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On the concept of general solution for impulsive differential equations of fractional-order $q \in (2, 3)$

Abstract: In this paper, we find the formula of general solution for a generalized impulsive differential equations of fractional-order $q \in (2, 3)$.

1 Introduction

Fractional differential equations play an important part in modeling of many phenomena in various fields of science and engineering, and the subject of fractional differential equations is extensively researched (see [1–19] and the references therein).

On the other hand, impulsive differential equation is a key tool to describe some systems and processes with impulsive effects. There have appeared many papers focused on the subject of impulsive differential equations with Caputo fractional derivative [20–31].

Recently, we have found that there exist general solutions for several kinds of impulsive fractional differential equations in [32–38]. Based on these works, we will further study the general solution of the generalized impulsive differential equations of fractional-order $q \in (2, 3)$.

$$
\begin{cases}
{}_aD_t^q x(t) = f(t, x(t)), \quad q \in (2, 3), \ t \in J = [a, T], \ t \neq t_i \ (i = 1, 2, ..., L) \ and \\
\qquad\qquad t \neq \bar{t}_j \ (j = 1, 2, ..., M) \ and \ t \neq \hat{t}_l \ (l = 1, 2, ..., N), \\
\Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-) = I_i\left(x(t_i^-)\right), \qquad i = 1, 2, ..., L, \\
\Delta x'|_{t=\bar{t}_j} = x'(\bar{t}_j^+) - x'(\bar{t}_j^-) = \bar{I}_j\left(x(\bar{t}_j^-)\right), \quad j = 1, 2, ..., M, \\
\Delta x''|_{t=\hat{t}_l} = x''(\hat{t}_l^+) - x''(\hat{t}_l^-) = \hat{I}_l\left(x(\hat{t}_l^-)\right), \quad l = 1, 2, ..., N, \\
x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a.
\end{cases}
\tag{1}
$$

where ${}_aD_t^q$ denote Caputo fractional derivative of order q in interval $[a, t]$, $f : J \times \mathbb{R} \to \mathbb{R}$ and $I_i, \bar{I}_j, \hat{I}_l : \mathbb{R} \to \mathbb{R}$ are appropriate functions (here $i = 1, 2, \ldots, L$ and $j = 1, 2, \ldots, M$ and $l = 1, 2, \ldots, N$, respectively), $a = t_0 <$

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$t_1 < ... < t_L < t_{L+1} = T, a = \bar{t}_0 < \bar{t}_1 < ... < \bar{t}_M < \bar{t}_{M+1} = T, a = \hat{t}_0 < \hat{t}_1 < ... < \hat{t}_N < \hat{t}_{N+1} = T$. Here $x(t_i^+) = \lim_{\varepsilon \to 0+} x(t_i + \varepsilon)$ and $x(t_i^-) = \lim_{\varepsilon \to 0-} x(t_i + \varepsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$, respectively, $(x'(\bar{t}_j^+), x'(\bar{t}_j^-)$ and $x''(\hat{t}_l^+), x''(\hat{t}_l^-)$ have similar meaning for $x'(t)$ at $t = \bar{t}_j$ and $x''(t)$ at $t = \hat{t}_l$, respectively).

Next, take $a, t_1, t_2, \cdots, t_L, \bar{t}_1, \bar{t}_2, \cdots, \bar{t}_M, \hat{t}_1, \hat{t}_2, \cdots, \hat{t}_N, T$ to $a = t_0' < t_1' < ... < t_K' < t_{K+1}' = T$ such that

$$\text{set } \{t_1, t_2, \cdots, t_L, \bar{t}_1, \bar{t}_2, \cdots, \bar{t}_M, \hat{t}_1, \hat{t}_2, \cdots, \hat{t}_N\} = \text{set } \{t_1', t_2', ..., t_K'\}$$

Let $J_0' = [a, t_1']$ and $J_k' = (t_k', t_{k+1}']$ $(k = 0, 1, 2, ..., K)$. For each $[a, t_k']$ (here $k = 0, 1, 2, ..., K$), assume $[a, t_{k_0}] \subseteq [a, t_k'] \subset [a, t_{k_0+1}]$ (here $k_0 \in \{1, 2, ..., L\}$) and $[a, \bar{t}_{k_1}] \subseteq [a, t_k'] \subset [a, \bar{t}_{k_1+1}]$ (here $k_1 \in \{1, 2, ..., M\}$) and $[a, \hat{t}_{k_2}] \subseteq [a, t_k'] \subset [a, \hat{t}_{k_2+1}]$ (here $k_2 \in \{1, 2, ..., N\}$) respectively.

With simplification of system (1), we get

$$\begin{cases} aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq t_i \ (k = 1, 2, ..., L), \\ \Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), & i = 1, 2, ..., L, \\ \Delta x'|_{t=t_i} = \bar{I}_i\left(x(t_i^-)\right), & i = 1, 2, ..., L, \\ \Delta x''|_{t=t_i} = \hat{I}_i\left(x(t_i^-)\right), & i = 1, 2, ..., L, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{2}$$

Let $J_0 = [a, t_1]$ and $J_i = (t_i, t_{i+1}]$ $(i = 1, 2, ..., L)$.

Considering some limiting cases in (1), we have

$$\lim_{\substack{I_i(x(t_i^-)) \to 0 \ for \ all \ i \in \{1,2,...,L\} \\ and \ \bar{I}_j\left(x(\bar{t}_j^-)\right) \to 0 \ for \ all \ j \in \{1,2,...,M\} \\ and \ \hat{I}_l(x(\hat{t}_l^-)) \to 0 \ for \ all \ l \in \{1,2,...,N\}}} \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} aD_t^q x(t) = f(t, x(t)), q \in (2,3), t \in J = [a, T], \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{3}$$

$$\lim_{\substack{\bar{I}_j\left(x(\bar{t}_j^-)\right) \to 0 \ for \ all \ j \in \{1,2,...,M\} \\ and \ \hat{I}_l(x(\hat{t}_l^-)) \to 0 \ for \ all \ l \in \{1,2,...,N\}}} \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq t_i \ (i = 1, 2, ..., L), \\ \Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), & i = 1, 2, ..., L, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{4}$$

$$\lim_{\substack{I_i(x(t_i^-)) \to 0 \ for \ all \ i \in \{1,2,...,L\} \\ and \ \hat{I}_l(x(\hat{t}_l^-)) \to 0 \ for \ all \ l \in \{1,2,...,N\}}} \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq \bar{t}_j \ (j = 1, 2, ..., M), \\ \Delta x'|_{t=\bar{t}_j} = \bar{I}_j\left(x(\bar{t}_j^-)\right), & j = 1, 2, ..., M, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{5}$$

$$\lim_{\substack{I_i(x(t_i^-)) \to 0 \ for \ all \ i \in \{1,2,...,L\} \\ and \ \bar{I}_j\left(x(\bar{t}_j^-)\right) \to 0 \ for \ all \ j \in \{1,2,...,M\}}} \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq \hat{t}_l \ (l = 1, 2, ..., N), \\ \Delta x''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right), & l = 1, 2, ..., N, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{6}$$

$$\underset{\hat{I}_l(x(\hat{t}_l^-))\to 0 \ for \ all \ l\in\{1,2,...,N\}}{\lim} \quad \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} _aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq t_i \ (i = 1, 2, ..., L) \\ \qquad\qquad and \ t \neq \bar{t}_j \ (j = 1, 2..., M), \\ \Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), \quad i = 1, 2, ..., L, \\ \Delta x'|_{t=\bar{t}_j} = \bar{I}_j\left(x(\bar{t}_j^-)\right), \quad j = 1, 2, ..., M, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{7}$$

$$\underset{\bar{I}_j\left(x(\bar{t}_j^-)\right)\to 0 \ for \ all \ j\in\{1,2,...,M\}}{\lim} \quad \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} _aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq t_i \ (i = 1, 2, ..., L) \\ \qquad\qquad and \ t \neq \hat{t}_l \ (l = 1, 2, ..., N), \\ \Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), \quad i = 1, 2, ..., L, \\ \Delta x''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right), \quad l = 1, 2, ..., N, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{8}$$

$$\underset{I_i(x(t_i^-))\to 0 \ for \ all \ i\in\{1,2,...,L\}}{\lim} \quad \{\text{impulsive system (1)}\}$$

$$\to \begin{cases} _aD_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \ t \neq \bar{t}_j \ (j = 1, 2, ..., M) \\ \qquad\qquad and \ t \neq \hat{t}_l \ (l = 1, 2, ..., N), \\ \Delta x'|_{t=\bar{t}_j} = \bar{I}_j\left(x(\bar{t}_j^-)\right), \quad j = 1, 2, ..., M, \\ \Delta x''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right), \quad l = 1, 2, ..., N, \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases} \tag{9}$$

