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Research Article

Yuhua Long, Yuanbiao Zhang, and Haiping Shi*

Homoclinic solutions of 2nth-order difference equations containing both advance and retardation

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Abstract: By using the critical point method, some new criteria are obtained for the existence and multiplicity of homoclinic solutions to a $2n$th-order nonlinear difference equation. The proof is based on the Mountain Pass Lemma in combination with periodic approximations. Our results extend and improve some known ones.

Keywords: Homoclinic solutions, $2n$th-order, Nonlinear difference equations, Discrete variational theory

MSC: 34C37, 37J45, 39A12

1 Introduction

The problem of homoclinic orbits for differential equations has been the subject of many investigations. As is known to us, homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon. So homoclinic orbits have been extensively investigated since the time of Poincaré, see for instance [1–8] and the references therein.

On the other hand, difference equations [9] are closely related to differential equations in the sense that a differential equation model is often derived from a difference equation, and numerical solutions of a differential equation have to be obtained by discretizing the differential equation. Therefore, the study of homoclinic orbits [10–21] of difference equation is meaningful.

Let $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. For $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a,b) = \{a, a+1, \cdots, b\}$ when $a < b$. l^2 denotes the space of all real functions whose second powers are summable on $\mathbf{Z}$. Also, $*$ denotes the transpose of a vector.

The present paper considers the $2n$th-order difference equation

$$\Delta^n \left(\gamma_{k-n}\Delta^n u_{k-n}\right) + (-1)^n \chi_k u_k = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ k \in \mathbf{Z}, \tag{1}$$

where n is a fixed positive integer, Δ is the forward difference operator $\Delta u_k = u_{k+1} - u_k$, $\Delta^n u_k = \Delta(\Delta^{n-1} u_k)$, γ_k and χ_k are positive real valued for each $k \in \mathbf{Z}$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, γ_k, χ_k and $f(k, v_1, v_2, v_3)$ are T-periodic in k for a given positive integer T.

Yuhua Long: School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China and Key Laboratory of Mathematics and Interdisciplinary Sciences of Guangdong Higher Education Institute, Guangzhou University, Guangzhou 510006, China

Yuanbiao Zhang: Packaging Engineering Institute, Jinan University, Zhuhai 519070, China

***Corresponding Author: Haiping Shi:** Modern Business and Management Department, Guangdong Construction Polytechnic, Guangzhou 510440, China, E-mail: shp7971@163.com

Difference equations containing both advance and retardation have many applications in theory and practice [9–11, 22]. We may think of (1) as a discrete analogue of the following $2n$th-order functional differential equation

$$\frac{d^n}{dt^n}\left[\gamma(t)\frac{d^n u(t)}{dt^n}\right]+(-1)^n\chi(t)u(t)=(-1)^n f(t,u(t+1),u(t),u(t-1)),\ t\in\mathbf{R}. \tag{2}$$

Equations similar in structure to (2) arise in the study of homoclinic orbits [2, 4–6] of functional differential equations for which the evolution of the function depends on its current state, its history, and its future as well. Such a problem is of special significance for the study of master equations in stochastic process [2–6].

Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [23] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. Compared to first-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, for example, [10–25] and the references contained therein). Peil and Peterson [26] in 1994 studied the asymptotic behavior of solutions of $2n$th-order difference equation

$$\sum_{i=0}^{n}\Delta^i\left(r_i(k-i)\Delta^i u(k-i)\right)=0 \tag{3}$$

with $r_i(k)\equiv 0$ for $1\le i\le n-1$. In 1998, Anderson [27] considered (3) for $k\in\mathbf{Z}(a)$, and obtained a formulation of generalized zeros and (n,n)-disconjugacy for (3). Migda [28] in 2004 studied an mth-order linear difference equation. Cai, Yu [24] in 2007 and Zhou, Yu, Chen [25] in 2010 obtained some criteria for the existence of periodic solutions of the following difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right)+f(k,u_k)=0. \tag{4}$$

In 2011, Chen and Tang [11] established some new existence criteria to guarantee that the $2n$th-order nonlinear difference equation

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right)+q(k)u(k)=f(k,u_{k+n},\cdots,u_k,\cdots,u_{k-n}) \tag{5}$$

has at least one or infinitely many homoclinic solutions.

