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Research Article

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Oscillation of impulsive conformable fractional differential equations

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Abstract: In this paper, we investigate oscillation results for the solutions of impulsive conformable fractional differential equations of the form

$$\begin{cases} {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)\right]\right)+q(t)x(t)=0, & t\geq t_0,\ t\neq t_k, \\ x(t_k^+)=a_k x(t_k^-), \quad {}_{t_k}D^{\alpha}x(t_k^+)=b_k\ {}_{t_{k-1}}D^{\alpha}x(t_k^-), & k=1,2,\ldots. \end{cases}$$

Some new oscillation results are obtained by using the equivalence transformation and the associated Riccati techniques.

Keywords: Fractional differential equations, Impulsive differential equations, Conformable fractional derivative, Oscillation

MSC: 34A08, 34A37, 34C10

1 Introduction

Fractional differential equations are generalizations of classical differential equations of integer order, and can find their applications in many fields of science and engineering. Research on the theory and applications of fractional differential and integral equations has been the focus of many studies due to their frequent appearance in various applications in physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics, and has attracted much attention of more and more scholars. The books on the subject of fractional integrals and fractional derivatives by Diethelm [1], Miller and Ross [2], Podlubny [3] and Kilbas *et al.* [4] summarize and organize much of fractional calculus and many of theories and applications of fractional differential equations. Initial and boundary value problems, stability of solutions, explicit and numerical solutions and many other properties have obtained significant development [5]-[11].

The oscillation of fractional differential equations as a new research field has received significant attention, and some interesting results have already been obtained. We refer to [12]-[18] and the references quoted therein.

The definition of the fractional order derivative used is either the Caputo or the Riemann-Liouville fractional order derivative involving an integral expression and the gamma function. Recently, Khalil *et al.* [19] introduced a new well-behaved definition of local fractional derivative, called *the conformable fractional derivative*, depending

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just on the basic limit definition of the derivative. This new theory is improved by Abdeljawad [20]. For recent results on conformable fractional derivatives we refer the reader to [21]-[25].

Impulsive differential equations are recognized as adequate mathematical models for studying evolution processes that are subject to abrupt changes in their states at certain moments. Many applications in physics, biology, control theory, economics, applied sciences and engineering exhibit impulse effects, see [26]-[28].

Recently, in [29] Tariboon and Thiramanus considered the following second-order linear impulsive differential equation of the form

$$\begin{cases} \left(a(t)\left[x'(t)+\lambda x(t)\right]\right)' + p(t)x(t) = 0, & t \ge t_0,\ t \ne t_k, \\ x(t_k^+) = b_k x(t_k^-), \quad x'(t_k^+) = c_k x'(t_k^-), & k = 1,2,\ldots, \end{cases} \tag{1}$$

where $0 \le t_0 < t_1 < \ldots < t_k < \ldots$, $\lim_{k\to\infty} t_k = +\infty$, $a \in C([t_0,\infty),(0,\infty))$, $p \in C([t_0,\infty),\mathbb{R})$, $\{b_k\},\{c_k\}$ are two known sequences of positive real numbers and $\lambda \in \mathbb{R}$. By using the equivalence transformation and the associated Riccati techniques, some interesting oscillation results were obtained.

In this paper, we investigate some new oscillation results for the solutions of impulsive conformable fractional differential equations of the form

$$\begin{cases} {}_{t_k}D^\alpha\left(p(t)\left[{}_{t_k}D^\alpha x(t) + r(t)x(t)\right]\right) + q(t)x(t) = 0, & t \ge t_0,\ t \ne t_k, \\ x(t_k^+) = a_k x(t_k^-), \quad k = 1,2,\ldots, \\ {}_{t_k}D^\alpha x(t_k^+) = b_k\ {}_{t_{k-1}}D^\alpha x(t_k^-), \quad k = 1,2,\ldots, \end{cases} \tag{2}$$

where ${}_\phi D^\alpha$ denotes the conformable fractional derivative of order $0 < \alpha \le 1$ starting from $\phi \in \{t_0,\ldots,t_k,\ldots\}$, $0 \le t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, $p \in C([t_0,\infty),(0,\infty))$, $q, r \in C([t_0,\infty),\mathbb{R})$, $\{a_k\}$ and $\{b_k\}$ are two known sequences of positive real numbers and $x(t_k^+) = \lim_{\theta\to 0^+} x(t_k+\theta)$, $x(t_k^-) = \lim_{\theta\to 0^+} x(t_k-\theta)$.

Some new oscillation results are obtained, generalizing the results of [29] to impulsive conformable fractional differential equations. Note that if $a_k = 1$ for all $k = 1,2,\ldots$, then x is continuous on $[t_0,\infty)$. However, if $a_k = b_k = 1$ for all $k = 1,2,\ldots$, then it does not guarantee that x' is also continuous function, as by the definition of conformable fractional derivative (see Definition 2.1 below) we have

$${}_{t_k}D^\alpha x(t_k^+) = \lim_{\varepsilon\to 0} \frac{x(t_k^+ + \varepsilon(t_k^+ - t_k)^{1-\alpha}) - x(t_k^+)}{\varepsilon},$$

and

$${}_{t_{k-1}}D^\alpha x(t_k^-) = \lim_{\varepsilon\to 0} \frac{x(t_k^- + \varepsilon(t_k^- - t_{k-1})^{1-\alpha}) - x(t_k^-)}{\varepsilon}.$$

We organize this paper as follows: In Section 2, we present some useful preliminaries from fractional calculus. In Section 3, we prove some auxiliary lemmas. The main oscillation results are established in Section 4. Examples illustrating the results are presented in Section 5.

