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Dynamics of doubly stochastic quadratic operators on a finite-dimensional simplex

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Abstract: The present paper focuses on the dynamics of doubly stochastic quadratic operators (d.s.q.o) on a finite-dimensional simplex. We prove that if a d.s.q.o. has no periodic points then the trajectory of any initial point inside the simplex is convergent. We show that if d.s.q.o. is not a permutation then it has no periodic points on the interior of the two dimensional (2D) simplex. We also show that this property fails in higher dimensions. In addition, the paper also discusses the dynamics classifications of extreme points of d.s.q.o. on two dimensional simplex. As such, we provide some examples of d.s.q.o. which has a property that the trajectory of any initial point tends to the center of the simplex. We also provide an example of d.s.q.o. that has infinitely many fixed points and has infinitely many invariant curves. We therefore came-up with a number of evidences. Finally, we classify the dynamics of extreme points of d.s.q.o. on 2D simplex.

Keywords: Doubly stochastic quadratic operators, Fixed point, Trajectory, Extreme point, Simplex

MSC: 15A51, 15A63, 46T99, 46A55

1 Introduction

Without a doubt, many biological processes can be considered as some nonlinear dynamical systems. From this point of view, the main problem associated with the state of the process in certain time frame is the same as when studying the limit behavior of the trajectories of corresponding dynamical systems. One of the important dynamical systems is the one generated by quadratic stochastic operators (q.s.o. henceforth) on finite-dimensional simplex, which appear in many problems of population genetics [1]. The central problem in this theory is to study the limit behavior of the trajectory(dynamics) of a given initial point. There are many papers that study such operators (see [2] for review). One of the important classes of q.s.o. are Volterra operators [3]. It is proven for this operators that the ω -limit set of any non-fixed initial point from the interior of the simplex belong to the boundary of the simplex. Moreover, ω -limits set (we will definite it later) of any initial point is either a countable set or is a singleton. There are other classes of q.s.o. that are well studied [4–6]. Another interesting class of q.s.o. is the class of doubly stochastic (d.s.q.o. henceforth) ones. It was firstly defined in [3] in terms of majorization. It was later developed in papers [7, 8] and infinite-dimensional case in [9]. These papers basically studied the structure of the class of d.s.q.o., that is, necessary and sufficient conditions for doubly stochasticity and extreme points of the set of doubly stochastic

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operators. In particular, it is shown that, up to permutation, there are 37 extreme d.s.q.o. on 2D simplex. It is also worth mentioning that d.s.q.o. are very different than Volterra operators since from the classification theorem for d.s.q.o. [8] it follows that the class of d.s.q.o. and Volterra intersect at the identity operator. Nonetheless, the study of the limit behavior (dynamics) of d.s.q.o. is still wide open, except for ergodic theorem for these operators [10]. In fact, the paper [2] asks to investigate the above problems as the class of d.s.q.o. Namely, it asks if a given d.s.q.o. in general is regular, i.e. has convergent trajectory for any initial point. We will answer positively to this question under small assumption. This question is interesting because d.s.q.o. have rich applications. One application is that these kinds of maps appear in population genetics problems as being a sub-class of the general q.s.o. In addition, doubly stochastic maps are widely used in economics and statistics. One can see it in [11]. The present paper, therefore, aims to provide a couple of convergence theorems for general d.s.q.o. Importantly, it does not give any clue where the trajectory converges. Therefore, in addition, we classify the dynamics of extreme d.s.q.o. on 2D simplex. A motivation to study extreme d.s.q.o. comes from the fact that any d.s.q.o. can be written as a convex combination of the extremes [7, 8]. Thus we deem we should start with extreme d.s.q.o. Another motivation comes from the paper [12] where general quadratic stochastic operators having coefficients 1 or 0 were classified on 2D simplex. Extreme d.s.q.o.'s have coefficients 1, $\frac{1}{2}$ or 0 [7, 8]. So our result will extend the results in [12] in the class of d.s.q.o. on 2D simplex. The paper is organized as follows. In the next section we give some preliminaries concerning majorization and d.s.q.o. In Section 3, we provide a class of Lyapunov functions for d.s.q.o. and provide a general convergence theorem for d.s.q.o. on finite-dimensional simplex. Section 4 focuses on dynamics classifications of extreme d.s.q.o. on 2D simplex. We note that a couple of examples of extreme d.s.q.o. on 2D simplex have been considered in [13].

2 Preliminaries

In this section we provide some definition based on majorization theory and define doubly stochastic operator.