This means that the solution of (1) satisfies:

(i) $\underset{\substack{I_i\left(x(t_i^-)\right)\to 0 \ for \ all \ i\in\{1,2,...,L\} \\ and \ \bar{I}_j\left(x(\bar{t}_j^-)\right)\to 0 \ for \ all \ j\in\{1,2,...,M\} \\ and \ \hat{I}_l(x(\hat{t}_l^-))\to 0 \ for \ all \ l\in\{1,2,...,N\}}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (3)}\}$,

(ii) $\underset{\substack{\bar{I}_j\left(x(\bar{t}_j^-)\right)\to 0 \ for \ all \ j\in\{1,2,...,M\} \\ and \ \hat{I}_l(x(\hat{t}_l^-))\to 0 \ for \ all \ l\in\{1,2,...,N\}}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (4)}\}$,

(iii) $\underset{\substack{I_i\left(x(t_i^-)\right)\to 0 \ for \ all \ i\in\{1,2,...,L\} \\ and \ \hat{I}_l(x(\hat{t}_l^-))\to 0 \ for \ all \ l\in\{1,2,...,N\}}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (5)}\}$,

(iv) $\underset{\substack{I_i\left(x(t_i^-)\right)\to 0 \ for \ all \ i\in\{1,2,...,L\} \\ and \ \bar{I}_j\left(x(\bar{t}_j^-)\right)\to 0 \ for \ all \ j\in\{1,2,...,M\}}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (6)}\}$,

(v) $\underset{\hat{I}_l(x(\hat{t}_l^-))\to 0 \ for \ all \ l\in\{1,2,...,N\}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (7)}\}$,

(vi) $\underset{\bar{I}_j\left(x(\bar{t}_j^-)\right)\to 0 \ for \ all \ j\in\{1,2,...,M\}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (8)}\}$,

(vii) $\underset{I_i\left(x(t_i^-)\right)\to 0 \ for \ all \ i\in\{1,2,...,L\}}{\lim} \{\text{the solution of system (1)}\} = \{\text{the solution of system (9)}\}$.

Thus, we present the definition of solution for (1) as follows

Definition 1.1. *A function* $z(t) : [a, T] \rightarrow \mathbb{R}$ *is said to be the solution of impulsive system* (1) *if* $z(a) = x_a$, $z'(a) = \bar{x}_a$ *and* $z''(a) = \hat{x}_a$, *the equation condition* $_aD_t^q z(t) = f(t, z(t))$ *for each* $t \in (a, T]$ *is verified, impulsive conditions* $\Delta z|_{t=t_i} = I_i\left(z(t_i^-)\right)$ (*here* $i = 1, 2, \ldots, L$), $\Delta z'|_{t=\bar{t}_j} = \bar{I}_j\left(z(\bar{t}_j^-)\right)$ (*here* $j = 1, 2, \ldots, M$) *and* $\Delta z''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right)$ (*here* $l = 1, 2, \ldots, N$) *are satisfied, the restriction of to the interval* J_k' (*here* $k = 0, 1, 2, \ldots, K$) *is continuous, and conditions* (i)-(vii) *hold.*

Define a function

$$\tilde{x}(t) = x(t_k^+) + x'(t_k^+)(t - t_k) + \frac{x''(t_k^+)}{2!}(t - t_k)^2 + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s)) ds \quad for \ t \in (t_k, t_{k+1}].$$

By the definition of Caputo fractional derivative, we have

$$\left[_aD_t^q \tilde{x}(t)\right]_{t \in (t_k, t_{k+1}]}$$

$$= \left[_aD_t^q \left(x(t_k^+) + x'(t_k^+)(t - t_k) + \frac{x''(t_k^+)}{2!}(t - t_k)^2 + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s)) ds \right) \right]_{t \in (t_k, t_{k+1}]}$$

$$= \left[_aD_t^q \left(\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s)) ds \right) \right]_{t \in (t_k, t_{k+1}]}$$

$$= \left[_{t_k}D_t^q \left(\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} f(s, x(s)) ds \right) \right]_{t \in (t_k, t_{k+1}]}$$

$$= f(t, x(t))|_{t \in (t_k, t_{k+1}]}.$$

Therefore, $\tilde{x}(t)$ can meet the condition of fractional derivative and impulsive conditions in (1). But, $\tilde{x}(t)$ is only considered as an approximate solution of (1) since it doesn't satisfy conditions (i)-(vii).

Next, we provide some definitions and conclusions in Section 2, and prove the formula of general solution for (1) in Section 3. Finally, an example is provided to expound the main result in Section 4.

2 Preliminaries

Definition 2.1 ([2])**.** *The fractional integral of order q for function x is defined as*

$$_aI_t^q x(t) = \frac{1}{\Gamma(q)} \int_{a}^{t} \frac{x(s)}{(t - s)^{1-q}} ds, \quad t > a, \ q > 0,$$

where Γ is the gamma function.

Definition 2.2 ([2])**.** *The Caputo fractional derivative of order q for a function x can be written as*

$$_aD_t^q x(t) = \frac{1}{\Gamma(n - q)} \int_{a}^{t} \frac{x^{(n)}(s)}{(t - s)^{q+1-n}} ds = {_aI_t^{n-q}} x^{(n)}(t), \quad t > a, \ 0 \leq n - 1 < q < n.$$

Lemma 2.3 ([39])**.** *If the function $h(t, x)$ is continuous, then the initial value problem*

$$\begin{cases} _aD_t^q x(t) = h(t, x(t)), & q \in (n, n + 1], n \in \mathbb{R}^+ \cup \{0\}, \\ x^{(k)}(a) = x_a^k, & k = 0, 1, 2, \ldots, n. \end{cases}$$

is equivalent to the following nonlinear Volterra integral equation of the second kind,

$$x(t) = \sum_{k=0}^{n} \frac{x_a^k}{k!}(t-a)^k + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} h(s, x(s)) ds,$$

and its solutions are continuous.

Lemma 2.4 ([32]). *Let $q \in (0,1)$ and ξ is a constant. Impulsive system*

$$\begin{cases} {}_0 D_t^q u(t) = h(t, u(t)), & t \in J = [0, T], \ t \neq t_k, \ k = 1, 2, ..., m, \\ \Delta u|_{t=t_k} = I_k \left(u(t_k^-) \right), & t = t_k, \ k = 1, 2, ..., m, \\ u(0) = u_0. \end{cases}$$

is equivalent to the fractional integral equation

$$u(t) = \begin{cases} u_0 + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} h(s, u(s)) ds, & for \ t \in [0, t_1], \\[2ex] u_0 + \displaystyle\sum_{k=1}^{n} I_k(u(t_k^-)) + \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-s)^{q-1} h(s, u(s)) ds \\[2ex] \quad + \dfrac{\xi}{\Gamma(q)} \displaystyle\sum_{k=1}^{n} I_k(u(t_k^-)) \left[\displaystyle\int_0^{t_k} (t_k-s)^{q-1} h(s, u(s)) ds + \displaystyle\int_{t_k}^t (t-s)^{q-1} h(s, u(s)) ds \right. \\[2ex] \quad \left. - \displaystyle\int_0^t (t-s)^{q-1} h(s, u(s)) ds \right] & for \ t \in (t_n, t_{n+1}], 1 \le n \le m, \end{cases} \tag{10}$$

provided that the integral in (10) exists.

3 Main results

For convenience, let $f = f(s, x(s))$ and $\sum_{i=1}^0 y_i = 0$ in this section.