2 Conformable Fractional Calculus

In this section, we recall some definitions, notations and results which will be used in our main results.

Definition 2.1. *[20] The conformable fractional derivative starting from a point ϕ of a function $f : [\phi,\infty) \to \mathbb{R}$ of order $0 < \alpha \le 1$ is defined by*

$${}_\phi D^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon(t-\phi)^{1-\alpha}) - f(t)}{\varepsilon}, \tag{3}$$

provided that the limit exists.

If f is differentiable then ${}_{\phi}D^{\alpha} f(t) = (t-\phi)^{1-\alpha} f'(t)$. In addition, if the conformable fractional derivative of f of order α exists on $[\phi, \infty)$, then we say that f is α-differentiable on $[\phi, \infty)$.

It is easy to prove the following results.

Lemma 2.2. *Let $\alpha \in (0,1]$, $k_1, k_2, p, \lambda \in \mathbb{R}$ and functions f, g be α-differentiable on $[\phi, \infty)$. Then:*
(i) ${}_{\phi}D^{\alpha}(k_1 f + k_2 g) = k_1 {}_{\phi}D^{\alpha}(f) + k_2 {}_{\phi}D^{\alpha}(g)$;
(ii) ${}_{\phi}D^{\alpha}(t-\phi)^{p} = p(t-\phi)^{p-\alpha}$;
(iii) ${}_{\phi}D^{\alpha}\lambda = 0$ *for all constant functions* $f(t) = \lambda$;
(iv) ${}_{\phi}D^{\alpha}(fg) = f {}_{\phi}D^{\alpha} g + g {}_{\phi}D^{\alpha} f$;
(v) ${}_{\phi}D^{\alpha}\left(\frac{f}{g}\right) = \frac{g {}_{\phi}D^{\alpha} f - f {}_{\phi}D^{\alpha} g}{g^2}$ *for all functions* $g(t) \neq 0$.

Definition 2.3 ([20])**.** *Let $\alpha \in (0,1]$. The conformable fractional integral starting from a point ϕ of a function $f : [\phi, \infty) \to \mathbb{R}$ of order α is defined as*

$$ {}_{\phi}I^{\alpha} f(t) = \int_{\phi}^{t} (s-\phi)^{\alpha-1} f(s)ds. \tag{4}$$

Remark 2.4. *If $\phi = 0$, the definitions of the conformable fractional derivative and integral above will be reduced to the results in [19].*

Definition 2.5. *A nontrivial solution of Eq. (2) is said to be nonoscillatory if the solution is eventually positive or eventually negative. Otherwise, it is said to be oscillatory. Eq. (2) is said to be oscillatory if all solutions are oscillatory.*

3 Auxiliary results

Let $J_k = [t_k, t_{k+1})$, $k = 0, 1, 2, \ldots$ be subintervals of $[t_0, \infty)$. $PC([t_0, \infty)) = \{x : [t_0, \infty) \to \mathbb{R} : x$ be continuous everywhere except at some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$.

Lemma 3.1. *Let $w \in PC([t_0, \infty), \mathbb{R})$ be integrable on J_k, $k = 0, 1, 2, \ldots$ and $z \in C([t_0, \infty), \mathbb{R}) \cap PC([t_0, \infty), \mathbb{R})$ such that*

$$ z(t) = \exp\left(\sum_{t_0}^{t} {}_{t_k}I^{\alpha} w(t) \right) := \exp({}_{t_0}I^{\alpha} w(t_1) + {}_{t_1}I^{\alpha} w(t_2) + \cdots + {}_{t_k}I^{\alpha} w(t)), \tag{5}$$

for some $t \in J_k$, $k = 0, 1, 2, \ldots$. Then

$$ z(t) = \exp\left(\sum_{t_0}^{t} {}_{t_k}I^{\alpha} w(t) \right) \Longleftrightarrow {}_{t_k}D^{\alpha} z(t) = w(t)z(t), \ \ t \in J_k, \ \ k = 0, 1, 2, \ldots. \tag{6}$$

Proof. Observe that $z(t) > 0$ for all $t \in [t_0, \infty)$ and $z(t_0) = 1$. Assume that ${}_{t_k}D^{\alpha} z(t) = w(t)z(t)$ holds. For $t \in J_0$, we have

$$ {}_{t_0}D^{\alpha}[\exp(-{}_{t_0}I^{\alpha} w(t))z(t)] = \exp(-{}_{t_0}I^{\alpha} w(t))[{}_{t_k}D^{\alpha} z(t) - w(t)z(t)] = 0.$$

Taking the conformable fractional integral of order α to both sides of the above equation, we get

$$ z(t) = \exp({}_{t_0}I^{\alpha} w(t)).$$

In particular, $z(t_1) = \exp({}_{t_0}I^{\alpha} w(t_1))$. For $t \in J_1$ and following the above process, we have

$$ \exp(-{}_{t_1}I^{\alpha} w(t))z(t) - z(t_1) = 0,$$

which implies

$$z(t) = \exp({}_{t_0}I^{\alpha} w(t_1) + {}_{t_1}I^{\alpha} w(t)), \quad t \in J_1.$$

Repeating the above method, for any $t \in J_k$, $k = 0, 1, 2, \ldots$, we obtain (5).