We define the $(m - 1)$ -dimensional simplex as follows.

$$S^{m-1} = \{x = (x_1, x_2, \dots, x_m) \in R^m : x_i \geq 0, \forall i = \overline{1, m}, \sum_{i=1}^m x_i = 1\} \quad (1)$$

The set $\text{int}S^{m-1} = \{x \in S^{m-1} : x_i > 0\}$ is called the (relative) interior of the simplex. The points $e_k = (0, 0, \dots, \underbrace{1}_k, \dots, 0)$ are the vertices of the simplex and the scalar vector $(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m})$ is the center of the simplex.

A quadratic stochastic operator $V : S^{m-1} \rightarrow S^{m-1}$ is defined as:

$$(Vx)_k = \sum_{i,j=1}^m p_{ij,k} x_i x_j \quad (2)$$

Where the coefficients $p_{ij,k}$ satisfy the following conditions

$$p_{ij,k} = p_{ji,k} \geq 0, \sum_{k=1}^m p_{ij,k} = 1, \forall i, j, k = \overline{1, m} \quad (3)$$

If we let $A_k = (p_{ij,k})_{i,j}$, $k = \overline{1, m}$, then the operator can be given in terms of m matrices and we write

$$V = (A_1 | A_2 | \dots | A_m) \quad (4)$$

where matrices A_i are non-negative and symmetric. For any $x = (x_1, x_2, \dots, x_m) \in S^{m-1}$, we define $x_{\downarrow} = (x_{[1]}, x_{[2]}, \dots, x_{[m]})$ where $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[m]}$ - nonincreasing rearrangement x . Recall [11, 14] that for two elements x, y of the simplex S^{m-1} the element x is majorized by y and written $x < y$ (or $y > x$) if the following condition holds

$$\sum_{i=1}^m x_{[i]} \leq \sum_{i=1}^m y_{[i]} \quad (5)$$

for any $k = \overline{1, m-1}$. In fact, this definition is referred to as weak majorization [11], the definition of majorization requires $\sum_{i=1}^m x_{[i]} = \sum_{i=1}^m y_{[i]}$. However, since we consider vectors only on the simplex, we may drop this condition.

A matrix $P = (p_{ij})_{i,j=\overline{1,m}}$ is called doubly stochastic (sometimes bistochastic), if

$$\begin{aligned} p_{ij} &\geq 0, \forall i, j = \overline{1, m}, \\ \sum_{j=1}^m p_{ij} &= 1, \forall i = \overline{1, m}, \quad \sum_{i=1}^m p_{ij} = 1, \forall j = \overline{1, m}. \end{aligned} \quad (6)$$

For a doubly stochastic matrix $P = (p_{ij})$, if its entries consist of only 0's and 1's, then the matrix is a permutation matrix.

A linear map $T : S^{m-1} \rightarrow S^{m-1}$ is said to be T -transform, if $T = \lambda I + (1 - \lambda)P$ where I is an identity matrix, P is a permutation matrix which is obtained by swapping only two rows of I and $0 \leq \lambda \leq 1$.

Lemma 2.1 ([11], chapter 2). *For the concept of majorization, the following assertions are equivalent for any $x, y \in S^{m-1}$.*

- 1) $x < y$ that is $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}, k = \overline{1, m-1}$.
- 2) $x = Py$ for some doubly stochastic matrix P .
- 3) The vector x belongs to the convex hull of all $m!$ permutation vectors of y .
- 4) The vector x can be obtained by a finite compositions of T -transforms of the vector y , that is, there exist T -transforms T_1, T_2, \dots, T_k such that $x = T_1 T_2 \dots T_k y$.
- 5) The inequality $\varphi(x) \leq \varphi(y)$ holds for any Schur-convex function.

So from the above lemma, it follows that doubly stochasticity of a matrix P is equivalent to $Px < x$ for all $x \in S^{m-1}$. Motivated by this, in [11], the definition of doubly stochastic operator is given as follows.

Definition 2.2 ([15]). *A continuous stochastic operator $V : S^{m-1} \rightarrow S^{m-1}$ is called doubly stochastic, if*

$$Vx < x \quad \text{for all } x \in S^{m-1}$$

Hence if a q.s.o. satisfies the above property, then we call it d.s.q.o.

For example, the identity operator, permutation operators (that is the linear operators with permutation matrix), T -transforms are all doubly stochastic. The following is an example of the d.s.q.o.