However, to the best of our knowledge, the results on homoclinic solutions of higher-order nonlinear difference equations are scarce in the literature [10, 11, 15, 20, 24, 27]. Furthermore, since (1) contains both advance and retardation, there are very few manuscripts dealing with this subject. The purpose of this paper is two-folded. On one hand, we shall further demonstrate the powerfulness of critical point theory in the study of homoclinic orbits for difference equations. On the other hand, we shall extend and improve some existing results. In fact, one can see the following Remarks 1.3 and 1.4 for details. The proof is based on the notable Mountain Pass Lemma in combination with variational technique. The motivation for the present work stems from the recent papers [1, 11, 20].

Let

$$\underline{\gamma}=\min_{k\in\mathbf{Z}(1,T)}\{\gamma_k\},\ \bar{\gamma}=\max_{k\in\mathbf{Z}(1,T)}\{\gamma_k\},\ \underline{\chi}=\min_{k\in\mathbf{Z}(1,T)}\{\chi_k\},\ \bar{\chi}=\max_{k\in\mathbf{Z}(1,T)}\{\chi_k\}.$$

Our main results are as follows.

Theorem 1.1. *Assume that the following hypotheses are satisfied:*

(F_1) *there exists a function $F(k,v_1,v_2)\in C^1(\mathbf{Z}\times\mathbf{R}^2,\mathbf{R})$ with $F(k+T,v_1,v_2)=F(k,v_1,v_2)$ and it satisfies*

$$\frac{\partial F(k-1,v_2,v_3)}{\partial v_2}+\frac{\partial F(k,v_1,v_2)}{\partial v_2}=f(k,v_1,v_2,v_3);$$

(F_2) *there exist positive constants ϱ and $a<\frac{\underline{\chi}}{4}$ such that $|F(k,v_1,v_2)|\le a\left(v_1^2+v_2^2\right)$ for all $k\in\mathbf{Z}$ and $\sqrt{v_1^2+v_2^2}\le\varrho$;*

(F_3) *there exist constants $\rho, c>4^{n-1}\bar{\gamma}+\frac{\bar{\chi}}{4}$ and b such that $F(k,v_1,v_2)\ge c\left(v_1^2+v_2^2\right)+b$ for all $k\in\mathbf{Z}$ and $\sqrt{v_1^2+v_2^2}\ge\rho$;*

(F_4) $\frac{\partial F(k,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(k,v_1,v_2)}{\partial v_2}v_2 - 2F(k,v_1,v_2) > 0$, *for all* $(k,v_1,v_2) \in \mathbf{Z} \times \boldsymbol{R}^2 \setminus \{(0,0)\}$;
(F_5) $\frac{\partial F(k,v_1,v_2)}{\partial v_1}v_1 + \frac{\partial F(k,v_1,v_2)}{\partial v_2}v_2 - 2F(k,v_1,v_2) \to +\infty$ *as* $\sqrt{v_1^2+v_2^2} \to +\infty$.
Then (1) *has a nontrivial homoclinic solution.*

Remark 1.2. *By* (F_3), *it is easy to see that there exists a constant* $\zeta > 0$ *such that*

$$(F_3')\ F(k,v_1,v_2) \geq c\left(v_1^2+v_2^2\right) + b - \zeta,\ \forall (k,v_1,v_2) \in \mathbf{Z} \times \boldsymbol{R}^2.$$

As a matter of fact, let $\zeta = \max\left\{\left|F(k,v_1,v_2) - c\left(v_1^2+v_2^2\right) - b\right| : k \in \mathbf{Z}, \sqrt{v_1^2+v_2^2} \leq \rho\right\}$, we can easily get the desired result.

Remark 1.3. *Theorem 1.1 extends Theorem 1.1 in [19] which is the special case of our Theorem 1.1 by letting* $n = 1$.