Lemma 3.1. *Let $q \in (2,3)$ and ξ_0 is a constant. System (4) is equivalent to the fractional integral equation*

$$x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^k I_i \left(x(t_i^-) \right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds$$

$$+ \sum_{i=1}^k \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i-s)^{q-1} f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.$$

$$\left. + \frac{(t-t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i-s)^{q-2} f ds + \frac{(t-t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i-s)^{q-3} f ds \right\} \tag{11}$$

$$for \ t \in (t_k, t_{k+1}], \ k = 0, 1, 2, ..., L.$$

provided that the integral in (11) exists.

Proof. "Necessity", for system (4), there exist an implicit condition

$$\lim_{I_i(x(t_i^-)) \to 0 \ for \ all \ i \in \{1,2,...,L\}} \{\text{system (4)}\} \to \begin{cases} {}_a D_t^q x(t) = f(t, x(t)), & q \in (2,3), \ t \in J = [a, T], \\ x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a. \end{cases}$$

That is

$$\lim_{I_i(x(t_i^-))\to 0 \ for \ all \ i\in\{1,2,\dots,L\}} \{\text{the solution of system (4)}\} = \{\text{the solution of system (3)}\}. \qquad (12)$$

In fact, we can verify that Eq. (11) satisfies condition (12).

Next, we can obtain $x(t_k^+)-x(t_k^-) = I_k(x(t_k^-))$ That is, Eq. (11) satisfies the impulsive condition of system (4).

Finally, using Eq. (11) for each $t \in (t_k, t_{k+1}]$ (where $k = 0, 1, \dots, L$), we have

$$\left[_aD_t^q x(t) \right]_{t\in(t_k,t_{k+1}]} = {_aD_t^q} \left\{ x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^{k} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds \right.$$

$$+ \sum_{i=1}^{k} \xi_0 I_i\left(x(t_i^-)\right) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i-s)^{q-1} f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.$$

$$\left. \left. + \frac{(t-t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i-s)^{q-2} f ds + \frac{(t-t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i-s)^{q-3} f ds \right\} \right\}$$

$$= \left\{ f(t,x(t))|_{t\ge a} + \sum_{i=1}^{k} \frac{\xi_0 I_i(x(t_i^-))}{\Gamma(q)} \left[{_{t_i}D_t^q}\left(\int_{t_i}^t (t-s)^{q-1} f ds\right) - {_aD_t^q}\left(\int_a^t (t-s)^{q-1} f ds\right) \right] \right\}$$

$$= \left\{ f(t,x(t))|_{t\ge a} + \sum_{i=1}^{k} \xi_0 I_i(x(t_i^-)) \left[f(t,x(t))|_{t\ge t_i} - f(t,x(t))|_{t\ge a} \right] \right\}_{t\in(t_k,t_{k+1}]}$$

$$= f(t,x(t))_{t\in(t_k,t_{k+1}]}.$$

So, Eq. (11) satisfies the condition of fractional derivative in (4). Thus, Eq. (11) satisfies all conditions of system (4).

"Sufficiency", we will prove that the solutions of (4) satisfy Eq. (11) by using mathematical induction. For $t \in [a, t_1]$, it is certain that the solution of system (4) satisfies Eq. (11) by Lemma 2.3 and

$$x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds. \qquad (13)$$

Using (13), we have

$$x(t_1^+) = x(t_1^-) + I_1\left(x(t_1^-)\right) = x_a + \bar{x}_a(t_1-a) + \frac{\hat{x}_a}{2!}(t_1-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \int_a^{t_1} (t_1-s)^{q-1} f ds,$$

$$x'(t_1^+) = x'(t_1^-) = \bar{x}_a + \hat{x}_a(t_1-a) + \frac{1}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds,$$

$$x''(t_1^+) = x''(t_1^-) = \hat{x}_a + \frac{1}{\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds.$$

Therefore, the approximate solution $\tilde{x}(t)$ is given by

$$\tilde{x}(t) = x(t_1^+) + x'(t_1^+)(t-t_1) + \frac{x''(t_1^+)}{2!}(t-t_1)^2 + \frac{1}{\Gamma(q)} \int_{t_1}^t (t-s)^{q-1} f ds$$

$$= x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds \right] \quad (14)$$

$$+ \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds \quad for \ t \in (t_1, t_2].$$

Let $e_1(t) = x(t) - \tilde{x}(t)$, for $t \in (t_1, t_2]$. Due to

$$\lim_{I_1(x(t_1^-)) \to 0} x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds \quad for \ t \in (t_1, t_2],$$

we get

$$\lim_{I_1(x(t_1^-)) \to 0} e_1(t) = \lim_{I_1(x(t_1^-)) \to 0} \{x(t) - \tilde{x}(t)\}$$

$$= \frac{-1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \tag{15}$$

$$- \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds - \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds.$$

Then, by (15), we assume

$$e_1(t) = \sigma\left(I_1(x(t_1^-))\right) \left\{ \frac{-1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.$$

$$\left. - \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds - \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds \right\}. \tag{16}$$

where σ is an undetermined function with $\sigma(0) = 1$. Therefore

$$x(t) = \tilde{x}(t) + e_1(t)$$

$$= x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds$$

$$+ \left[1 - \sigma\left(I_1(x(t_1^-))\right)\right] \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.$$

$$\left. + \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds \right\} \quad for \ t \in (t_1, t_2].$$

Let $\theta(z) = 1 - \sigma(z)$ for $z \in \mathbb{R}$ in the above equation, then

$$x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds$$

$$+ \theta\left(I_1(x(t_1^-))\right) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right. \tag{17}$$

$$\left. + \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds \right\} \quad for \ t \in (t_1, t_2].$$

Using (17), we get

$$x(t_2^+) = x(t_2^-) + I_2\left(x(t_2^-)\right)$$

$$= x_a + \bar{x}_a(t_2 - a) + \frac{\hat{x}_a}{2!}(t_2 - a)^2 + \sum_{i=1,2} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)} \int\limits_a^{t_2} (t_2 - s)^{q-1} f ds$$

$$+ \theta\left(I_1(x(t_1^-))\right) \left\{ \frac{1}{\Gamma(q)} \left[\int\limits_a^{t_1} (t_1 - s)^{q-1} f ds + \int\limits_{t_1}^{t_2} (t_2 - s)^{q-1} f ds - \int\limits_a^{t_2} (t_2 - s)^{q-1} f ds \right] \right.$$

$$\left. + \frac{(t_2 - t_1)}{\Gamma(q-1)} \int\limits_a^{t_1} (t_1 - s)^{q-2} f ds + \frac{(t_2 - t_1)^2}{2!\Gamma(q-2)} \int\limits_a^{t_1} (t_1 - s)^{q-3} f ds \right\},$$

$$x'(t_2^+) = x'(t_2^-) = \bar{x}_a + \hat{x}_a(t_2 - a) + \frac{1}{\Gamma(q-1)} \int\limits_a^{t_2} (t_2 - s)^{q-2} f ds$$

$$+ \theta\left(I_1(x(t_1^-))\right) \left\{ \frac{1}{\Gamma(q-1)} \left[\int\limits_a^{t_1} (t_1 - s)^{q-2} f ds + \int\limits_{t_1}^{t_2} (t_2 - s)^{q-2} f ds - \int\limits_a^{t_2} (t_2 - s)^{q-2} f ds \right] \right.$$

$$\left. + \frac{(t_2 - t_1)}{\Gamma(q-2)} \int\limits_a^{t_1} (t_1 - s)^{q-3} f ds \right\},$$

$$x''(t_2^+) = x''(t_2^-) = \hat{x}_a + \frac{1}{\Gamma(q-2)} \int\limits_a^{t_2} (t_2 - s)^{q-3} f ds$$

$$+ \frac{\theta\left(I_1(x(t_1^-))\right)}{\Gamma(q-2)} \left[\int\limits_a^{t_1} (t_1 - s)^{q-3} f ds + \int\limits_{t_1}^{t_2} (t_2 - s)^{q-3} f ds - \int\limits_a^{t_2} (t_2 - s)^{q-3} f ds \right].$$