Example 2.3. Consider $V : S^2 \rightarrow S^2$ given by

$$\begin{aligned} V(x) &= x^2 + 2yz \\ V(y) &= y^2 + 2xz \\ V(z) &= z^2 + 2xy \end{aligned}$$

It is straightforward to check that this operator satisfies the condition of doubly stochasticity in the Definition 2.2.

Let V be doubly stochastic operator and $x^0 \in S^{m-1}$. The set

$$\{x^0, V(x^0), V^2(x^0), \dots, V^n(x^0), \dots\}$$

is called the trajectory starting at x^0 . Here, $V^0(x^0) = x^0$ and $V^n(x^0) = V(V^{n-1}(x^0))$. We denote by $\omega(x^0)$ the set of limit points of the trajectory starting at x^0 and it is said to be the ω -limit set of the trajectory starting at x^0 . Notice that the ω -limit set of any point is invariant for V by definition.

The point x^0 is called p -periodic, if there is a positive integer p such that $V^p(x^0) = x^0$ and $V^i(x^0) \neq x^0 \quad \forall i = \overline{1, p-1}$. If $p = 1$, we say that the point is fixed. We just say periodic if the period is irrelevant. By periodic point we always mean a periodic point of a period strictly greater than one.

Notice that the center $C = (\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m})$ is always fixed for a doubly stochastic V . Indeed, by the definition of the double stochasticity, we have $V(C) < C$. On the other hand, $C < x$ for any vector x from the simplex.

Therefore, $C \prec V(C) \prec C$ implies $(V(C))_{\downarrow} = C_{\downarrow} = C$. Hence, $V(C) = C$ and C is a fixed point. We will use this fact in subsequent sections.

3 The limit behavior of d.s.q.o.

In this section we first provide a class of functionals on the simplex that are Lyapunov functions for a d.s.q.o. Then we prove the convergence theorem for d.s.q.o.

A continuous functional given $\varphi : S^{m-1} \rightarrow R$ is said to be a Lyapunov function for the operator V if the limit $\lim_{n \rightarrow \infty} \varphi(V^n(x^0))$ exists along the trajectory $\{x^0, V(x^0), V^2(x^0), \dots, V^n(x^0), \dots\}$.

The Lyapunov function is considered to be very useful in the study of limit behavior of (discrete) dynamical systems.

Theorem 3.1. *A continuous functional given by*

$$\varphi(x) = \sum_{i=1}^m x_i^2$$

is a Lyapunov function for doubly stochastic operator V . Moreover, if $x \in S^{m-1}$ and Vx is not the permutation of x , then $\varphi(Vx) < \varphi(x)$.

Proof. Use the fact (see [11], prop. F.1, page 78.) that the function $\psi(x) = \sum_{i < j} x_i x_j$, defined on the simplex, is Schur-concave, i.e. satisfies $\psi(x) \geq \psi(y)$ whenever $x \prec y$. Moreover, if $x, y \in S^{m-1}$, and x is not the permutation of y , then $\psi(x) > \psi(y)$. Taking into consideration $\varphi(x) = 1 - 2\psi(x)$, we obtain that $x \prec y$ implies $\varphi(x) \leq \varphi(y)$. Since V is doubly stochastic, then $Vx \prec x$ hence $\varphi(Vx) \prec \varphi(x)$ is bounded. Note that the sequence $a_n = \varphi(V^n(x^0))$, where V is doubly stochastic and $x^0 \in S^{m-1}$, is monotone. Hence the $\lim_{n \rightarrow \infty} \varphi(V^n(x^0))$ exists.

Now we show the second part. If $x \in \text{int} S^{m-1}$, then according to Lemma it follows that $V(x) \in \text{int} S^{m-1}$. Since all components of $V(x)$ and x are positive and Vx is not the permutation of x , then by applying the above mentioned fact, we get $\varphi(x) < \varphi(y)$. \square

Based on the above method one can provide larger class of Lyapunov functions.

Theorem 3.2. *Any continuous symmetric convex (or concave) functional is a Lyapunov function for a doubly stochastic operator V .*

These theorems are very useful to study the dynamics of individual doubly stochastic operators. We will provide an application of these theorems in the next section.