On the other hand, for any $t \in J_k$, $k = 0, 1, 2, \ldots$, we have

$${}_{t_k}D^{\alpha} z(t) = \exp\left(\sum_{t_0}^{t} {}_{t_k}I^{\alpha} w(t)\right) {}_{t_k}D^{\alpha}\left({}_{t_k}I^{\alpha} w(t)\right) = z(t)w(t).$$

This completes the proof. □

Let $T \geq t_0$ be a positive constant. We denote $t_r = \min_k\{t_k : T \leq t_k, k = 0, 1, 2, \ldots\}$.

Lemma 3.2. *Let $g, h \in C([t_0, \infty), \mathbb{R})$ be two given functions. The linear conformable fractional differential equation*

$${}_{t_k}D^{\alpha} z(t) - g(t)z(t) = h(t), \quad t \in J_k, \ k = r, r+1, r+2, \ldots, \tag{7}$$

has a solution given by

$$z(t) = z(t_r)\exp\left(\sum_{r}^{t} I^{\alpha} g\right) + \sum_{r}^{t}\left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_{r}^{t} I^{\alpha} g\right)\right]. \tag{8}$$

Proof. For $t \in J_r$, we have

$${}_{t_r}D^{\alpha}\left(e^{-{}_{t_r}I^{\alpha}g(t)} z(t)\right) = e^{-{}_{t_r}I^{\alpha}g(t)} h(t),$$

which leads to

$$z(t) = z(t_r)e^{{}_{t_r}I^{\alpha}g(t)} + e^{{}_{t_r}I^{\alpha}g(t)} {}_{t_r}I^{\alpha}\left(e^{-{}_{t_r}I^{\alpha}g}h\right)(t).$$

Therefore, (8) is true for $[t_r, t_{r+1})$. Now assume that (8) holds for $t \in [t_r, t_{r+n})$ for some integer $n > 1$. Then, for $t \in [t_{r+n}, t_{r+n+1})$, it follows from (7) that

$$z(t) = z(t_{r+n})e^{{}_{t_{r+n}}I^{\alpha}g(t)} + e^{{}_{t_{r+n}}I^{\alpha}g(t)} {}_{t_{r+n}}I^{\alpha}\left(e^{-{}_{t_{r+n}}I^{\alpha}g}h\right)(t).$$

Using (8), we deduce that

$$\begin{aligned} z(t) &= \left\{z(t_r)\exp\left(\sum_{r}^{r+n} I^{\alpha} g\right) + \sum_{r}^{r+n}\left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_{r}^{r+n} I^{\alpha} g\right)\right]\right\} e^{{}_{t_{r+n}}I^{\alpha}g(t)} \\ &\quad + e^{{}_{t_{r+n}}I^{\alpha}g(t)} {}_{t_{r+n}}I^{\alpha}\left(e^{-{}_{t_{r+n}}I^{\alpha}g}h\right)(t) \\ &= z(t_r)\exp\left(\sum_{r}^{t} I^{\alpha} g\right) + \sum_{r}^{r+n}\left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_{r}^{t} I^{\alpha} g\right)\right] \\ &\quad + e^{{}_{t_{r+n}}I^{\alpha}g(t)} {}_{t_{r+n}}I^{\alpha}\left(e^{-{}_{t_{r+n}}I^{\alpha}g}h\right)(t) \\ &= z(t_r)\exp\left(\sum_{r}^{t} I^{\alpha} g\right) + \sum_{r}^{t}\left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_{r}^{t} I^{\alpha} g\right)\right], \end{aligned}$$

which gives (8) for $t \in [t_r, t_{r+n+1})$. The proof is completed. □

4 Main results

Theorem 4.1. *If the following conformable fractional differential equation*

$${}_{t_k}D^{\alpha}\left(p^*(t)\left[{}_{t_k}D^{\alpha} y(t) + r^*(t)y(t)\right]\right) + q^*(t)y(t) = 0, \quad t > t_0, \tag{9}$$

is oscillatory, then Eq. (2) *is oscillatory, where* $c_k = b_k/a_k$, $d_k = p(t_k)r(t_k)(a_k - b_k)/a_k$, *for* $k = 1, 2, 3, \ldots$, *and*

$$p^*(t) = \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right) p(t), \tag{10}$$

$$r^*(t) = r(t) - \frac{2}{p(t)} \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k, \tag{11}$$

$$q^*(t) = \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right)\left[q(t) + \frac{1}{p(t)}\left(\sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right)^2 - r(t) \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right]. \tag{12}$$