Thus,

$$\tilde{x}(t) = x(t_2^+) + x'(t_2^+)(t - t_2) + \frac{x''(t_2^+)}{2!}(t - t_2)^2 + \frac{1}{\Gamma(q)} \int\limits_{t_2}^{t} (t - s)^{q-1} f ds$$

$$= x_a + \bar{x}_a(t - a) + \frac{\hat{x}_a}{2!}(t - a)^2 + I_1\left(x(t_1^-)\right) + I_2\left(x(t_2^-)\right) + \frac{(t - t_2)}{\Gamma(q-1)} \int\limits_a^{t_2} (t_2 - s)^{q-2} f ds$$

$$+ \frac{(t - t_2)^2}{2!\Gamma(q-2)} \int\limits_a^{t_2} (t_2 - s)^{q-3} f ds + \frac{1}{\Gamma(q)} \left[\int\limits_a^{t_2} (t_2 - s)^{q-1} f ds + \int\limits_{t_2}^{t} (t - s)^{q-1} f ds \right]$$

$$+ \theta\left(I_1(x(t_1^-))\right) \left\{ \frac{1}{\Gamma(q)} \left[\int\limits_a^{t_1} (t_1 - s)^{q-1} f ds + \int\limits_{t_1}^{t_2} (t_2 - s)^{q-1} f ds - \int\limits_a^{t_2} (t_2 - s)^{q-1} f ds \right] \right.$$

$$+ \frac{(t - t_1)}{\Gamma(q-1)} \int\limits_a^{t_1} (t_1 - s)^{q-2} f ds + \frac{(t - t_1)^2}{2!\Gamma(q-2)} \int\limits_a^{t_1} (t_1 - s)^{q-3} f ds$$

$$\text{(18)}$$

$$+ \frac{(t - t_2)}{\Gamma(q-1)} \int\limits_{t_1}^{t_2} (t_2 - s)^{q-2} f ds + \frac{(t - t_2)^2}{2!\Gamma(q-2)} \int\limits_{t_1}^{t_2} (t_2 - s)^{q-3} f ds$$

$$\left. - \frac{(t - t_2)}{\Gamma(q-1)} \int\limits_a^{t_2} (t_2 - s)^{q-2} f ds - \frac{(t - t_2)^2}{2!\Gamma(q-2)} \int\limits_a^{t_2} (t_2 - s)^{q-3} f ds \right\} \quad \text{for } t \in (t_2, t_3].$$

Let $e_2(t) = x(t) - \tilde{x}(t)$, for $t \in (t_2, t_3]$. By (17), the exact solution $x(t)$ of system (4) satisfies

$$
\lim_{I_1(x(t_1^-)) \to 0, I_2(x(t_2^-)) \to 0} x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds \quad for\ t \in (t_2, t_3],
$$

$$
\lim_{I_1(x(t_1^-)) \to 0} x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_2\left(x(t_2^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds
$$

$$
+ \theta\left(I_2(x(t_2^-))\right) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_2} (t_2-s)^{q-1} f ds + \int_{t_2}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t-t_2)}{\Gamma(q-1)} \int_a^{t_2} (t_2-s)^{q-2} f ds + \frac{(t-t_2)^2}{2!\Gamma(q-2)} \int_a^{t_2} (t_2-s)^{q-3} f ds \right\} \quad for\ t \in (t_2, t_3],
$$

$$
\lim_{I_2(x(t_2^-)) \to 0} x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds
$$

$$
+ \theta\left(I_1(x(t_1^-))\right) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f ds + \int_{t_1}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f ds \right\} \quad for\ t \in (t_2, t_3].
$$

Thus,

$$
\lim_{\substack{I_1(x(t_1^-)) \to 0, \\ I_2(x(t_2^-)) \to 0}} e_2(t) = \lim_{\substack{I_1(x(t_1^-)) \to 0, \\ I_2(x(t_2^-)) \to 0}} \{x(t) - \tilde{x}(t)\}
$$

$$
= \frac{-1}{\Gamma(q)} \left[\int_a^{t_2} (t_2-s)^{q-1} f ds + \int_{t_2}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \tag{19}
$$

$$
- \frac{(t-t_2)}{\Gamma(q-1)} \int_a^{t_2} (t_2-s)^{q-2} f ds - \frac{(t-t_2)^2}{2!\Gamma(q-2)} \int_a^{t_2} (t_2-s)^{q-3} f ds \quad for\ t \in (t_2, t_3],
$$

$$
\lim_{I_1(x(t_1^-)) \to 0} e_2(t) = \lim_{I_1(x(t_1^-)) \to 0} \{x(t) - \tilde{x}(t)\}
$$

$$
= \left[-1 + \theta\left(I_2(x(t_2^-))\right)\right] \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_2} (t_2-s)^{q-1} f ds + \int_{t_2}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right. \tag{20}
$$

$$
\left. + \frac{(t-t_2)}{\Gamma(q-1)} \int_a^{t_2} (t_2-s)^{q-2} f ds + \frac{(t-t_2)^2}{2!\Gamma(q-2)} \int_a^{t_2} (t_2-s)^{q-3} f ds \right\} \quad for\ t \in (t_2, t_3],
$$

$$
\lim_{I_2(x(t_2^-)) \to 0} e_2(t) = \lim_{I_2(x(t_2^-)) \to 0} \{x(t) - \tilde{x}(t)\}
$$

$$
= \left[-1 + \theta\left(I_1(x(t_1^-))\right)\right] \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_2} (t_2-s)^{q-1} f ds + \int_{t_2}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t-t_2)}{\Gamma(q-1)} \int_a^{t_2} (t_2-s)^{q-2} f ds + \frac{(t-t_2)^2}{2!\Gamma(q-2)} \int_a^{t_2} (t_2-s)^{q-3} f ds \right\}
$$

$$
-\theta\left(I_1(x(t_1^-))\right)\left\{\frac{1}{\Gamma(q)}\left[\int_{t_1}^{t_2}(t_2-s)^{q-1}f\,ds+\int_{t_2}^{t}(t-s)^{q-1}f\,ds-\int_{t_1}^{t}(t-s)^{q-1}f\,ds\right]\right.
$$

$$
\left.+\frac{(t-t_2)}{\Gamma(q-1)}\int_{t_1}^{t_2}(t_2-s)^{q-2}f(s,x(s))ds+\frac{(t-t_2)^2}{2!\Gamma(q-2)}\int_{t_1}^{t_2}(t_2-s)^{q-3}f(s,x(s))ds\right\} \tag{21}
$$

$$
for\ t\in(t_2,t_3].
$$

By (19)-(21), we obtain

$$
e_2(t)=\left[-1+\theta\left(I_1(x(t_1^-))\right)+\theta\left(I_2(x(t_2^-))\right)\right]\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}(t_2-s)^{q-1}f\,ds+\int_{t_2}^{t}(t-s)^{q-1}f\,ds\right.\right.
$$

$$
\left.-\int_a^{t}(t-s)^{q-1}f\,ds\right]+\frac{(t-t_2)}{\Gamma(q-1)}\int_a^{t_2}(t_2-s)^{q-2}f\,ds+\frac{(t-t_2)^2}{2!\Gamma(q-2)}\int_a^{t_2}(t_2-s)^{q-3}f\,ds\right\}
$$

$$
-\theta\left(I_1(x(t_1^-))\right)\left\{\frac{1}{\Gamma(q)}\left[\int_{t_1}^{t_2}(t_2-s)^{q-1}f\,ds+\int_{t_2}^{t}(t-s)^{q-1}f\,ds-\int_{t_1}^{t}(t-s)^{q-1}f\,ds\right]\right.
$$

$$
\left.+\frac{(t-t_2)}{\Gamma(q-1)}\int_{t_1}^{t_2}(t_2-s)^{q-2}f\,ds+\frac{(t-t_2)^2}{2!\Gamma(q-2)}\int_{t_1}^{t_2}(t_2-s)^{q-3}f\,ds\right\}\quad for\ t\in(t_2,t_3].
$$