Theorem 3.3. *Assume that a d.s.q.o. V does not have any periodic points on $\text{int} S^{m-1}$. Then the trajectory of any initial point on the interior of simplex is convergent under any d.s.q.o.*

Proof. Let $V : S^{m-1} \rightarrow S^{m-1}$ be d.s.q.o. Then we have

$$x \succ Vx \succ V^2x \succ \dots$$

It means that

$$\begin{aligned} x_{[1]} &\geq (Vx)_{[1]} \geq (V^2x)_{[1]} \geq \dots, \\ x_{[1]} + x_{[2]} &\geq (Vx)_{[1]} + (Vx)_{[2]} \geq (V^2x)_{[1]} + (V^2x)_{[2]} \geq \dots, \\ \dots, \sum_{i=1}^k x_{[i]} &\geq \sum_{i=1}^k (Vx)_{[i]} \geq \sum_{i=1}^k (V^2x)_{[i]} \geq \dots \end{aligned}$$

The sequences $\{\sum_{i=1}^k (V^n(x))_{[i]}, n = 1, 2, \dots \forall k = \overline{1, m}\}$ are increasing and bounded, consequently convergent. The last suggests that the following sequences are also convergent

$$\{(V^n(x))_{[k]}, n = 1, 2, \dots \forall k = \overline{1, m}\}$$

Let us denote $y_k = \lim_{n \rightarrow \infty} (V^n x)_{[k]}$ and $y = (y_1, y_2, \dots, y_m)$.

If $z = (z_1, z_2, \dots, z_m) \in \omega(x^0)$ then there exists $\{x^{(n_j)}\}$, such that $(V^{n_j} x) \rightarrow z$.

Therefore we have $(V^{n_j} x)_\downarrow \rightarrow z_\downarrow$. On the other hand $(V^{n_j} x)_\downarrow \rightarrow y$, since y is a limit of the sequence $(V^n x)$, $n = 1, 2, \dots$. That's why $z_\downarrow = (y_1, y_2, \dots, y_m) = y$. This means that the cardinality of $\omega(x^0)$ cannot be greater than $m!$.

Note that since the simplex is compact, then $\omega(x^0) \neq \emptyset$. Now, assume that the cardinality of it is greater than 1. Then there is a point z in $\omega(x^0)$ that is not fixed by V . Because the set $\omega(x^0)$ is V -invariant then any iterate of z under must belong to $\omega(x^0)$. But this set is finite and therefore there exist natural numbers p, q (and let us assume $p > q$) such that $V^p(z) = V^q(z)$. But then the point $V^q(z)$ is periodic with period $p - q$. However, by the assumption V does not have any periodic points. Hence m must be 1 and $|\omega(x^0)| = 1$. Therefore, $\omega(x^0)$ may consist of a single point only and this implies that the trajectory starting at x^0 tends to that point. \square

Remark 3.4. In the proof of this theorem we did not use the fact that V is d.s.q.o. In fact, it does not have to be quadratic. This theorem holds for any (continuous) doubly stochastic map on a finite-dimensional simplex.

An operator $V : S^{m-1} \rightarrow S^{m-1}$ is a permutation if there exists a permutation matrix such that $Vx = Px \quad \forall x \in S^{m-1}$. It is evident that there are exactly $m!$ permutation operators on S^{m-1} .

Theorem 3.5. Any non-permutation d.s.q.o. $V : S^2 \rightarrow S^2$ on 2D simplex does not have periodic points on the interior of the simplex.

Proof. Assume V has a periodic point $x^0 \in \text{int} S^2$ and $V^p(x^0) = x^0$ for $p > 1$. Then

$$V \succ V(x^0) \succ \dots \succ V^p(x^0) = x^0$$

Hence $V(x^0)$ is some permutation of x^0 .

Let $x^0 = (x, y, z)$, $x, y, z > 0$, $x + y + z = 1$ and

$$V(x^0) = ((A_1 x^0, x^0), (A_2 x^0, x^0), (A_3 x^0, x^0))$$

where (\cdot, \cdot) denotes the usual inner product. Without loss of generality we may assume $x \geq y \geq z$ and $(A_1 x^0, x^0) \geq (A_2 x^0, x^0) \geq (A_3 x^0, x^0)$. Then $V(x^0) = x^0$ (not some permutation of x^0). Because $x + y + z = 1$ then one can assume $x \geq y \geq z$ and one can also assume the second assumption by changing matrices A_1, A_2, A_3 if necessary. By applying Theorems 2.6 and 2.4 of [8] we can represent each A_k , $k = 1, 2, 3$ as

$$A_k = \frac{S_k + S_k^T}{2}$$

where S_k is a row-stochastic matrix and S_k^T is its transpose. Not that

$$\begin{aligned} (A_k x^0, x^0) &= \left(\frac{S_k + S_k^T}{2} x^0, x^0 \right) \\ &= \frac{1}{2} (S_k x^0, x^0) + \frac{1}{2} (S_k^T x^0, x^0) \\ &= \frac{1}{2} (S_k x^0, x^0) + \frac{1}{2} (x^0, S_k x^0) \\ &= (S_k x^0, x^0) \end{aligned}$$

Hence $V(x^0) = ((S_1 x^0, x^0), (S_2 x^0, x^0), (S_3 x^0, x^0)) = (x, y, z)$.