Remark 1.4. *In many studies (see e.g. [16, 18, 19, 22, 23]) of second order difference equations, the following classical Ambrosetti-Rabinowitz condition is assumed.*
(AR) *there exists a constant* $\beta > 2$ *such that* $0 < \beta F(k,u) \leq uf(k,u)$ *for all* $k \in \mathbf{Z}$ *and* $u \in \boldsymbol{R} \setminus \{0\}$.
Note that $(F_3) - (F_5)$ *are much weaker than* $(\boldsymbol{AR})$. *Thus our result improves the existing ones.*

Theorem 1.5. *Assume that* $(F_1) - (F_5)$ *and the following hypothesis are satisfied:*
(F_6) $\gamma_{-k} = \gamma_k,\ \chi_{-k} = \chi_k,\ F(-k,v_1,v_2) = F(k,v_1,v_2)$.
Then (1) *has a nontrivial even homoclinic solution.*

For the basic knowledge of variational methods, the reader is referred to [29, 30].

2 Preliminaries

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1) and give some lemmas which will be of fundamental importance in proving our results. We start by some basic notations.

Let S be the set of sequences $u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty}$, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R},\ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, E_m is defined as a subspace of S by

$$E_m = \{u \in S | u_{k+2mT} = u_k,\ \forall k \in \mathbf{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbf{R}^{2mT}$. E_m can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=-mT}^{mT-1} u_j v_j,\ \forall u, v \in E_m, \tag{6}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT-1} u_j^2\right)^{\frac{1}{2}},\ \forall u \in E_m. \tag{7}$$

It is obvious that E_m with the inner product (6) is a finite dimensional Hilbert space and linearly homeomorphic to $\mathbf{R}^{2mT}$.

For all $u \in E_m$, define the functional J on E_m as follows:

$$J(u) = \frac{1}{2}\sum_{k=-mT}^{mT-1} \gamma_{k-1}\left(\Delta^n u_{k-1}\right)^2 + \frac{1}{2}\sum_{k=-mT}^{mT-1} \chi_k u_k^2 - \sum_{k=-mT}^{mT-1} F(k, u_{k+1}, u_k). \tag{8}$$

Clearly, $J \in C^1(E_m, \mathbf{R})$ and for any $u = \{u_k\}_{k\in\mathbf{Z}} \in E_m$, by the periodicity of $\{u_k\}_{n\in\mathbf{Z}}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}\right) + \chi_k u_k - f(k, u_{k+1}, u_k, u_{k-1}). \tag{9}$$

Thus, u is a critical point of J on E_m if and only if

$$\Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}\right) + (-1)^n \chi_k u_k = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}),\ \forall k \in \mathbf{Z}(-mT, mT-1).$$

Due to the periodicity of $u = \{u_k\}_{k\in\mathbf{Z}} \in E_m$ and $f(k, v_1, v_2, v_3)$ in the first variable k, we reduce the existence of periodic solutions of (1) to the existence of critical points of J on E_m. That is, the functional J is just the variational framework of (1).

In what follows, we define a norm $\|\cdot\|_\infty$ in E_m by

$$\|u\|_\infty = \max_{j\in\mathbf{Z}(-mT,mT-1)} |u_j|,\ \forall u \in E_m.$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(i)}\} \subset E$ for which $\{J(u^{(i)})\}$ is bounded and $J'(u^{(i)}) \to 0$ $(i \to \infty)$ possesses a convergent subsequence in E.

Let P be the $2mT \times 2mT$ matrix defined by

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of P are

$$\lambda_j = 2\left(1 - \cos\frac{j}{mT}\pi\right), j = 0, 1, 2, \cdots, 2mT-1. \tag{10}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{2mT-1} > 0$. Therefore, $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{2mT-1}\} = 4$.