Thus,

$$
x(t)=\tilde{x}(t)+e_2(t)
$$

$$
=x_a+\bar{x}_a(t-a)+\frac{\hat{x}_a}{2!}(t-a)^2+I_1\left(x(t_1^-)\right)+I_2\left(x(t_2^-)\right)+\frac{1}{\Gamma(q)}\int_a^{t}(t-s)^{q-1}f\,ds
$$

$$
+\theta\left(I_1(x(t_1^-))\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_1}(t_1-s)^{q-1}f\,ds+\int_{t_1}^{t}(t-s)^{q-1}f\,ds-\int_a^{t}(t-s)^{q-1}f\,ds\right]\right.
$$

$$
\left.+\frac{(t-t_1)}{\Gamma(q-1)}\int_a^{t_1}(t_1-s)^{q-2}f\,ds+\frac{(t-t_1)^2}{2!\Gamma(q-2)}\int_a^{t_1}(t_1-s)^{q-3}f\,ds\right\} \tag{22}
$$

$$
+\theta\left(I_2(x(t_2^-))\right)\left\{\frac{1}{\Gamma(q)}\left[\int_a^{t_2}(t_2-s)^{q-1}f\,ds+\int_{t_2}^{t}(t-s)^{q-1}f\,ds-\int_a^{t}(t-s)^{q-1}f\,ds\right]\right.
$$

$$
\left.+\frac{(t-t_2)}{\Gamma(q-1)}\int_a^{t_2}(t_2-s)^{q-2}f\,ds+\frac{(t-t_2)^2}{2!\Gamma(q-2)}\int_a^{t_2}(t_2-s)^{q-3}f\,ds\right\}\quad for\ t\in(t_2,t_3].
$$

Letting $t_2\to t_1$, we have

$$
\lim_{t_2\to t_1}\begin{cases}{}_aD_t^q x(t)=f(t,x(t)),\ q\in(2,3),\\\quad t\in J=[a,t_3],\ and\ t\neq t_1,t_2,\\\Delta x|_{t=t_k}=I_k\left(x(t_k^-)\right),\quad k=1,2,\\x(a)=x_a,\ x'(a)=\bar{x}_a,\ x''(a)=\hat{x}_a.\end{cases}\to\begin{cases}{}_aD_t^q x(t)=f(t,x(t)),\ q\in(2,3),\\\quad t\in J=[a,t_3],\ and\ t\neq t_1,\\\Delta x|_{t=t_1}=I_1\left(x(t_1^-)\right)+I_2\left(x(t_2^-)\right),\\x(a)=x_a,\ x'(a)=\bar{x}_a,\ x''(a)=\hat{x}_a.\end{cases} \tag{23}
$$

Using (17) and (22) for (23), we obtain

$$
\theta\left(I_1(x(t_1^-))+I_2(x(t_2^-))\right)=\theta\left(I_1(x(t_1^-))\right)+\theta\left(I_2(x(t_2^-))\right)\ for\ \forall I_1(x(t_1^-)),I_2(x(t_2^-))\in\mathbb{R}. \tag{24}
$$

Therefore $\theta(z) = \xi_0 z$ for $\forall z \in \mathbb{R}$ (where ξ_0 is a constant). Thus

$$
\begin{aligned}
x(t) = {} & x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f\,ds \\
& + \xi_0 I_1(x(t_1^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f\,ds + \int_{t_1}^t (t-s)^{q-1} f\,ds - \int_a^t (t-s)^{q-1} f\,ds \right] \right. \\
& \left. + \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f\,ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f\,ds \right\} \quad \text{for } t \in (t_1, t_2].
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
x(t) = {} & x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + I_1\left(x(t_1^-)\right) + I_2\left(x(t_2^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f\,ds \\
& + \xi_0 I_1(x(t_1^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_1} (t_1-s)^{q-1} f\,ds + \int_{t_1}^t (t-s)^{q-1} f\,ds - \int_a^t (t-s)^{q-1} f\,ds \right] \right. \\
& \left. + \frac{(t-t_1)}{\Gamma(q-1)} \int_a^{t_1} (t_1-s)^{q-2} f\,ds + \frac{(t-t_1)^2}{2!\Gamma(q-2)} \int_a^{t_1} (t_1-s)^{q-3} f\,ds \right\} \\
& + \xi_0 I_2(x(t_2^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_2} (t_2-s)^{q-1} f\,ds + \int_{t_2}^t (t-s)^{q-1} f\,ds - \int_a^t (t-s)^{q-1} f\,ds \right] \right. \\
& \left. + \frac{(t-t_2)}{\Gamma(q-1)} \int_a^{t_2} (t_2-s)^{q-2} f\,ds + \frac{(t-t_2)^2}{2!\Gamma(q-2)} \int_a^{t_2} (t_2-s)^{q-3} f\,ds \right\} \quad \text{for } t \in (t_2, t_3].
\end{aligned}
\tag{26}
$$

Next, suppose

$$
\begin{aligned}
x(t) = {} & x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^{k} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f\,ds \\
& + \sum_{i=1}^{k} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i-s)^{q-1} f\,ds + \int_{t_i}^t (t-s)^{q-1} f\,ds - \int_a^t (t-s)^{q-1} f\,ds \right] \right. \\
& \left. + \frac{(t-t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i-s)^{q-2} f\,ds + \frac{(t-t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i-s)^{q-3} f\,ds \right\} \quad \text{for } t \in (t_k, t_{k+1}].
\end{aligned}
\tag{27}
$$

Using (27), we get

$$
\begin{aligned}
x(t_{k+1}^+) = {} & x(t_{k+1}^-) + I_{k+1}\left(x(t_{k+1}^-)\right) \\
= {} & x_a + \bar{x}_a(t_{k+1}-a) + \frac{\hat{x}_a}{2!}(t_{k+1}-a)^2 + \sum_{i=1}^{k+1} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)} \int_a^{t_{k+1}} (t_{k+1}-s)^{q-1} f\,ds \\
& + \sum_{i=1}^{k} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i-s)^{q-1} f\,ds + \int_{t_i}^{t_{k+1}} (t_{k+1}-s)^{q-1} f\,ds - \int_a^{t_{k+1}} (t_{k+1}-s)^{q-1} f\,ds \right] \right. \\
& \left. + \frac{(t_{k+1}-t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i-s)^{q-2} f\,ds + \frac{(t_{k+1}-t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i-s)^{q-3} f\,ds \right\},
\end{aligned}
$$

$$x'(t_{k+1}^+) = x'(t_{k+1}^-)$$

$$= \bar{x}_a + \hat{x}_a(t_{k+1} - a) + \frac{1}{\Gamma(q-1)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds$$

$$+ \sum_{i=1}^{k} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q-1)} \left[\int_a^{t_i} (t_i - s)^{q-2} f ds + \int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds \right] \right.$$

$$\left. + \frac{(t_{k+1} - t_i)}{\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \right\},$$

$$x''(t_{k+1}^+) = x''(t_{k+1}^-) = \hat{x}_a + \frac{1}{\Gamma(q-2)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds$$

$$+ \sum_{i=1}^{k} \frac{\xi_0 I_i(x(t_i^-))}{\Gamma(q-2)} \left[\int_a^{t_i} (t_i - s)^{q-3} f ds + \int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds \right].$$

Thus,

$$\tilde{x}(t) = x(t_{k+1}^+) + x'(t_{k+1}^+)(t - t_{k+1}) + \frac{x''(t_{k+1}^+)}{2!}(t - t_{k+1})^2 + \frac{1}{\Gamma(q)} \int_{t_{k+1}}^{t} (t - s)^{q-1} f ds$$

$$= x_a + \bar{x}_a(t - a) + \frac{\hat{x}_a}{2!}(t - a)^2 + \sum_{i=1}^{k+1} I_i\left(x(t_i^-)\right) + \frac{(t - t_{k+1})}{\Gamma(q-1)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds$$

$$+ \frac{(t - t_{k+1})^2}{2! \Gamma(q-2)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds + \frac{1}{\Gamma(q)} \left[\int_a^{t_{k+1}} (t_{k+1} - s)^{q-1} f ds + \int_{t_{k+1}}^{t} (t - s)^{q-1} f ds \right]$$

$$+ \sum_{i=1}^{k} \xi_0 I_i(y(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-1} f ds \right. \right. \tag{28}$$

$$\left. - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-1} f ds \right] + \frac{(t - t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds + \frac{(t - t_i)^2}{2! \Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds$$

$$+ \frac{(t - t_{k+1})}{\Gamma(q-1)} \left[\int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds \right]$$

$$+ \frac{(t - t_{k+1})^2}{2! \Gamma(q-2)} \left[\int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds \right] \right\} \quad for \ t \in (t_{k+1}, t_{k+2}].$$