Let $S_k = (s_{ij,k})_{i,j=\overline{1,3}}$. Since $V(x^0) = x^0$ then $(S_1 x^0, x^0) = x$. On the other hand, since S_1 is row-stochastic we must have

$$s_{11,1}x + s_{12,1}y + s_{13,1}z \leq x \quad (7)$$

Therefore

$$\begin{aligned}
 x &= (s_{11}x^0, x^0) \\
 &= (s_{11,1}x + s_{12,1}y + s_{13,1}z)x + (s_{21,1}x + s_{22,1}y + s_{23,1}z)y + (s_{31,1}x + s_{32,1}y + s_{33,1}z)z \\
 &\leq xx + xy + xz \\
 &= x(x + y + z) \\
 &= x,
 \end{aligned}$$

and we must have the equality in (7). Recall that $x^0 = (x, y, z) \neq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ as $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a fixed point for V . Hence the equality in (7) is possible only in the following cases:

$$S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (8)$$

or

$$x = y, \text{ and } S_1 = \begin{pmatrix} s_{11,1} & s_{12,1} & 0 \\ s_{21,1} & s_{22,1} & 0 \\ s_{31,1} & s_{32,1} & 0 \end{pmatrix} \quad (9)$$

Similar arguments for $S_3 = (s_{ij,3})_{i,j=\overline{1,3}}$ show that

$$s_{i1,3}x + s_{i2,3}y + s_{i3,3}z \geq z \quad (10)$$

with equality is only possible in the following cases:

$$S_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (11)$$

or

$$y = z \text{ and } S_3 = \begin{pmatrix} 0 & s_{12,3} & s_{13,3} \\ 0 & s_{22,3} & s_{23,3} \\ 0 & s_{32,3} & s_{33,3} \end{pmatrix} \quad (12)$$

Not that (9) and (12) can not hold simultaneously, as it would imply $x = y = z$ hence $x^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

If (8) and (12) hold, then V would fix the first and the third components of $x^0 = (x, y, z)$ which implies that it would also fix the second component of $x^0 = (x, y, z)$. By the same reasons (9) and (11) can not hold

simultaneously. Finally (8) and (11) together would imply that $S_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and hence $V = Id$ which is not

impossible as V is not a permutation operator. \square

Remark 3.6. In the assumption of the theorem we assumed V to be non-permutation but only used the fact that V is not an identity operator. This is because we assumed $(A_1x^0, x^0) \geq (A_2x^0, x^0) \geq (A_3x^0, x^0)$.

Remark 3.7. It is essential that $x^0 \in \text{int}S^2$, that is, a d.s.q.o. can have periodic points on the boundary of the simplex.

Consider $V : S^2 \rightarrow S^2$ given by

$$\begin{aligned}
 V(x) &= y^2 + 2xz \\
 V(y) &= z^2 + 2xy \\
 V(z) &= x^2 + 2yz
 \end{aligned}$$

This operator is d.s.q.o. by the classification theorem 2.6 of [8] or one can easily check that $Vx \prec x$. One can see that $V^3(1, 0, 0) = (1, 0, 0)$, that is, $(1, 0, 0)$ is a 3-periodic point.

Combining the previous two theorems we obtain a very interesting corollary.

Corollary 3.8. *Any non-permutation d.s.q.o. has convergent trajectory on the interior of the simplex.*

Theorem 3.5 fails in higher dimension. Consider the following example in 3D simplex.