For convenience, we identify $u \in E_m$ with $u = (u_{-mT}, u_{-mT+1}, \cdots, u_{mT-1})^*$. Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [29, 30]). *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*
(J_1) there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$, and
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.
Then J possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]} J(g(s)), \tag{11}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{12}$$

Lemma 2.2. *The following inequality is true:*

$$\frac{1}{2}\sum_{k=-mT}^{mT-1}\gamma_{k-1}\left(\Delta^n u_{k-1}\right)^2 \le \frac{4^n\bar{\gamma}}{2}\|u\|^2. \tag{13}$$

Proof. From the definition of P,

$$\frac{1}{2}\sum_{k=-mT}^{mT-1}\gamma_{k-1}\left(\Delta^n u_{k-1}\right)^2 = \frac{1}{2}\sum_{k=-mT}^{mT-1}\gamma_k\left(\Delta^n u_k, \Delta^n u_k\right) \le \frac{\bar{\gamma}}{2}x^*Px \le \frac{\bar{\gamma}}{2}\cdot 4\|x\|^2,$$

where $x = (\Delta^{n-1}u_{-mT}, \Delta^{n-1}u_{-mT+1}, \cdots, \Delta^{n-1}u_{mT-1})^*$. Since

$$\|x\|^2 = \sum_{k=-mT}^{mT-1}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \le 4\sum_{k=-mT}^{mT-1}\left(\Delta^{n-2}u_k\right)^2 \le 4^{n-1}\|u\|^2,$$

we have

$$\frac{1}{2}\sum_{k=-mT}^{mT-1}\gamma_{k-1}\left(\Delta^n u_{k-1}\right)^2 \le \frac{4^n\bar{\gamma}}{2}\|u\|^2.$$

□

3 Proofs of theorems

In this section, we shall prove our main results by using the critical point method.

Lemma 3.1. *Assume that $(F_1)-(F_5)$ are satisfied. Then J satisfies the P.S. condition.*

Proof. Assume that $\{u^{(i)}\}_{i\in\mathbf{N}}$ in E_m is a sequence such that $\{J(u^{(i)})\}_{i\in\mathbf{N}}$ is bounded. Then there exists a constant $K>0$ such that $-K \le J(u^{(i)})$. By (13) and (F_3'), we have

$$\begin{aligned}-K \le J\left(u^{(i)}\right) &\le \frac{4^n\bar{\gamma}}{2}\left\|u^{(i)}\right\|^2 + \frac{\bar{\chi}}{2}\left\|u^{(i)}\right\|^2 - \sum_{k=-mT}^{mT-1}\left[c\left(\left|u_{k+1}^{(i)}\right|^2 + \left|u_k^{(i)}\right|^2\right) + b - \zeta\right] \\ &\le \left(\frac{4^n\bar{\gamma}}{2} + \frac{\bar{\chi}}{2} - 2c\right)\left\|u^{(i)}\right\|^2 + 2mT(\zeta - b).\end{aligned}$$

Therefore,

$$\left(2c - \frac{4^n\bar{\gamma}}{2} - \frac{\bar{\chi}}{2}\right)\left\|u^{(i)}\right\|^2 \le 2mT(\zeta - b) + K. \tag{14}$$

Since $c > 4^{n-1}\bar{\gamma} + \frac{\bar{\chi}}{4}$, (14) implies that $\{u^{(i)}\}_{i\in\mathbf{N}}$ is bounded in E_m. Thus, $\{u^{(i)}\}_{i\in\mathbf{N}}$ possesses a convergence subsequence in E_m. The desired result follows. □

Lemma 3.2. *Assume that $(F_1)-(F_5)$ are satisfied. Then for any given positive integer m,* (1) *possesses a $2mT$-periodic solution $u^{(m)} \in E_m$.*

Proof. In our case, it is clear that $J(0)=0$. By Lemma 3.1, J satisfies the P.S. condition. By (F_2), we have

$$J(u) \ge \frac{\chi}{2}\|u\|^2 - a\sum_{k=-mT}^{mT-1}\left(|u_{k+1}|^2 + |u_k|^2\right) \ge \frac{\chi}{2}\|u\|^2 - 2a\|u\|^2 = \left(\frac{\chi}{2} - 2a\right)\|u\|^2.$$

Taking $\alpha = \left(\frac{\chi}{2} - 2a\right)\varrho^2 > 0$, we obtain

$$J(u)|_{\partial B_\varrho} \ge \alpha > 0,$$

which implies that J satisfies the condition (J_1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J_2).