Proof. Without loss of generality, we assume that Eq. (2) has an eventually positive solution x. This means that there exists a constant $T \ge t_0$ such that $x(t) > 0$ for all $t \ge T$. For any $t \in J_k$ with $t_k \ge t_r \ge T$, setting

$$u(t) = \frac{p(t)\,{}_{t_k}D^\alpha\left(e^{{}_{t_k}I^\alpha r(t)} x(t)\right)}{e^{{}_{t_k}I^\alpha r(t)} x(t)}, \tag{13}$$

it is easy to verify that

$$u(t) = \frac{p(t)\left({}_{t_k}D^\alpha x(t) + r(t)x(t)\right)}{x(t)}, \quad t \in J_k.$$

Therefore, we obtain

$$\begin{aligned}
{}_{t_k}D^\alpha u(t) &= \frac{{}_{t_k}D^\alpha\left[p(t)\,{}_{t_k}D^\alpha\left(e^{{}_{t_k}I^\alpha r(t)} x(t)\right)\right]}{e^{{}_{t_k}I^\alpha r(t)} x(t)} - p(t)\frac{\left({}_{t_k}D^\alpha\left(e^{{}_{t_k}I^\alpha r(t)} x(t)\right)\right)^2}{\left(e^{{}_{t_k}I^\alpha r(t)} x(t)\right)^2} \\
&= \frac{{}_{t_k}D^\alpha\left[e^{{}_{t_k}I^\alpha r(t)} p(t)\left({}_{t_k}D^\alpha x(t) + r(t)x(t)\right)\right]}{e^{{}_{t_k}I^\alpha r(t)} x(t)} - \frac{u^2(t)}{p(t)} \\
&= \frac{{}_{t_k}D^\alpha\left[p(t)\left({}_{t_k}D^\alpha x(t) + r(t)x(t)\right)\right]}{x(t)} + r(t)\frac{p(t)\left({}_{t_k}D^\alpha x(t) + r(t)x(t)\right)}{x(t)} - \frac{u^2(t)}{p(t)} \\
&= -q(t) + r(t)u(t) - \frac{u^2(t)}{p(t)},
\end{aligned}$$

which leads to

$${}_{t_k}D^\alpha u(t) - r(t)u(t) + \frac{u^2(t)}{p(t)} + q(t) = 0, \quad t \ge t_r,\ t \ne t_k. \tag{14}$$

For $t = t_k^+ \ge t_r$, $k = 1, 2, \ldots$, we have

$$\begin{aligned}
u(t_k^+) &= \frac{p(t_k^+)\left({}_{t_k}D^\alpha x(t_k^+) + r(t_k^+)x(t_k^+)\right)}{x(t_k^+)} \\
&= \frac{p(t_k)\left(b_k\,{}_{t_{k-1}}D^\alpha x(t_k^-) + r(t_k)a_k x(t_k^-)\right)}{a_k x(t_k^-)} \\
&= \frac{b_k}{a_k} p(t_k)\frac{\left({}_{t_{k-1}}D^\alpha x(t_k^-) + r(t_k)x(t_k^-)\right)}{x(t_k^-)} + p(t_k)r(t_k)\frac{(a_k - b_k)}{a_k} \\
&= c_k u(t_k^-) + d_k.
\end{aligned}$$

Define a function $v(t)$ for $t \ge t_r$ by

$$v(t) = \prod_{t_r \le t_k < t} \frac{1}{c_k}\left[u(t) - \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right].$$

For each $t_n \ge t_r$, we see that

$$\begin{aligned} v(t_n^+) &= \prod_{t_r \le t_k \le t_n} \frac{1}{c_k}\left[u(t_n^+) - \sum_{t_r \le t_k \le t_n} \prod_{t_k < t_j \le t_n} c_j d_k\right] \\ &= \prod_{t_r \le t_k \le t_n} \frac{1}{c_k}\left[c_n u(t_n^-) + d_n - \sum_{t_r \le t_k < t_n} \prod_{t_k < t_j < t_n} c_j c_n d_k - d_n\right] \\ &= \prod_{t_r \le t_k < t_n} \frac{1}{c_k}\left[u(t_n^-) - \sum_{t_r \le t_k < t_n} \prod_{t_k < t_j < t_n} c_j d_k\right] \\ &= v(t_n^-), \end{aligned}$$

which implies that $v(t)$ is continuous on (t_r, ∞).

Denote $t_{n,\alpha} = (t - t_n)^{1-\alpha}$ for some $t \in J_n$ and $t_n \ge t_r$. From the definition of conformable fractional derivative, for $t \in J_n$ we obtain

$$\begin{aligned} {}_{t_n}D^{\alpha} v(t) &= \lim_{\varepsilon \to 0}\left(\prod_{t_r \le t_k < t + \varepsilon t_{n,\alpha}} \frac{1}{c_k}\left[u(t + \varepsilon t_{n,\alpha}) - \sum_{t_r \le t_k < t + \varepsilon t_{n,\alpha}} \prod_{t_k < t_j < t + \varepsilon t_{n,\alpha}} c_j d_k\right]\right. \\ &\quad \left. - \prod_{t_r \le t_k < t} \frac{1}{c_k}\left[u(t) - \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right]\right) \Big/ \varepsilon \\ &= \lim_{\varepsilon \to 0}\left(\prod_{t_r \le t_k < t + \varepsilon t_{n,\alpha}} \frac{1}{c_k} u(t + \varepsilon t_{n,\alpha}) - \prod_{t_r \le t_k < t} \frac{1}{c_k} u(t)\right) \Big/ \varepsilon \\ &= \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right) \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t_{n,\alpha}) - u(t)}{\varepsilon} = \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right) {}_{t_n}D^{\alpha} u(t). \end{aligned}$$