Let $e_{k+1}(t) = x(t) - \tilde{x}(t)$, for $t \in (t_{k+1}, t_{k+2}]$. By (27), the exact solution $x(t)$ of system (4) satisfies

$$\lim_{\substack{I_i(x(t_i^-)) \to 0, \\ for \ all \ i \in \{1, \dots, k+1\}}} x(t) = x_a + \bar{x}_a(t - a) + \frac{\hat{x}_a}{2!}(t - a)^2 + \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} f ds \quad for \ t \in (t_{k+1}, t_{k+2}],$$

$$\lim_{\substack{I_p(y(t_p^-))\to 0,\\ for\ \forall p\in\{1,2,...,k+1\}}} x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{\substack{1\leq i\leq k+1\\ and\ i\neq p}} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)}\int_a^t (t-s)^{q-1} f ds$$

$$+ \sum_{\substack{1\leq i\leq k+1\\ and\ i\neq p}} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)}\left[\int_a^{t_i}(t_i-s)^{q-1}f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds\right]\right.$$

$$\left.+ \frac{(t-t_i)}{\Gamma(q-1)}\int_a^{t_i}(t_i-s)^{q-2}f ds + \frac{(t-t_i)^2}{2!\Gamma(q-2)}\int_a^{t_i}(t_i-s)^{q-3}f ds\right\}\quad for\ t\in(t_{k+1},t_{k+2}].$$

Then,

$$\lim_{\substack{I_i(x(t_i^-))\to 0,\\ for\ all\ i\in\{1,...,k+1\}}} e_{k+1}(t) = \lim_{\substack{I_i(x(t_i^-))\to 0,\\ for\ all\ i\in\{1,...,k+1\}}} \{x(t)-\hat{x}(t)\}$$

$$= \frac{-1}{\Gamma(q)}\left[\int_a^{t_{k+1}}(t_{k+1}-s)^{q-1}f ds + \int_{t_{k+1}}^t (t-s)^{q-1}f ds - \int_a^t (t-s)^{q-1}f ds\right] \tag{29}$$

$$- \frac{(t-t_{k+1})}{\Gamma(q-1)}\int_a^{t_{k+1}}(t_{k+1}-s)^{q-2}f ds - \frac{(t-t_{k+1})^2}{2!\Gamma(q-2)}\int_a^{t_{k+1}}(t_{k+1}-s)^{q-3}f ds,$$

$$\lim_{\substack{I_p(y(t_p^-))\to 0,\\ for\ \forall p\in\{1,2,...,k+1\}}} e_{k+1}(t) = \lim_{\substack{I_p(y(t_p^-))\to 0,\\ for\ \forall p\in\{1,2,...,k+1\}}} \{x(t)-\hat{x}(t)\}$$

$$= \frac{-1}{\Gamma(q)}\left[\int_a^{t_{k+1}}(t_{k+1}-s)^{q-1}f ds + \int_{t_{k+1}}^t (t-s)^{q-1}f ds - \int_a^t (t-s)^{q-1}f ds\right]$$

$$+ \sum_{\substack{1\leq i\leq k+1\\ and\ i\neq p}} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)}\left[\int_a^{t_{k+1}}(t_{k+1}-s)^{q-1}f ds + \int_{t_{k+1}}^t (t-s)^{q-1}f ds\right.\right.$$

$$\left.- \int_a^t (t-s)^{q-1}f ds - \int_{t_i}^{t_{k+1}}(t_{k+1}-s)^{q-1}f ds - \int_{t_{k+1}}^t (t-s)^{q-1}f ds + \int_{t_i}^t (t-s)^{q-1}f ds\right]$$

$$- \frac{(t-t_{k+1})}{\Gamma(q-1)}\left[\int_{t_i}^{t_{k+1}}(t_{k+1}-s)^{q-2}f ds - \int_a^{t_{k+1}}(t_{k+1}-s)^{q-2}f ds\right]$$

$$\left.- \frac{(t-t_{k+1})^2}{2!\Gamma(q-2)}\left[\int_{t_i}^{t_{k+1}}(t_{k+1}-s)^{q-3}f ds - \int_a^{t_{k+1}}(t_{k+1}-s)^{q-3}f ds\right]\right\} \tag{30}$$

$$- \frac{(t-t_{k+1})}{\Gamma(q-1)}\int_a^{t_{k+1}}(t_{k+1}-s)^{q-2}f ds - \frac{(t-t_{k+1})^2}{2!\Gamma(q-2)}\int_a^{t_{k+1}}(t_{k+1}-s)^{q-3}f ds.$$

By (29) and (30), we obtain

$$e_{k+1}(t)$$

$$= \frac{-1 + \sum_{i=1}^{k+1} \xi_0 I_i(x(t_i^-))}{\Gamma(q)} \left[\int_a^{t_{k+1}} (t_{k+1} - s)^{q-1} f ds + \int_{t_{k+1}}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right]$$

$$- \sum_{i=1}^{k+1} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-1} f ds + \int_{t_{k+1}}^t (t-s)^{q-1} f ds - \int_{t_i}^t (t-s)^{q-1} f ds \right] \right.$$

$$+ \frac{(t - t_{k+1})}{\Gamma(q-1)} \left[\int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds \right] \tag{31}$$

$$+ \left. \frac{(t - t_{k+1})^2}{2!\Gamma(q-2)} \left[\int_{t_i}^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds - \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds \right] \right\}$$

$$- \frac{(t - t_{k+1})}{\Gamma(q-1)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-2} f ds - \frac{(t - t_{k+1})^2}{2!\Gamma(q-2)} \int_a^{t_{k+1}} (t_{k+1} - s)^{q-3} f ds.$$

Thus,

$$x(t) = \tilde{x}(t) + e_{k+1}(t)$$

$$= x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^{k+1} I_i\left(x(t_i^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds$$

$$+ \sum_{i=1}^{k+1} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.$$

$$+ \left. \frac{(t - t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds + \frac{(t - t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \right\} \quad for \ t \in (t_{k+1}, t_{k+2}].$$

Therefore, the solutions of system (4) satisfy Eq. (11). So, system (4) is equivalent to Eq. (11). The proof is now completed. □

With similarity to Lemma 3.1, the following two conclusions can be proved.

Lemma 3.2. *Let* $q \in (2, 3)$ *and* ξ_1 *is a constant. System (5) is equivalent to the fractional integral equation*

$$x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{j=1}^{k} \bar{I}_j\left(x(\bar{t}_j^-)\right)(t-\bar{t}_j) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds$$

$$+ \sum_{j=1}^{k} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-1} f ds + \int_{\bar{t}_j}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right. \tag{32}$$

$$+ \left. \frac{(t - \bar{t}_j)}{\Gamma(q-1)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-2} f ds + \frac{(t - \bar{t}_j)^2}{2!\Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \right\}$$

$$for \ t \in (\bar{t}_k, \bar{t}_{k+1}], \ k = 0, 1, 2, ..., M.$$

provided that the integral in (32) exists.

Lemma 3.3. *Let $q \in (2, 3)$ and ξ_2 is a constant. System (6) is equivalent to the fractional integral equation*

$$
x(t) = x_a + \bar{x}_a(t - a) + \frac{\hat{x}_a}{2!}(t - a)^2 + \frac{1}{2!} \sum_{l=1}^{k} \hat{I}_l\left(x(\hat{t}_l^-)\right)(t - \hat{t}_l)^2 + \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} f ds
$$

$$
+ \sum_{l=1}^{k} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-1} f ds + \int_{\hat{t}_l}^t (t - s)^{q-1} f ds - \int_a^t (t - s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t - \hat{t}_l)}{\Gamma(q - 1)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-2} f ds + \frac{(t - \hat{t}_l)^2}{2!\Gamma(q - 2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \right\}
$$

$$
for \; t \in (\hat{t}_k, \hat{t}_{k+1}], \; k = 0, 1, 2, ..., N.
$$

\(33\)

provided that the integral in (33) exists.