There exists a sufficiently large number $\varepsilon > \max\{\varrho, \rho\}$ such that

$$\left(2c - \frac{4^n \bar{\gamma}}{2} - \frac{\bar{\chi}}{2}\right)\varepsilon^2 \geq |b|. \tag{15}$$

Let $e \in E_m$ and

$$e_k = \begin{cases} \varepsilon, \text{ if } k = 0, \\ 0, \text{ if } k \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}, \end{cases}$$

$$e_{k+1} = \begin{cases} \varepsilon, \text{ if } k = 0, \\ 0, \text{ if } k \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

Then

$$F(k, e_{k+1}, e_k) = \begin{cases} F(0, \varepsilon, \varepsilon), \text{ if } k = 0, \\ 0, \qquad \text{if } k \in \{j \in \mathbf{Z} : -mT \leq j \leq mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

With (15) and (F_3), we have

$$\begin{aligned} J(e) &= \frac{1}{2}\sum_{k=-mT}^{mT-1} \gamma_{k-1}\left(\Delta^n e_{k-1}\right)^2 + \frac{1}{2}\sum_{k=-mT}^{mT-1} \chi_k e_k^2 - \sum_{k=-mT}^{mT-1} F(k, e_{k+1}, e_k) \\ &\leq \frac{4^n \bar{\gamma}}{2}\|e\|^2 + \frac{\bar{\chi}}{2}\|e\|^2 - 2c\|e\|^2 - b \\ &= -\left(2c - \frac{4^n \bar{\gamma}}{2} - \frac{\bar{\chi}}{2}\right)\varepsilon^2 - b \leq 0. \end{aligned} \tag{16}$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently, J possesses a critical value c_m given by (11) and (12) with $E = E_m$ and $\Gamma = \Gamma_m$, where $\Gamma_m = \{g_m \in C([0,1], E_m) | g_m(0) = 0,\ g_m(1) = e,\ e \in E_m \backslash B_\varepsilon\}$.

Let $u^{(m)}$ denote the corresponding critical point of J on E_m. Note that $\left\|u^{(m)}\right\| \neq 0$ since $c_m > 0$. □

Lemma 3.3. *Assume that $(F_1) - (F_5)$ are satisfied. Then there exist positive constants ϱ and η independent of m such that*

$$\varrho \leq \left\|u^{(m)}\right\|_\infty \leq \eta. \tag{17}$$

Proof. The continuity of $F(0, v_1, v_2)$ with respect to the second and third variables implies that there exists a constant $\tau > 0$ such that $|F(0, v_1, v_2)| \leq \tau$ for $\sqrt{v_1^2 + v_2^2} \leq \varrho$. It is clear that

$$\begin{aligned} J\left(u^{(m)}\right) &\leq \max_{0 \leq s \leq 1}\left\{\frac{1}{2}\sum_{k=-mT}^{mT-1}\left(\gamma_{k-1}\left(\Delta^n (se)_{k-1}\right)^2 + \chi_k (se)_k^2\right) - \sum_{k=-mT}^{mT-1} F\left(k, (se)_{k+1}, (se)_k\right)\right\} \\ &\leq \left(\frac{4^n \bar{\gamma}}{2} + \frac{\bar{\chi}}{2}\right)\|e\|^2 + \tau = \left(\frac{4^n \bar{\gamma}}{2} + \frac{\bar{\chi}}{2}\right)\varepsilon^2 + \tau. \end{aligned}$$