From (14), we have

$$\begin{aligned} {}_{t_n}D^{\alpha} v(t) &= \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right)\left[r(t)u(t) - \frac{u^2(t)}{p(t)} - q(t)\right] \\ &= \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right)\left[r(t)\left\{v(t)\left(\prod_{t_r \le t_k < t} c_k\right) + \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right\}\right. \\ &\quad \left. - \frac{1}{p(t)}\left\{v(t)\left(\prod_{t_r \le t_k < t} c_k\right) + \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right\}^2 - q(t)\right] \\ &= r(t)v(t) + r(t) \sum_{t_r \le t_k < t} \prod_{t_r \le t_j \le t_k} \frac{d_k}{c_j} - \left(\prod_{t_r \le t_k < t} c_k\right) \frac{v^2(t)}{p(t)} \\ &\quad - 2\frac{v(t)}{p(t)} \sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k - \frac{1}{p(t)}\left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right)\left(\sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right)^2 \\ &\quad - \left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right) q(t), \end{aligned}$$

by using the fact that

$$\left(\prod_{t_r \le t_k < t} \frac{1}{c_k}\right)\left(\sum_{t_r \le t_k < t} \prod_{t_k < t_j < t} c_j d_k\right) = \sum_{t_r \le t_k < t} \prod_{t_r \le t_j \le t_k} \frac{d_k}{c_j}.$$

Setting

$$\xi(t) = \exp\left(\sum_r^t {}_{t_k}I^\alpha \sigma(t)\right), \quad t \geq t_r,$$

where

$$\sigma(t) = -r(t) + \left(\prod_{t_r \leq t_k < t} c_k\right) \frac{v(t)}{p(t)} + \frac{2}{p(t)} \sum_{t_r \leq t_k < t} \prod_{t_k < t_j < t} c_j d_k,$$

we have $\xi(t) > 0$ for $t \geq t_r$ and from Lemma 3.1 that

$$\begin{aligned} {}_{t_k}D^\alpha \xi(t) &= \xi(t)\sigma(t) \\ &= \xi(t)\left[-r(t) + \left(\prod_{T \leq t_k < t} c_k\right) \frac{v(t)}{p(t)} + \frac{2}{p(t)} \sum_{T \leq t_k < t} \prod_{t_k < t_j < t} c_j d_k\right]. \end{aligned}$$

Hence

$$\begin{aligned} \xi(t)v(t) &= \left(\prod_{t_r \leq t_k < t} \frac{1}{c_k}\right) p(t)_{t_k}D^\alpha \xi(t) + \left(\prod_{T \leq t_k < t} \frac{1}{c_k}\right) r(t)p(t)\xi(t) - 2 \sum_{t_r \leq t_k < t} \prod_{T \leq t_j \leq t_k} \frac{d_k}{c_j}\xi(t) \\ &= p^*(t)\left[{}_{t_k}D^\alpha \xi(t) + r^*(t)\xi(t)\right]. \end{aligned} \tag{15}$$

Applying the conformable fractional derivative of order α on any interval J_k to the above equation, we get

$$\begin{aligned} {}_{t_k}D^\alpha \xi(t)v(t) &= \xi(t)_{t_k}D^\alpha v(t) + v(t)_{t_k}D^\alpha \xi(t) \\ &= \xi(t)\left[{}_{t_k}D^\alpha v(t) + v(t)\sigma(t)\right] \\ &= \xi(t)\Bigg[r(t) \sum_{t_r \leq t_k < t} \prod_{T \leq t_j \leq t_k} \frac{d_k}{c_j} - \left(\prod_{t_r \leq t_k < t} \frac{1}{c_k}\right) q(t) \\ &\quad - \frac{1}{p(t)}\left(\prod_{t_r \leq t_k < t} \frac{1}{c_k}\right)\left(\sum_{t_r \leq t_k < t} \prod_{t_k < t_j < t} c_j d_k\right)^2\Bigg]. \end{aligned} \tag{16}$$

From (15) and (16), we obtain

$${}_{t_k}D^\alpha \left(p^*(t)\left[{}_{t_k}D^\alpha \xi(t) + r^*(t)\xi(t)\right]\right) + q^*(t)\xi(t) = 0, \quad t \geq t_r.$$

This means that $\xi(t)$ is an eventually positive solution of Eq. (9) which is a contradiction. In the same way, we can prove that Eq. (9) could not have an eventually negative solution. Therefore, the solution of Eq. (9) is oscillatory. This completes the proof. □

Theorem 4.2. *Assume that $g, h \in C([t_0, \infty), \mathbb{R})$ and $f \in C([t_0, \infty), \mathbb{R}_+)$. If*

$$\sum_r^\infty I^\alpha \left(e^{-I^\alpha g} h\right) = \infty \quad \text{and} \quad \sum_r^\infty I^\alpha \left(e^{I^\alpha g} \frac{1}{f}\right) = \infty,$$

then

$${}_{t_k}D^\alpha \left(f(t)\left[{}_{t_k}D^\alpha x(t) + g(t)x(t)\right]\right) + h(t)x(t) = 0, \quad t > t_0, \tag{17}$$

is oscillatory.

