered set. Then the following are equivalent:*
(1) *P is a complete lattice;*
(2) *$\bigwedge S$ exists in P for every subset S of P;*
(3) *P has a top element, and $\bigwedge S$ exists in P for every non-empty subset S of P.*

Definition 4.3 ([35])**.** *Let P be an ordered set. A closure operator is a mapping $c : P \to P$ satisfying for every $a, b \in P$,*
(1) *$a \leq c(a)$;*
(2) *$a \leq b \Longrightarrow c(a) \leq c(b)$;*
(3) *$c(c(a)) = c(a)$.*

Denote $P_c = \{x \in P \mid c(x) = x\}$.

Proposition 4.4 ([35])**.** *Let c be a closure operator on an ordered set P. Then*
(1) *$P_c = \{c(x) \mid x \in P\}$;*
(2) *for any $x \in P$, $c(x) = \bigwedge_P\{y \in P_c \mid x \leq y\}$;*
(3) *P_c is a complete lattice, under the order inherited from P, such that, for every subset S of P_c:*

$$\bigwedge_{P_c} S = \bigwedge_P S, \quad \bigvee_{P_c} S = c(\bigvee_P S).$$

Theorem 4.5. *Let μ be an L-fuzzy ideal of R and $\nu \in L^R$. We have*
(1) *if R has identity e, then $(\varepsilon_\mu, \subseteq)$ is a complete lattice with the least element $\varepsilon_\mu(\chi_e)$ and the greatest element $\varepsilon_\mu(\bar{0})$. Moreover, for any $\{\varepsilon_\mu(\nu_i)\}_{i \in \tau} \subseteq \varepsilon_\mu$:*

$$\bigwedge_{i \in \tau} \varepsilon_\mu(\nu_i) = \varepsilon_\mu(\bigcup_{i \in \tau} \nu_i), \ \bigvee_{i \in \tau} \varepsilon_\mu(\nu_i) = \bigcap\{\varepsilon_\mu(\nu_i) \mid i \in \tau\}^u.$$

(2) $(\varepsilon^{\nu}, \subseteq)$ *is a complete lattice with the least element* $\varepsilon_{\bar{0}}(\nu)$ *and the greatest element* $\varepsilon_{\bar{1}}(\nu)$*, and for any* $\{\varepsilon_{\mu_i}(\nu)\}_{i\in\tau} \subseteq \varepsilon^{\nu}$*:*

$$\bigwedge_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \varepsilon_{\bigcap\limits_{i\in\tau} \mu_i}(\nu), \quad \bigvee_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \bigcap\{\varepsilon_{\mu_i}(\nu) \mid i \in \tau\}^u.$$

(3) $(S(\nu), \subseteq)$ *is a sub-complete lattice of* $(\varepsilon^{\nu}, \subseteq)$ *with the greatest element* $\varepsilon_{\bar{1}}(\nu)$*. Furthermore, for any* $\{\mu_i\}_{i\in\tau} \subseteq S(\nu)$*,*

$$\bigwedge_{i\in\tau} \mu_i = \bigcap_{i\in\tau} \mu_i, \quad \bigvee_{i\in\tau} \mu_i = \bigcap_{i\in\tau}\{\mu_i \mid i \in \tau\}^u.$$

Proof. (1) It is trivial that $\bar{1} = \varepsilon_{\mu}(\bar{0}) \in \varepsilon_{\mu}$. According to (9) (a) in Proposition 3.2, for any $\{\varepsilon_{\mu}(\nu_i)\}_{i\in\tau} \subseteq \varepsilon_{\mu}$, $\bigcap\limits_{i\in\tau} \varepsilon_{\mu}(\nu_i) = \varepsilon_{\mu}(\bigcup\limits_{i\in\tau} \nu_i) \in \varepsilon_{\mu}$. This completes the proof by (1) in Proposition 3.2, Theorem 4.1 and Proposition 4.1.

(2) It is obvious that for any $\mu \in LI(R)$, $\varepsilon_{\bar{0}}(\nu) \subseteq \varepsilon_{\mu}(\nu)$ by (3) in Proposition 3.2, and $\bar{1} = \varepsilon_{\bar{1}}(\nu) \in \varepsilon^{\nu}$. For any $\{\varepsilon_{\mu_i}(\nu)\}_{i\in\tau} \subseteq \varepsilon^{\nu}$, from (9) (b) in Proposition 3.2, it follows that $\bigcap_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \varepsilon_{\bigcap\limits_{i\in\tau} \mu_i}(\nu) \in \varepsilon^{\nu}$. Similar to the proof of (1), the proof is completed.