Example 3.9. *Consider the operator $V : S^3 \rightarrow S^3$ given by*

$$Vx = Dx = \left(\frac{x_2 + x_4}{2}, \frac{x_1 + x_3}{2}, \frac{x_2 + x_4}{2}, \frac{x_1 + x_3}{2} \right)$$

where

$$D = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

For this operator, one has

$$V^{2k+1} = V^1 = \left(\frac{x_2 + x_4}{2}, \frac{x_1 + x_3}{2}, \frac{x_2 + x_4}{2}, \frac{x_1 + x_3}{2} \right)$$

and

$$V^{2k} = V^2 = \left(\frac{x_1 + x_3}{2}, \frac{x_2 + x_4}{2}, \frac{x_1 + x_3}{2}, \frac{x_2 + x_4}{2} \right)$$

Therefore, any point is either periodic or becomes periodic after one iteration. Notice that this linear operator is given by a doubly stochastic matrix. Hence $Vx \prec x$ and so V is doubly stochastic. One can make this operator a d.s.q.o. by multiplying each component to $x_1 + x_2 + x_3 + x_4 = 1$.

4 The classification of extreme points of d.s.q.o. on S^2

In this section we give classification of dynamics of extreme d.s.q.o.

Definition 4.1. *A d.s.q.o. is extreme if it cannot be written as a convex hull of two different d.s.q.o.*

In [7, 8] it was shown that the set of all d.s.q.o. form a convex polytope and the extreme points in 2D simplex were fully described in the following way considering a d.s.q.o. $V : S^2 \rightarrow S^2$ given by $V = (A_1|A_2|A_3)$ where $(A_1|A_2|A_3)$ are non-negative symmetric matrices such that (this follows from the definition of q.s.o.) their sum is a matrix whose all entries equal to 1.

$$\text{Let } \mathcal{U}_3 = \{A = (a_{ij})_{i,j=1,3} \mid a_{ij} = a_{ji} \geq 0, \sum_{i,j \in \alpha} a_{ij} \leq |\alpha| \text{ for } \alpha \subset \{1, 2, 3\}, \sum_{i,j=1}^3 a_{ij} = 3\}.$$

The set of extreme points of a set A we denote by $\text{Extr}A$.

We recall a couple of facts from [7] and [8].

It is known [7, 8] that d.s.q.o. is extreme $A = (a_{ij}) \in \text{Extr}\mathcal{U}_3$ if and only if $a_{ii} = 1$ or 0 , $a_{ij} =$

1 or $\frac{1}{2}$ or 0 and $A \neq \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ and there are 25 extreme matrices.

In [7, 8] A d.s.q.o. $V = (A_1|A_2|A_3)$ is extreme if and only if at least two of matrices A_1, A_2, A_3 are extreme in \mathcal{U}_3 .

Using these facts one can find triples $(A_1|A_2|A_3)$ such that V is extreme. There are 37 such triples. For a given extreme d.s.q.o. $(A_1|A_2|A_3)$ if we rearrange matrices A_1, A_2, A_3 we still get extreme d.s.q.o.

Thus, counting permutations, there are $37 * 3! = 222$ extreme d.s.q.o.

Now our objective is to classify them in terms of their dynamics. As we mentioned all 222 extreme d.s.q.o. are obtained by permuting component of 37 extreme operators. Thus, to study the dynamics we will be dealing with the same collection of matrices even if we change the components. This implies that it is enough to consider only 37 extreme d.s.q.o. It is worth noting that in general a permuted operator does not have the same dynamics, but it can only be studied as the original operator. For instance, we if consider the identity operator, then all points are fixed. However, if we permute the identity operator (see operator below) then all points are going to be periodic except the center of the simplex. Let's consider the simplest case - permutation operator given by

$$V(x, y, z) = (y, z, x)$$

The matrix correspondence for this operator is

$$A_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \quad (13)$$

Note that $V^3(x, y, z) = (x, y, z)$ and therefore every initial point is 3-periodic. Thus, it is not difficult to study the dynamics of this operator. We split the remaining 36 operators into two classes: those who have exactly one matrix from (13) in their matrix representation are in class 2 and the rest in class 1.

It is important that operators from the class 2 must have exactly one matrix from (13). Referring to Ganikhodzhaev and Shahidi (2010)'s Theorem 3.4 if an operator has more than 1 matrix from (13) then it becomes a permutation operator.

Because extreme d.s.q.o. are classified in [8] (Theorem 3.4), then it is not difficult to see that in fact the class 2 has 3 (or 18 up to permutation) operators while the class 1 has 33.

We are going to choose one representative from each class and study the limit behavior of the trajectories. We will see that their dynamics is different.

Example 4.2 (Class 1). Consider $V : S^2 \rightarrow S^2$ given by

$$V_1(x) = z^2 + xy + yz$$

$$V_1(y) = x^2 + xz + yz$$

$$V_1(z) = y^2 + xy + xz$$

Figure 1 shows that the trajectory of initial values of x, y and z for V_1 converges to the center $(\frac{1}{3})$.