Proof. Let a function x be a nonoscillatory solution of Eq. (17). Then we can assume that there exists a constant $T \geq t_0$ such that $x(t) > 0$ for all $t \geq T$. For the case $x(t) < 0$, $t \geq T$, the proof is similar and we omit it.

Using the Riccati transform for conformable fractional differential equation

$$w(t) = \frac{f(t)_{t_k}D^\alpha \left(e^{t_k I^\alpha g(t)} x(t)\right)}{e^{t_k I^\alpha g(t)} x(t)}, \quad t \in J_k,\ k \geq r, \tag{18}$$

we get that the equation

$$_{t_k}D^{\alpha}w(t) - g(t)w(t) + \frac{1}{f(t)}w^2(t) + h(t) = 0, \tag{19}$$

has a solution $w(t)$ on $[t_r, \infty)$. From Lemma 3.2, we get

$$\begin{aligned} w(t) &= w(t_r)\exp\left(\sum_r^t I^{\alpha} g\right) - \sum_r^t \left[I^{\alpha}\left(e^{-I^{\alpha}g}\frac{w^2}{f}\right)\exp\left(\sum_r^t I^{\alpha} g\right)\right] \\ &\quad - \sum_r^t \left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_r^t I^{\alpha} g\right)\right]. \end{aligned} \tag{20}$$

Since $\sum_r^{\infty} I^{\alpha}\left(e^{-I^{\alpha}g}h\right) = \infty$, then there exists a fixed point $\eta > t_r$ such that

$$w(t_r)\exp\left(\sum_r^t I^{\alpha} g\right) - \sum_r^t \left[I^{\alpha}\left(e^{-I^{\alpha}g}h\right)\exp\left(\sum_r^t I^{\alpha} g\right)\right] < 0, \quad \forall t \in [\eta, \infty).$$

Thus (20) implies

$$w(t) < -\sum_r^t \left[I^{\alpha}\left(e^{-I^{\alpha}g}\frac{w^2}{f}\right)\exp\left(\sum_r^t I^{\alpha} g\right)\right], \quad \forall t \in [\eta, \infty).$$

Let $\eta \leq t \in J_j$ for some j. We observe that

$$\begin{aligned} &\sum_r^t \left[I^{\alpha}\left(e^{-I^{\alpha}g}\frac{w^2}{f}\right)\exp\left(\sum_r^t I^{\alpha} g\right)\right] \\ &= e^{t_j I^{\alpha}g(t)}\Bigg[\sum_{t_r \leq t_k \leq t_{j-1}} {}_{t_k}I^{\alpha}\left(e^{-{}_{t_k}I^{\alpha}g}\frac{w^2}{f}\right)(t_{k+1})\exp\left(\sum_{t_r \leq t_k \leq t_{j-1}} {}_{t_k}I^{\alpha}g(t_{k+1})\right) \\ &\quad + {}_{t_j}I^{\alpha}\left(e^{-{}_{t_j}I^{\alpha}g}\frac{w^2}{f}\right)(t)\Bigg] \\ &:= e^{t_j I^{\alpha}g(t)}u(t). \end{aligned}$$

Then we have $w(t) < -e^{t_j I^{\alpha}g(t)}u(t)$ and

$$_{t_j}D^{\alpha}u(t) = e^{-{}_{t_j}I^{\alpha}g(t)}\frac{w^2(t)}{f(t)} > e^{t_j I^{\alpha}g(t)}\frac{u^2(t)}{f(t)}, \quad t \in J_j,$$

which yields for $t = t_{j+1}$

$$-\frac{1}{u(t_{j+1})} + \frac{1}{u(t_j)} > {}_{t_j}I^{\alpha}\left(e^{t_j I^{\alpha}g}\frac{1}{f}\right)(t_{j+1}),$$

and also

$$-\frac{1}{u(t_{j+2})} + \frac{1}{u(t_{j+1})} > {}_{t_{j+1}}I^{\alpha}\left(e^{t_{j+1} I^{\alpha}g}\frac{1}{f}\right)(t_{j+2}), \ldots.$$

Summing up the above inequalities, we obtain

$$-\frac{1}{u(\infty)} + \frac{1}{u(t_j)} > \sum_{t_j}^{\infty} I^{\alpha}\left(e^{I^{\alpha}g}\frac{1}{f}\right).$$

Therefore,

$$\sum_r^{\infty} I^{\alpha}\left(e^{I^{\alpha}g}\frac{1}{f}\right) = \infty < \frac{1}{u(t_j)} < \infty,$$

which is a contradiction. Hence, the solution x is oscillatory. This completes the proof. □