(3) Immediately, $S(\nu) \subseteq \varepsilon^{\nu}$ and $\varepsilon_{\bar{1}}(\nu) \in S(\nu)$. For any $\{\mu_i\}_{i\in\tau} \subseteq S(\nu)$, $\bigcap\limits_{i\in\tau} \mu_i = \bigcap\limits_{i\in\tau} \varepsilon_{\mu_i}(\nu) = \varepsilon_{\bigcap\limits_{i\in\tau} \mu_i}(\nu)$ by Theorem 2.1 and (9) (b) in Proposition 3.2, i.e., $\bigcap\limits_{i\in\tau} \mu_i \in S(\nu)$. Analogously, the proof is completed. □

For an L-fuzzy subset ν of R, the following example indicates that $(S(\nu), \subseteq)$ is not always a complete sublattice of $(\varepsilon^{\nu}, \subseteq)$.

Example 4.6. *Let* R *be the Boolean ring,* L *the complete residuated lattice and* ν *the* L*-fuzzy subset defined as in Example 3.1, respectively. Considering the* L*-fuzzy ideals*

$$g_0 = \left\{\frac{b}{0}, \frac{c}{p}, \frac{b}{q}, \frac{c}{r}\right\}$$

and f_2*, where* f_2 *is defined as in Example 3.3, we have*

$$\varepsilon_{g_0}(\nu) = g_0 = \left\{\frac{b}{0}, \frac{c}{p}, \frac{b}{q}, \frac{c}{r}\right\}, \quad \varepsilon_{f_2}(\nu) = f_2 = \left\{\frac{a}{0}, \frac{c}{p}, \frac{a}{q}, \frac{c}{r}\right\},$$

i.e., $g_0, f_2 \in S(\nu)$*. Put* $\varrho = g_0 \bigvee f_2 = \bigcap\{g_0, f_2\}^u$*, we can calculate*

$$\varrho = \left\{\frac{d}{0}, \frac{c}{p}, \frac{d}{q}, \frac{c}{r}\right\},$$

however,

$$\varepsilon_{\varrho}(\nu) = \left\{\frac{1}{0}, \frac{c}{p}, \frac{1}{q}, \frac{c}{r}\right\} \neq \varrho, \text{ i.e., } \varrho \notin S(\nu).$$

Theorem 4.7. *Let* L *be a complete Heyting algebra. Then* $(LI(R), \wedge, \vee, \rightsquigarrow, \bar{0}, \bar{1})$ *is a complete Heyting algebra, thus a distributive lattice, where for any* $\{\mu_i\}_{i\in\tau} \subseteq LI(R)$*,* $\wedge, \vee$ *are defined as in Theorem 2.1 and for any* $\mu, \varrho \in LI(R)$*,* $\rightsquigarrow$ *is defined as:*

$$\mu \rightsquigarrow \varrho = \varepsilon_{\varrho}(\mu).$$

Proof. It is sufficient to prove that for any $\mu, \varrho, \sigma \in LI(R)$, $\mu \wedge \varrho \subseteq \sigma \Longleftrightarrow \mu \subseteq \varrho \rightsquigarrow \sigma$.

($\Longrightarrow$) For any $x \in R$, we get

$$\begin{aligned}(\varrho \rightsquigarrow \sigma)(x) &= \varepsilon_{\sigma}(\varrho)(x)\\ &= \bigwedge_{b\in R} (\varrho(b) \rightarrow \sigma(xb))\\ &\geq \bigwedge_{b\in R} (\varrho(b) \rightarrow (\mu \cap \varrho)(xb))\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{b\in R} (\varrho(b) \to (\mu(xb) \wedge \varrho(xb))) \\
&\geq \bigwedge_{b\in R} (\varrho(xb) \to (\mu(xb) \wedge \varrho(xb))) \\
&\geq \bigwedge_{b\in R} \mu(xb) \\
&\geq \mu(x),
\end{aligned}
$$

i.e., $\mu \subseteq \varrho \rightsquigarrow \sigma$.