Let $\xi = \left(\frac{4^n \bar{\gamma}}{2} + \frac{\bar{\chi}}{2}\right)\varepsilon^2 + \tau$, we have that $J\left(u^{(m)}\right) \leq \xi$, which is independent of m. From (8) and (9), we have

$$\begin{aligned} J\left(u^{(m)}\right) &= \frac{1}{2}\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k-1, u_k^{(m)}, u_{k-1}^{(m)})}{\partial v_2} u_k^{(m)} + \frac{\partial F(k, u_{k+1}^{(m)}, u_k^{(m)})}{\partial v_2} u_k^{(m)}\right] - \sum_{k=-mT}^{mT-1} F(k, u_{k+1}^{(m)}, u_k^{(m)}) \\ &= \frac{1}{2}\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k, u_{k+1}^{(m)}, u_k^{(m)})}{\partial v_1} u_{k+1}^{(m)} + \frac{\partial F(k, u_{k+1}^{(m)}, u_k^{(m)})}{\partial v_2} u_k^{(m)}\right] - \sum_{k=-mT}^{mT-1} F(k, u_{k+1}^{(m)}, u_k^{(m)}) \leq \xi. \end{aligned}$$

By (F_4) and (F_5), there exists a constant $\eta > 0$ such that $\frac{1}{2}\left(\frac{\partial F(k, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(k, v_1, v_2)}{\partial v_2} v_2\right) - F(k, v_1, v_2) > \xi$, for all $k \in \mathbf{Z}$ and $\sqrt{v_1^2 + v_2^2} \geq \eta$, which implies that $\left|u_k^{(m)}\right| \leq \eta$ for all $k \in \mathbf{Z}$, that is $\left\|u^{(m)}\right\|_\infty \leq \eta$.

From the definition of J, we have

$$0=\left\langle J'(u^{(m)}),u^{(m)}\right\rangle \geq \underline{\chi}\sum_{k=-mT}^{mT-1}\left|u_n^{(m)}\right|^2-\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k-1,u_k^{(m)},u_{k-1}^{(m)})}{\partial v_2}u_k^{(m)}+\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}u_k^{(m)}\right]$$

$$\geq \underline{\chi}\|u^{(m)}\|^2-\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}u_{k+1}^{(m)}+\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}u_k^{(m)}\right].$$

Therefore, combined with (F_2), we get

$$\underline{\chi}\|u^{(m)}\|^2\leq \sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}u_{k+1}^{(m)}+\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}u_k^{(m)}\right]$$

$$\leq \left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}\|u^{(m)}\|+\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}\|u^{(m)}\|.$$

That is,

$$\underline{\chi}\|u^{(m)}\|\leq \left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}.$$

Thus,

$$\underline{\chi}^2\|u^{(m)}\|^2\leq \left\{\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}\right\}^2. \quad (18)$$

Combined with (F_2), we get

$$\underline{\chi}^2\|u^{(m)}\|^2\leq \left\{\left\{\sum_{k=-mT}^{mT-1}\left[2a\left|u_{k+1}^{(m)}\right|\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=-mT}^{mT-1}\left[2a\left|u_k^{(m)}\right|\right]^2\right\}^{\frac{1}{2}}\right\}^2\leq 16a^2\|u^{(m)}\|^2.$$

Thus, we have $u^{(m)}=0$. But this contradicts $\|u^{(m)}\|\neq 0$, which shows that

$$\|u^{(m)}\|_\infty \geq \varrho,$$

and the proof of Lemma 3.3 is finished. □

Proof of Theorem 1.1. In the following, we shall give the existence of a nontrivial homoclinic solution.

Consider the sequence $\left\{u_k^{(m)}\right\}_{k\in\mathbf{Z}}$ of $2mT$-periodic solutions found in Lemma 3.2. First, by (17), for any $m\in\mathbf{N}$, there exists a constant $k_m\in\mathbf{Z}$ independent of m such that

$$\left|u_{k_m}^{(m)}\right|\geq \varrho. \quad (19)$$

Since γ_k, χ_k and $f(k,v_1,v_2,v_3)$ are all T-periodic in k, $\left\{u_{k+jT}^{(m)}\right\}$ $(\forall j\in\mathbf{N})$ is also $2mT$-periodic solution of (1). Hence, making such shifts, we can assume that $k_m\in\mathbf{Z}(0,T-1)$ in (19). Moreover, passing to a subsequence of ms, we can even assume that $k_m=k_0$ is independent of m.