Theorem 4.3. For V_1 , the trajectory of any initial point from the interior of the simplex converges to the center of the simplex. Moreover, V_1 has no periodic points on $\text{int}S^2$.

We note that the convergence part of this theorem automatically follows from Theorems 3.3 and 3.5 of previous section. However, we will show how one apply Theorem 3.1 of previous section.

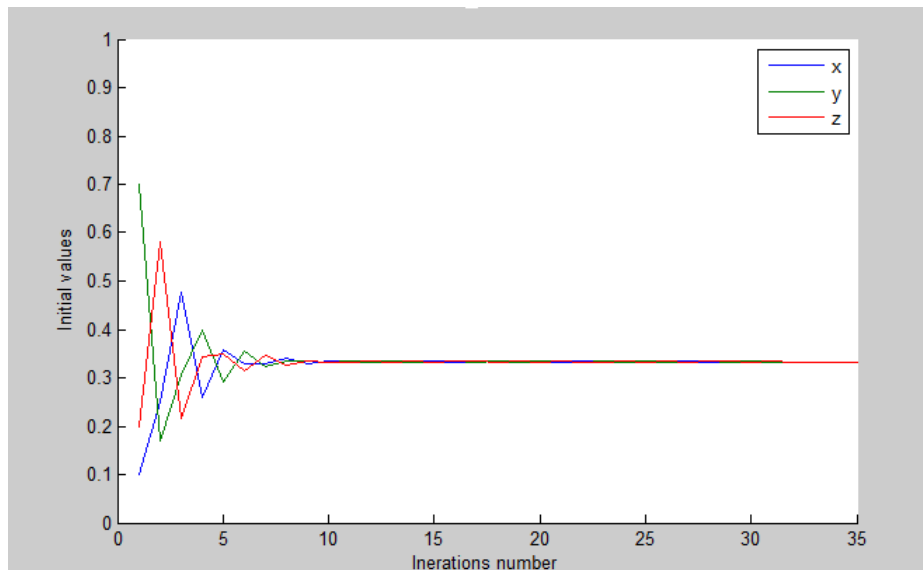
Proof. We first show that V_1 has no periodic point on $\text{int}S^2$. Suppose on the contrary, that $\text{int}S^2$ is a p -periodic ($p > 1$) point of V_1 . Then by definition

$$x^0 \succ V_1 x^0 \succ V_1^2 x^0 \succ \dots \succ V_1^p x^0 = x$$

It follows that each $V^k x^0, k = \overline{1, p}$ interchange the components of x^0 . Therefore, if we define $\varphi(x^0) = x^2 + y^2 + z^2$ for $x^0 = (x, y, z)$ we must have $\varphi(Vx^0) = \varphi(x^0)$ by Theorem 3.1.

One can easily compute that

$$\varphi(Vx^0) - \varphi(x^0) = -zy(x-z)^2(x+z)$$

Fig. 1. The trajectory of operator V_1 .

hence $-zy(x-z)^2(x+z) = 0$.

Since $x, y, z > 0$ the above holds only if $x = z$. Assume $x = z = c$, $0 < c < 1$ then $x^0 = (c, 1-2c, c)$. One can see that $V_1(x) = 2c - 3c^2$ so in order for V_1 to permute the components of x^0 we must have $2c - 3c^2 = c$ or $2c - 3c^2 = 1 - 2c$ which only happens if $c = \frac{1}{3}$ that is when $x^0 = (x, y, z) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. But $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is fixed point and we assumed x^0 to be periodic (of period greater than one). Thus, as one can recall Theorem 3.3 from the previous section, that the trajectory of any $x^0 \in \text{int}S^2$ is convergent. Notice that a convergent point is always fixed by V_1 .