($\Longleftarrow$) L is a complete Heyting algebra, which implies that for any $x \in R$,

$$
\begin{aligned}
(\mu \wedge \varrho)(x) &= \mu(x) \wedge \varrho(x) \\
&\leq (\varrho \rightsquigarrow \sigma)(x) \wedge \varrho(x) \\
&= \varepsilon_{\sigma}(\varrho)(x) \wedge \varrho(x) \\
&= \bigwedge_{b\in R} (\varrho(b) \to \sigma(xb)) \wedge \varrho(x) \\
&\leq (\varrho(x) \to \sigma(xx)) \wedge \varrho(x) \\
&= (\varrho(x) \to \sigma(x)) \wedge \varrho(x) \\
&\leq \sigma(x),
\end{aligned}
$$

i.e., $\mu \wedge \varrho \subseteq \sigma$. □

Theorem 4.8. *Let L be a complete Heyting algebra and $\nu \in L^R$. Then*
(1) *$\varepsilon(\nu)$ is a closure operator on $LI(R)$;*
(2) *$S(\nu) = \varepsilon^{\nu}$;*
(3) *in the complete lattice $(S(\nu)$ $(resp., \varepsilon^{\nu}), \subseteq)$, the least element is $\varepsilon_{\bar{0}}(\nu)$ and for any $\{\mu_i\}_{i\in\tau} \subseteq S(\nu)$,*

$$\bigvee_{i\in\tau} \mu_i = \varepsilon_{\langle \cup_{i\in\tau}\mu_i\rangle}(\nu);$$

(4) *if ν is an L-fuzzy ideal, then $(S(\nu)(resp., \varepsilon^{\nu}), \subseteq)$ is a complete Heyting algebra.*

Proof. (1) It follows from (2) in Proposition 3.1 and (3), (10) in Proposition 3.2.

(2) It is trivial from (1) in this theorem and (1) in Proposition 4.2.

(3) Obviously, for any $\mu \in LI(R)$, $\varepsilon_{\bar{0}}(\nu) \subseteq \varepsilon_{\mu}(\nu) \subseteq \bar{1} = \varepsilon_{\bar{1}}(\nu)$ according to (2) in Proposition 3.1. By (1) in this theorem and (3) in Proposition 4.1, for any $\{\mu_i\}_{i\in\tau} \subseteq S(\nu)$ $(resp., \varepsilon^{\nu})$, $\bigvee_{i\in\tau} \mu_i = \varepsilon_{\langle \cup_{i\in\tau}\mu_i\rangle}(\nu)$.

(4) We only need to prove that for any $\mu, \varrho \in S(\nu)$, $\mu \rightsquigarrow \varrho \in S(\nu)$.

For any $\mu, \varrho \in S(\nu)$, we get $\mu = \varepsilon_{\mu}(\nu)$ and $\varrho = \varepsilon_{\varrho}(\nu)$. Then $\mu \rightsquigarrow \varrho = \mu \rightsquigarrow \varepsilon_{\varrho}(\nu) = \mu \rightsquigarrow (\nu \rightsquigarrow \varrho) = (\nu \wedge \mu) \rightsquigarrow \varrho = \nu \rightsquigarrow (\mu \rightsquigarrow \varrho) = \varepsilon_{\mu\rightsquigarrow\varrho}(\nu) \in \varepsilon^{\nu} = S(\nu)$ by Theorem 4.3 and (2) in this theorem. □

Remark 4.9. *Let L be a complete Heyting algebra and $\nu \in LI(R)$. Then (1) in Example 3.3 illustrates that $(S(\nu)$ $(resp., \varepsilon^{\nu}), \subseteq)$ may not be a subalgebra of $(LI(R), \sqcap, \sqcup, \rightsquigarrow, \bar{0}, \bar{1})$.*

5 Conclusions

By the aid of L-fuzzy extended ideals, which are firstly introduced in this work, we conclude that if R is a Boolean ring, then the lattice of all its L-fuzzy ideals $(LI(R), \subseteq)$ is distributive. We will consider whether the converse is affirmative or not in our further work. In this paper, we have also obtained some other results such as: the family of L-fuzzy extended ideals of an L-fuzzy ideal μ $(\varepsilon_{\mu}, \subseteq)$ forms a complete lattice, all L-fuzzy extended ideals relative to an L-fuzzy subset ν $(\varepsilon^{\nu}, \subseteq)$ generate a complete lattice and the lattice of L-fuzzy stable ideals relative to an L-fuzzy subset ν $(S(\nu), \subseteq)$ is a sub-complete lattice rather than a complete sublattice of $(\varepsilon^{\nu}, \subseteq)$. In particular, if L is a complete Heyting algebra, then ε^{ν} and $S(\nu)$ coincide, and the class of L-fuzzy stable ideals relative to an L-fuzzy ideal produces a complete Heyting algebra, but it is not a subalgebra of $(LI(R), \subseteq)$.

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