Next, we extract a subsequence, still denote by $u^{(m)}$, such that

$$u_k^{(m)}\to u_k,\ m\to\infty,\ \forall k\in\mathbf{Z}.$$

Inequality (19) implies that $\left|u_{k_0}\right|\geq \xi$ and, hence, $u=\{u_k\}$ is a nonzero sequence. Moreover,

$$\Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}\right)+(-1)^n\chi_k u_k-(-1)^n f(k,u_{k+1},u_k,u_{k-1})$$

$$= \lim_{n\to\infty}\left[\Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}^{(m)}\right)+(-1)^n\chi_k u_k^{(m)}-(-1)^n f\left(k,u_{k+1}^{(m)},u_k^{(m)},u_{k-1}^{(m)}\right)\right]=0.$$

So $u=\{u_k\}$ is a solution of (1).

Finally, we show that $u\in l^2$. For $u_m\in E_m$, let

$$P_m=\left\{k\in\mathbf{Z}:\left|u_k^{(m)}\right|<\frac{\sqrt{2}}{2}\varrho,-mT\le k\le mT-1\right\},$$

$$Q_m=\left\{k\in\mathbf{Z}:\left|u_k^{(m)}\right|\ge\frac{\sqrt{2}}{2}\varrho,-mT\le k\le mT-1\right\}.$$

Since $F(k,v_1,v_2)\in C^1(\mathbf{Z}\times\mathbf{R}^2,\mathbf{R})$, there exist constants $\bar{\xi}>0$, $\underline{\xi}>0$ such that

$$\max\left\{\left\{\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,v_1,v_2)}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k=-mT}^{mT-1}\left[\frac{\partial F(k,v_1,v_2)}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}\right\}^2:\varrho\le\sqrt{v_1^2+v_2^2}\le\eta,k\in\mathbf{Z}\right\}\le\bar{\xi},$$

$$\min\left\{\frac{1}{2}\left[\frac{\partial F(k,v_1,v_2)}{\partial v_1}v_1+\frac{\partial F(k,v_1,v_2)}{\partial v_2}v_2\right]-F(k,v_1,v_2):\varrho\le\sqrt{v_1^2+v_2^2}\le\eta,k\in\mathbf{Z}\right\}\ge\underline{\xi}.$$

For $k\in Q_m$,

$$\left\{\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}$$
$$\le\frac{\bar{\xi}}{\underline{\xi}}\left\{\frac{1}{2}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}u_{k+1}^{(m)}+\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}u_k^{(m)})\right]-F(k,u_{k+1}^{(m)},u_k^{(m)})\right\}.$$

By (18), we have

$$\underline{\chi}^2\|u^{(m)}\|^2\le\left\{\left\{\sum_{k\in P_m}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k\in P_m}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}\right\}^2$$
$$+\left\{\left\{\sum_{k\in Q_m}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k\in Q_m}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}\right]^2\right\}^{\frac{1}{2}}\right\}^2$$
$$\le\left\{\left\{\sum_{k\in P_m}\left[2a\left|u_{k+1}^{(m)}\right|\right]^2\right\}^{\frac{1}{2}}+\left\{\sum_{k\in P_m}\left[2a\left|u_k^{(m)}\right|\right]^2\right\}^{\frac{1}{2}}\right\}^2$$
$$+\frac{\bar{\xi}}{\underline{\xi}}\left\{\frac{1}{2}\sum_{k\in Q_m}\left[\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_1}u_{k+1}^{(m)}+\frac{\partial F(k,u_{k+1}^{(m)},u_k^{(m)})}{\partial v_2}u_k^{(m)})\right]-F(k,u_{k+1}^{(m)},u_k^{(m)})\right\}$$
$$\le16a^2\|u^{(m)}\|^2+\frac{\bar{\xi}\xi}{\underline{\xi}}.$$

Thus,

$$\left\|u^{(m)}\right\|^2\le\frac{\bar{\xi}\xi}{\underline{\xi}\left(\underline{\chi}^2-16a^2\right)}.$$