Theorem 4.3. *If*

$$\sum_{r}^{\infty} I^{\alpha}\left(e^{-I^{\alpha} r^{*}} q^{*}\right)=\infty \quad \text{and} \quad \sum_{r}^{\infty} I^{\alpha}\left(e^{I^{\alpha} r^{*}} \frac{1}{p^{*}}\right)=\infty, \tag{21}$$

where functions p^*, r^* and q^* are defined by (10)-(12), *respectively, then Eq.* (2) *is oscillatory.*

If $a_k = b_k = e_k$ for all $k = 1, 2, 3, \ldots$, then $c_k = 1$ and $d_k = 0$ for $k = 1, 2, 3, \ldots$ and (2) can be written as

$$\begin{cases} {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)\right]\right)+q(t)x(t)=0, \quad t \geq t_0,\ t \neq t_k, \\ x(t_k^{+})=e_k x(t_k^{-}), \quad k=1,2,\ldots, \\ {}_{t_k}D^{\alpha}x(t_k^{+})=e_k\, {}_{t_{k-1}}D^{\alpha}x(t_k^{-}), \quad k=1,2,\ldots. \end{cases} \tag{22}$$

Theorem 4.4. *Assume that $a_k = b_k$ for all $k = 1, 2, 3, \ldots$. Eq.* (22) *is oscillatory, if and only if*

$$ {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}y(t)+r(t)y(t)\right]\right)+q(t)y(t)=0, \quad t>t_0, \tag{23}$$

is oscillatory.

Proof. From Theorem 4.1, we only need to prove that if Eq. (22) is oscillatory, then (23) is oscillatory.

Assume that the function y is an eventually positive solution of Eq. (18) such that $y(t) > 0$ for all $t \geq T \geq t_0$. We define

$$x(t)=\left(\prod_{t_r \leq t_k<t} e_k\right) y(t), \quad t \geq t_r \geq T.$$

Then, we obtain $x(t) > 0$ for $t \geq t_r$ and

$$x(t_n^{+})=\left(\prod_{t_r \leq t_k \leq t_n} e_k\right) y(t_n^{+})=e_n\left(\prod_{t_r \leq t_k<t_n} e_k\right) y(t_n)=e_n x(t_n),$$

for some $n > r$. In addition, for $t \in J_n$, $n > r$, we have

$${}_{t_n}D^{\alpha}x(t)=\left(\prod_{t_r \leq t_k<t} e_k\right) {}_{t_n}D^{\alpha}y(t),$$

and for $t = t_n$,

$${}_{t_n}D^{\alpha}x(t_n^{+})=\left(\prod_{t_r \leq t_k \leq t_n} e_k\right) {}_{t_n}D^{\alpha}y(t_n^{+})=e_n\left(\prod_{t_r \leq t_k<t_n} e_k\right) {}_{t_{n-1}}D^{\alpha}y(t_n^{-})=e_n\, {}_{t_{n-1}}D^{\alpha}x(t_n^{-}).$$

For any $t \in J_k$, $k \geq r$, we have

$$\begin{aligned} & {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)\right]\right) \\ & = {}_{t_k}D^{\alpha}\left(p(t)\left[\left(\prod_{t_r \leq t_k<t} e_k\right) {}_{t_k}D^{\alpha}y(t)+r(t)\left(\prod_{t_r \leq t_k<t} e_k\right) y(t)\right]\right) \\ & =\left(\prod_{t_r \leq t_k<t} e_k\right) {}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}y(t)+r(t)y(t)\right]\right) \\ & =-\left(\prod_{t_r \leq t_k<t} e_k\right) q(t)y(t)=-q(t)x(t). \end{aligned}$$

Therefore,

$${}_{t_k}D^{\alpha}\left(p(t)\left[{}_{t_k}D^{\alpha}x(t)+r(t)x(t)\right]\right)+q(t)x(t)=0,\ t \neq t_n,\ t>T.$$

This means that x is an eventually positive solution of Eq. (22) which is a contradiction. The proof is completed. □

Corollary 4.5. *If*

$$\sum_{r}^{\infty} I^{\alpha}\left(e^{-I^{\alpha}r}q\right)=\infty \quad \textit{and} \quad \sum_{r}^{\infty} I^{\alpha}\left(e^{I^{\alpha}r}\frac{1}{p}\right)=\infty,$$

then Eq. (22) *is oscillatory.*

5 Examples

Example 5.1. *Consider the following impulsive conformable fractional differential equation*

$$\begin{cases} {}_kD^{\frac{2}{3}}\left([t]\left[{}_kD^{\frac{2}{3}}x(t)+\frac{1}{[t]}x(t)\right]\right)+(t+2)3^{t^4}=0, & t\in(k,k+1),\\ x(k^+)=kx(k^-), \quad {}_kD^{\frac{2}{3}}x(k^+)=(k+1)_{k-1}D^{\frac{2}{3}}x(k^-), & k=1,2,3,\ldots. \end{cases} \tag{24}$$