By solving the system of equations

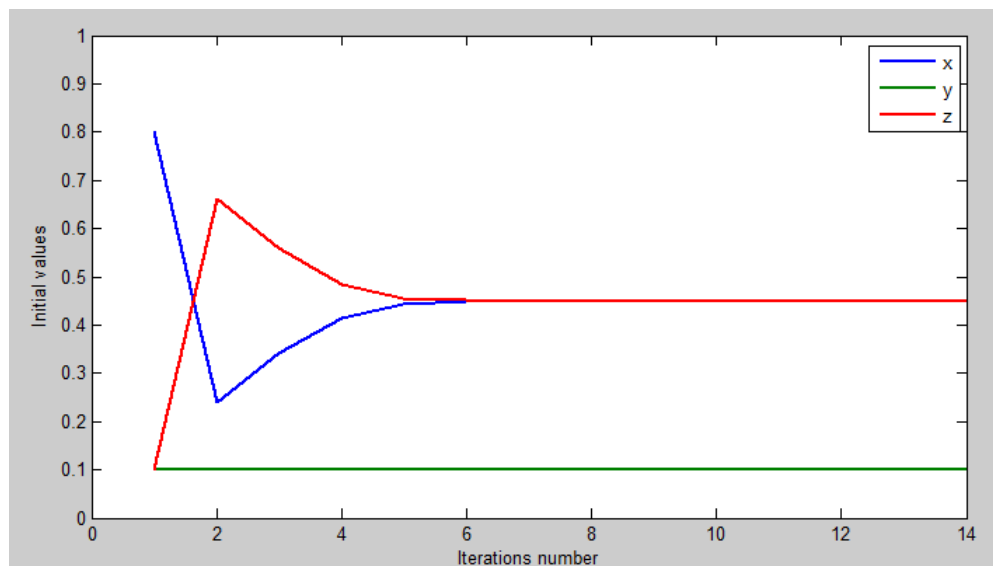
$$\begin{aligned}x &= z^2 + xy + yz \\y &= x^2 + xz + yz \\z &= y^2 + xy + xz\end{aligned}$$

we find that V_1 has a unique fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ on $\text{int}S^2$. Therefore $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ must be the convergent point for any $x^0 \in \text{int}S^2$. \square

Example 4.4 (Class 2). Consider $V : S^2 \rightarrow S^2$ given by

$$\begin{aligned}V_2(x) &= xy + 2xz \\V_2(y) &= y^2 + xy + yz = y \\V_2(z) &= x^2 + z^2 + yz.\end{aligned}$$

Accordingly, we see that under any iteration y does not change (see Figure 2). For a fixed, simple calculations we show that the trajectory of any point from the line $y = c$ tends to $(\frac{1-c}{2}, c, \frac{1-c}{2})$ (see Figure 2).

Fig. 2. The trajectory of operator V_2 .

Theorem 4.5. *The operator V_2 has infinitely many invariant curves (segments). Moreover, V_2 has a unique fixed point in each invariant curve and the trajectory of any initial point on a given invariant curve tends to the "center" of the curve.*

Proof. Let $\gamma_c = \{(x, y, z) \in \text{int}S^2 \mid y = c\}$ $0 < c < 1$. It is obvious that γ_c is V_2 -invariant. By solving the system of equations

$$\begin{aligned}x &= xc + 2xz \\ y &= y = c \\ z &= x^2 + z^2 + cz.\end{aligned}$$

where c is fixed we find a unique solution $(\frac{1-c}{2}, c, \frac{1-c}{2})$ which shows that V_2 has infinitely many fixed points.

Notice that $y = c$ defines a plane in R^3 and $\gamma_c = \text{int}S^2 \cap \{y = c\}$ is in fact a segment and the point $(\frac{1-c}{2}, c, \frac{1-c}{2})$ is the middle of this segment, so that is why we called it the center of γ_c .

Now, take an initial point on γ_c . We know from the previous section that the trajectory is convergent. At the same time it stays in γ_c (as γ_c is invariant) which has a fixed point. Thus, the trajectory must tend to its unique fixed point, that is to the center $(\frac{1-c}{2}, c, \frac{1-c}{2})$ of γ_c . \square

5 Conclusions and future work

In this paper, we discussed an important class of quadratic stochastic operators, which was defined in terms of majorization of vectors, that is, doubly stochastic quadratic operators (d.s.q.o) on finite-dimensional simplex. The theorem of the convergence of d.s.q.o. is proved. We also showed that it has no periodic point is on the interior of the simplex but it may have (in fact infinitely many) periodic points in higher dimensions of the simplex. Moreover, we classified the extreme points of d.s.q.o. on two dimensional 2D simplex. From the analysis, we found three different classes of 222 extreme points of d.s.q.o. on 2D simplex. We found out that 198 extreme d.s.q.o.'s converge to the center of the simplex. In the second class, we found 18 extreme d.s.q.o., which had property as in Theorem 4.5. In the third classification, we had 6 permutation operators which are extreme d.s.q.o. It is very interesting to study dynamics on infinite-dimensional simplex. The problem would be very different if the infinite-dimensional simplex

was not compact, the operator would not necessarily have a fixed point and one should choose a proper metric to study the dynamics.

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