Corollary 3.4. *Let $q \in (2, 3)$ and ξ_0 is a constant. If a function x is the general solution of system (4) then*

$$
x''(t) = \hat{x}_a + \frac{1}{\Gamma(q - 2)} \int_a^t (t - s)^{q-3} f(s, x(s)) ds
$$

$$
+ \xi_0 \sum_{i=1}^{k} \frac{I_i(x(t_i^-))}{\Gamma(q - 2)} \left[\int_a^{t_i} (t_i - s)^{q-3} f ds + \int_{t_i}^t (t - s)^{q-3} f ds - \int_a^t (t - s)^{q-3} f ds \right]
$$

$$
for \; t \in (t_k, t_{k+1}], \; k = 0, 1, 2, ..., L.
$$

Corollary 3.5. *Let $q \in (2, 3)$ and ξ_1 is a constant. If a function x is the general solution of system (5) then*

$$
x''(t) = \hat{x}_a + \frac{1}{\Gamma(q - 2)} \int_a^t (t - s)^{q-3} f ds
$$

$$
+ \xi_1 \sum_{j=1}^{k} \frac{\bar{I}_j(x(\bar{t}_j^-))}{\Gamma(q - 2)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds + \int_{\bar{t}_j}^t (t - s)^{q-3} f ds - \int_a^t (t - s)^{q-3} f ds \right]
$$

$$
for \; t \in (\bar{t}_k, \bar{t}_{k+1}], \; k = 0, 1, 2, ..., M.
$$

Corollary 3.6. *Let $q \in (2, 3)$ and ξ_2 is a constant. If a function x is the general solution of system (6) then*

$$
x''(t) = \hat{x}_a + \sum_{l=1}^{k} \hat{I}_l\left(x(\hat{t}_l^-)\right) + \frac{1}{\Gamma(q - 2)} \int_a^t (t - s)^{q-3} f ds
$$

$$
+ \xi_2 \sum_{l=1}^{k} \frac{\hat{I}_l(x(\hat{t}_l^-))}{\Gamma(q - 2)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds + \int_{\hat{t}_l}^t (t - s)^{q-3} f ds - \int_a^t (t - s)^{q-3} f ds \right]
$$

$$
for \; t \in (\hat{t}_k, \hat{t}_{k+1}], \; k = 0, 1, 2, ..., N.
$$

Remark 3.7. *Impulses $\Delta x|_{t=t_i}$, $\Delta x'|_{t=\bar{t}_j}$ and $\Delta x''|_{t=\hat{t}_l}$ have similar effect on $x''(t)$ of (3) by Corollaries 3.4-3.6. Thus, we will consider $\Delta x|_{t=t_i}$ and $\Delta x'|_{t=\bar{t}_j}$ as some special impulses $\Delta x''|_{t=\hat{t}_l}$ for system (3).*

Lemma 3.8. *Let $q \in (2, 3)$ and ξ_b (where $b \in \{0, 1, 2\}$) are three constants. If a function x is the general solution of system (1) then*

$$
x''(t) = \hat{x}_a + \sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right) + \frac{1}{\Gamma(q-2)} \int_a^t (t-s)^{q-3} f ds
$$

$$
+ \xi_0 \sum_{i=1}^{k_0} \frac{I_i(x(t_i^-))}{\Gamma(q-2)} \left[\int_a^{t_i} (t_i - s)^{q-3} f ds + \int_{t_i}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right]
$$

$$
+ \xi_1 \sum_{j=1}^{k_1} \frac{\bar{I}_j(x(\bar{t}_j^-))}{\Gamma(q-2)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds + \int_{\bar{t}_j}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right] \tag{34}
$$

$$
+ \xi_2 \sum_{l=1}^{k_2} \frac{\hat{I}_l(x(\hat{t}_l^-))}{\Gamma(q-2)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds + \int_{\hat{t}_l}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right]
$$

$$
for \ t \in J_k', \ k = 0, 1, 2, ..., K.
$$

Proof. By Definition 2.2, we have

$$
\{\text{system (1)}\} \Leftrightarrow
\begin{cases}
{}_a D_t^{q-2}\left(x''(t)\right) = f(t, x(t)), \ q \in (2, 3), \ t \in J = [a, T], \ t \neq t_i \, (i = 1, 2, ..., L) \\
\qquad\qquad\quad and \ t \neq \bar{t}_j \, (j = 1, 2, ..., M) \ and \ t \neq \hat{t}_l \, (l = 1, 2, ..., N), \\
\Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), \ i = 1, 2, ..., L, \\
\Delta x'|_{t=\bar{t}_j} = \bar{I}_j\left(x(\bar{t}_j^-)\right), \ j = 1, 2, ..., M, \\
\Delta x''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right), \ l = 1, 2, ..., N, \\
x(a) = x_a, \ x'(a) = \bar{x}_a, \ x''(a) = \hat{x}_a.
\end{cases}
$$

Moreover, $\Delta x|_{t=t_i}$ and $\Delta x'|_{t=\bar{t}_j}$ are considered as impulse $\Delta x''|_{t=\hat{t}_l}$ by Remark 3.7. Thus, using Lemma 2.4 and Corollaries 3.4-3.6, Eq. (34) can be obtained. □

Theorem 3.9. *Let $q \in (2, 3)$ and ξ_b (where $b \in \{0, 1, 2\}$) are three constants. System (1) is equivalent to the fractional integral equation*

$$
x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^{k_0} I_i\left(x(t_i^-)\right) + \sum_{j=1}^{k_1} \bar{I}_j\left(x(\bar{t}_j^-)\right)(t-\bar{t}_j)
$$

$$
+ \frac{1}{2!} \sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right)(t-\hat{t}_l)^2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds
$$

$$
+ \sum_{i=1}^{k_0} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t-t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds + \frac{(t-t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \right\}
$$

$$
+ \sum_{j=1}^{k_1} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-1} f ds + \int_{\bar{t}_j}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t-\bar{t}_j)}{\Gamma(q-1)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-2} f ds + \frac{(t-\bar{t}_j)^2}{2!\Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \right\}
$$

$$
+ \sum_{l=1}^{k_2} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-1} f ds + \int_{\hat{t}_l}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] \right.
$$

$$
\left. + \frac{(t - \hat{t}_l)}{\Gamma(q-1)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-2} f ds + \frac{(t - \hat{t}_l)^2}{2! \Gamma(q-2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \right\} \quad \text{for } t \in J_k',
$$

(35)

here $k \in \{0, 1, 2, ..., K\}$, *provided that the integral in* (35) *exists.*

Proof. For $t \in J_0'$, it is clear that system (1) is equivalent to

$$
x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds \quad \text{for } t \in J_0'.
$$

For $t \in J_k' (1 \le k \le K)$, by Lemma 3.8, we have

$$
x''(t) = \hat{x}_a + \sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right) + \frac{1}{\Gamma(q-2)} \int_a^t (t-s)^{q-3} f ds
$$

$$
+ \xi_0 \sum_{i=1}^{k_0} \frac{I_i(x(t_i^-))}{\Gamma(q-2)} \left[\int_a^{t_i} (t_i - s)^{q-3} f ds + \int_{t_i}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right]
$$

$$
+ \xi_1 \sum_{j=1}^{k_1} \frac{\bar{I}_j(x(\bar{t}_j^-))}{\Gamma(q-2)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds + \int_{\bar{t}_j}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right]
$$

$$
+ \xi_2 \sum_{l=1}^{k_2} \frac{\hat{I}_l(x(\hat{t}_l^-))}{\Gamma(q-2)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds + \int_{\hat{t}_l}^t (t-s)^{q-3} f ds - \int_a^t (t-s)^{q-3} f ds \right].
$$