Here $\alpha = 2/3$, $p(t) = [t]$, $r(t) = 1/[t]$, $t > 0$, where $[\cdot]$ denotes the greatest integer function, $q(t) = (t+2)3^{t^4}$, $a_k = k, b_k = k+1, k = 1,2,3,\ldots$. We find $c_k = b_k/a_k = (k+1/k)$, $d_k = p(t_k)r(t_k)(a_k-b_k)/a_k = -1/k$. Let $t_r \in (m, m+1]$ for some integer $m > 1$. Then we have

$$\prod_{t_r\le t_k<[t]+1}\frac{1}{c_k}=\prod_{t_r\le t_k<[t]+1}\frac{k}{k+1}=\frac{m+1}{m+2}\cdot\frac{m+2}{m+3}\cdot\cdots\cdot\frac{[t]}{[t]+1}=\frac{m+1}{[t]+1},$$

and

$$\begin{aligned}\sum_{t_r\le t_k<[t]}\prod_{t_k<t_j<[t]}c_jd_k&=-\sum_{t_r\le t_k<[t]}\prod_{t_k<t_j<[t]}\left(\frac{j+1}{j}\right)\left(\frac{1}{k}\right)\\&=-\left(\frac{1}{m+1}\cdot\frac{m+3}{m+2}\cdot\frac{m+4}{m+3}\cdot\cdots\cdot\frac{[t]}{[t]-1}\right.\\&\quad+\frac{1}{m+2}\cdot\frac{m+4}{m+3}\cdot\frac{m+5}{m+4}\cdot\cdots\cdot\frac{[t]}{[t]-1}+\cdots\\&\quad\left.+\frac{1}{[t]-2}\cdot\frac{[t]}{[t]-1}+\frac{[t]}{[t]-1}\right)\\&=1-\frac{[t]}{m+1}.\end{aligned}$$

Observe that $1-([t])/(m+1) < 0$. Hence, by direct computation, we have

$$\begin{aligned}&\sum_{r}^{\infty} I^{\frac{2}{3}}\left(q^*\exp\left(-I^{\frac{2}{3}}r^*\right)\right)\\&=\sum_{r}^{\infty} I^{\frac{2}{3}}\left(\frac{m+1}{[t]+1}\left[(t+2)3^{t^4}+\frac{1}{[t]}\left(1-\frac{[t]}{m+1}\right)^2-\frac{1}{[t]}\left(1-\frac{[t]}{m+1}\right)\right]\right.\\&\quad\left.\times\exp\left(-I^{\frac{2}{3}}\left\{\frac{1}{[t]}-\frac{2}{[t]}\left(1-\frac{[t]}{m+1}\right)\right\}\right)\right)\\&\ge\sum_{r}^{\infty} I^{\frac{2}{3}}\left((m+1)3^{t^4}\exp\left(I^{\frac{2}{3}}\left\{\frac{1}{[t]}\left(1-2\frac{[t]}{m+1}\right)\right\}\right)\right)=\infty,\end{aligned}$$

and

$$\begin{aligned}&\sum_{r}^{\infty} I^{\frac{2}{3}}\left(\frac{1}{p^*}\exp\left(I^{\frac{2}{3}}r^*\right)\right)\\&=\sum_{r}^{\infty} I^{\frac{2}{3}}\left(\frac{[t]+1}{m+1}\cdot\frac{1}{[t]}\exp\left(I^{\frac{2}{3}}\left\{\frac{1}{[t]}-\frac{2}{[t]}\left(1-\frac{[t]}{m+1}\right)\right\}\right)\right)\end{aligned}$$

$$\geq \frac{1}{m+1}\sum_{r}^{\infty} I^{\frac{2}{3}}\left(\exp\left(I^{\frac{2}{3}}\left\{\frac{1}{[t]}\left(\frac{2[t]}{m+1}-1\right)\right\}\right)\right)=\infty.$$

Therefore, by Theorem 4.3, we deduce that Eq. (24) is oscillatory.

Example 5.2. *Consider the following impulsive conformable fractional differential equation*

$$\begin{cases} {}_kD^{\frac{2}{3}}\left(\frac{1}{\sqrt{t^2+4}}\left[{}_kD^{\frac{2}{3}}x(t)+tx(t)\right]\right)+5^{4t+3}=0, \quad t\in(k,k+1), \\ x(k^+)=\frac{3k+1}{4k+2}x(k^-), \quad {}_kD^{\frac{2}{3}}x(k^+)=\frac{3k+1}{4k+2}{}_{k-1}D^{\frac{2}{3}}x(k^-), \quad k=1,2,3,\ldots. \end{cases} \tag{25}$$

Here $p(t)=1/\sqrt{t^2+4}$, $r(t)=t$, $q(t)=5^{4t+3}$, $\alpha=2/3$. We find that ${}_kI^{2/3}r(t)\Big|_{t=k+1}=((3k)/2)+(3/5)$. For each $t_r\in(m,m+1]$, $m>1$, we have

$$\sum_{r}^{\infty} I^{\frac{2}{3}}\left(5^{4t+3}\exp\left(-I^{\frac{2}{3}}t\right)\right)=\infty,$$

and

$$\sum_{r}^{\infty} I^{\frac{2}{3}}\left(\sqrt{t^2+4}\exp\left(I^{\frac{2}{3}}t\right)\right)=\infty.$$

Applying Corollary 4.5, we obtain that Eq. (25) is oscillatory.

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