For any fixed $D \in \mathbf{Z}$ and m large enough, we have that

$$\sum_{k=-D}^{D} \left|u_k^{(m)}\right|^2 \leq \left\|u^{(m)}\right\|^2 \leq \frac{\bar{\xi}\xi}{\underline{\xi}\left(\underline{\chi}^2 - 16a^2\right)}.$$

Since $\bar{\xi}$, $\underline{\xi}$, ξ, $\underline{\chi}$ and a are constants independent of m, passing to the limit, we have that

$$\sum_{k=-D}^{D} |u_k|^2 \leq \frac{\bar{\xi}\xi}{\underline{\xi}\left(\underline{\chi}^2 - 16a^2\right)}.$$

Due to the arbitrariness of D, $u \in l^2$. Therefore, u satisfies $u_k \to 0$ as $|k| \to \infty$. The existence of a nontrivial homoclinic solution is obtained. □

Proof of Theorem 1.5. Consider the following boundary problem:

$$\begin{cases} \Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}\right) + (-1)^n \chi_k u_k = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), & k \in \mathbf{Z}(-mT, mT), \\ \gamma_{-mT} = \gamma_{mT} = 0, \ \chi_{-mT} = \chi_{mT} = 0, & \\ \gamma_{-k} = \gamma_k, \ \chi_{-k} = \chi_k, & k \in \mathbf{Z}(-mT, mT). \end{cases}$$

Let S be the set of sequences $u = (\cdots, u_{-k}, \cdots, u_{-1}, u_0, u_1, \cdots, u_k, \cdots) = \{u_k\}_{k=-\infty}^{+\infty}$, that is

$$S = \{\{u_k\} | u_k \in \mathbf{R}, \ k \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_k + bv_k\}_{k=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T, $\tilde{E}_m$ is defined as a subspace of S by

$$\tilde{E}_m = \{u \in S | u_{-k} = u_k, \ \forall k \in \mathbf{Z}\}.$$

Clearly, $\tilde{E}_m$ is isomorphic to $\mathbf{R}^{2mT+1}$. $\tilde{E}_m$ can be equipped with the inner product

$$\langle u, v\rangle = \sum_{j=-mT}^{mT} u_j v_j, \ \forall u, v \in \tilde{E}_m,$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \tilde{E}_m.$$

It is obvious that $\tilde{E}_m$ is Hilbert space with $2mT + 1$-periodicity and linearly homeomorphic to $\mathbf{R}^{2mT+1}$.

Similarly to the proof of Theorem 1.1, we can also prove Theorem 1.5. For simplicity, we omit its proof. □

4 Example

As an application of Theorem 1.1, we give an example to illustrate our main result.

Example 4.1. *For all $k \in \mathbf{Z}$, assume that*

$$\Delta^n\left(\gamma_{k-n}\Delta^n u_{k-n}\right) + (-1)^n \chi_k u_k = (-1)^n \lambda u_k \left(\frac{u_{k-1}^2 + u_k^2}{u_{k-1}^2 + u_k^2 + 1} + \frac{u_k^2 + u_{k+1}^2}{u_k^2 + u_{k+1}^2 + 1}\right), \tag{20}$$

where $\lambda > 4^{n-1}\bar{\gamma} + \frac{\bar{\chi}}{4}$. *We have*

$$f(k, v_1, v_2, v_3) = \lambda v_2 \left(\frac{v_1^2 + v_2^2}{v_1^2 + v_2^2 + 1} + \frac{v_2^2 + v_3^2}{v_2^2 + v_3^2 + 1} \right)$$

and

$$F(k, v_1, v_2) = \frac{\lambda}{2} \left[v_1^2 + v_2^2 - \ln \left(v_1^2 + v_2^2 + 1 \right) \right].$$

It is easy to verify that all the assumptions of Theorem 1.1 are satisfied. Consequently, (20) *has a nontrivial homoclinic solution.*

Remark 4.2. *It is easy to check that f doesn't satisfy the classical Ambrosetti-Rabinowitz condition. Thus, our result improves the existing ones.*

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