Integrating both sides of the above equation twice, we get

$$
x(t) = C_0 + C_1 t + \frac{\hat{x}_a}{2!} t^2 + \frac{1}{2!} t^2 \sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds
$$

$$
+ \sum_{i=1}^{k_0} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_{t_i}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] + \frac{t^2}{2! \Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \right\}
$$

$$
+ \sum_{j=1}^{k_1} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_{\bar{t}_j}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] + \frac{t^2}{2! \Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \right\}
$$

$$
+ \sum_{l=1}^{k_2} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_{\hat{t}_l}^t (t-s)^{q-1} f ds - \int_a^t (t-s)^{q-1} f ds \right] + \frac{t^2}{2! \Gamma(q-2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \right\},
$$

here C, C_1 are two constants. Supposing $\bar{I}_j\left(x(t_j^-)\right) = 0$, $\hat{I}_l\left(x(\hat{t}_l^-)\right) = 0$; $I_i\left(x(t_i^-)\right) = 0$, $\hat{I}_l\left(x(\hat{t}_l^-)\right) = 0$ and $I_i\left(x(t_i^-)\right) = 0$, $\bar{I}_j\left(x(t_j^-)\right) = 0$ (here $i = 1, 2, ..., k_0$, $j = 1, 2, ..., k_1$, $l = 1, 2, ..., k_2$) respectively, by Lemmas 3.1-3.3, we obtain

$$C_0 = x_a - \bar{x}_a a + \frac{\hat{x}_a}{2!}a^2 + \sum_{i=1}^{k_0} I_i\left(x(t_i^-)\right) - \sum_{j=1}^{k_1} \bar{I}_j\left(x(\bar{t}_j^-)\right)\bar{t}_j + \frac{1}{2!}\sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right)\hat{t}_l^2$$

$$+ \sum_{i=1}^{k_0} \xi_0 I_i(x(t_i^-)) \left[\frac{1}{\Gamma(q)} \int_a^{t_i} (t_i - s)^{q-1} f ds - \frac{t_i}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds \right.$$

$$+ \frac{t_i^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \Bigg] + \sum_{j=1}^{k_1} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left[\frac{1}{\Gamma(q)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-1} f ds \right.$$

$$- \frac{\bar{t}_j}{\Gamma(q-1)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-2} f ds + \frac{\bar{t}_j^2}{2!\Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \Bigg]$$

$$+ \sum_{l=1}^{k_2} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left[\frac{1}{\Gamma(q)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-1} f ds - \frac{\hat{t}_l}{\Gamma(q-1)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-2} f ds \right.$$

$$+ \frac{\hat{t}_l^2}{2!\Gamma(q-2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \Bigg],$$

and

$$C_1 = \bar{x}_a - a\hat{x}_a + \sum_{j=1}^{k_1} \bar{I}_j\left(x(\bar{t}_j^-)\right) - \hat{t}_l \sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right)$$

$$+ \sum_{i=1}^{k_0} \xi_0 I_i(x(t_i^-)) \left[\frac{1}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds - \frac{t_i}{\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \right]$$

$$+ \sum_{j=1}^{k_1} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left[\frac{1}{\Gamma(q-1)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-2} f ds - \frac{\bar{t}_j}{\Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \right]$$

$$+ \sum_{l=1}^{k_2} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left[\frac{1}{\Gamma(q-1)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-2} f ds - \frac{\hat{t}_l}{\Gamma(q-2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \right].$$

Thus, for $t \in J_k'$ (here k = 1, 2,..., K), we get

$$x(t) = x_a + \bar{x}_a(t - a) + \frac{\hat{x}_a}{2!}(t - a)^2 + \sum_{i=1}^{k_0} I_i\left(x(t_i^-)\right) + \sum_{j=1}^{k_1} \bar{I}_j\left(x(\bar{t}_j^-)\right)(t - \bar{t}_j)$$

$$+ \frac{1}{2!}\sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right)(t - \hat{t}_l)^2 + \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} f ds$$

$$+ \sum_{i=1}^{k_0} \xi_0 I_i(x(t_i^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^t (t - s)^{q-1} f ds - \int_a^t (t - s)^{q-1} f ds \right] \right.$$

$$+ \frac{(t - t_i)}{\Gamma(q-1)} \int_a^{t_i} (t_i - s)^{q-2} f ds + \frac{(t - t_i)^2}{2!\Gamma(q-2)} \int_a^{t_i} (t_i - s)^{q-3} f ds \Bigg\}$$

$$+ \sum_{j=1}^{k_1} \xi_1 \bar{I}_j(x(\bar{t}_j^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-1} f ds + \int_{\bar{t}_j}^t (t - s)^{q-1} f ds - \int_a^t (t - s)^{q-1} f ds \right] \right.$$

$$
+ \frac{(t - \bar{t}_j)}{\Gamma(q-1)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-2} f ds + \frac{(t - \bar{t}_j)^2}{2!\Gamma(q-2)} \int_a^{\bar{t}_j} (\bar{t}_j - s)^{q-3} f ds \Bigg\}
$$

$$
+ \sum_{l=1}^{k_2} \xi_2 \hat{I}_l(x(\hat{t}_l^-)) \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-1} f ds + \int_{\hat{t}_l}^{t} (t-s)^{q-1} f ds - \int_a^{t} (t-s)^{q-1} f ds \right] \right.
$$

$$
+ \frac{(t - \hat{t}_l)}{\Gamma(q-1)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-2} f ds + \frac{(t - \hat{t}_l)^2}{2!\Gamma(q-2)} \int_a^{\hat{t}_l} (\hat{t}_l - s)^{q-3} f ds \Bigg\}.
$$

So, the solution of system (1) satisfies Eq. (35).

Next, we can verify that Eq. (35) satisfies all conditions (including conditions (i)-(vii)) in system (1). So, system (1) is equivalent to Eq. (35). The proof is completed. □

Remark 3.10. *For impulsive system (1), we have*

$$
\lim_{q \to 3^-} \{\text{impulsive system (1)}\}
$$

$$
\rightarrow \begin{cases}
x^{(3)}(t) = f(t, x(t)), & t \in J = [a, T],\ t \neq t_i\ (i = 1, 2, ..., L)\ and \\
\qquad\qquad t \neq \bar{t}_j\ (j = 1, 2, ..., M)\ and\ t \neq \hat{t}_l\ (l = 1, 2, ..., N), \\
\Delta x|_{t=t_i} = I_i\left(x(t_i^-)\right), & i = 1, 2, ..., L, \\
\Delta x'|_{t=\bar{t}_j} = \bar{I}_j\left(x(\bar{t}_j^-)\right), & j = 1, 2, ..., M, \\
\Delta x''|_{t=\hat{t}_l} = \hat{I}_l\left(x(\hat{t}_l^-)\right), & l = 1, 2, ..., N, \\
x(a) = x_a,\ x'(a) = \bar{x}_a,\ x''(a) = \hat{x}_a.
\end{cases}
\tag{36}
$$

On the other hand, using (35), we get

$$
\lim_{q \to 3^-} x(t) = \begin{cases}
x_a + \bar{x}_a(t-a) + \dfrac{\hat{x}_a}{2!}(t-a)^2 + \dfrac{1}{2}\displaystyle\int_a^t (t-s)^2 f ds, & for\ t \in J_0', \\[2mm]
x_a + \bar{x}_a(t-a) + \dfrac{\hat{x}_a}{2!}(t-a)^2 + \displaystyle\sum_{i=1}^{k_0} I_i\left(x(t_i^-)\right) + \sum_{j=1}^{k_1} \bar{I}_j\left(x(\bar{t}_j^-)\right)(t-\bar{t}_j) \\[2mm]
\quad + \dfrac{1}{2!}\displaystyle\sum_{l=1}^{k_2} \hat{I}_l\left(x(\hat{t}_l^-)\right)(t-\hat{t}_l)^2 + \dfrac{1}{2}\int_a^t (t-s)^2 f ds & for\ t \in J_k',\ k = 1, 2, ..., K.
\end{cases}
\tag{37}
$$

Moreover, we can verify that Eq. (37) is the solution of (36), and indirectly supports our results.

Corollary 3.11. *Let $q \in (2, 3)$ and ξ_b (where $b \in \{0, 1, 2\}$) are three constants. System (2) is equivalent to the fractional integral equation*

$$
x(t) = x_a + \bar{x}_a(t-a) + \frac{\hat{x}_a}{2!}(t-a)^2 + \sum_{i=1}^{k} \left[I_i\left(x(t_i^-)\right) + \bar{I}_i\left(x(\bar{t}_i^-)\right)(t-t_i) + \frac{1}{2!}\hat{I}_i\left(x(t_i^-)\right)(t-t_i)^2 \right]
$$

$$
+ \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f ds + \sum_{i=1}^{k} \left[\xi_0 I_i(x(t_i^-)) + \xi_1 \bar{I}_i(x(t_i^-)) + \xi_2 \hat{I}_i(x(t_i^-)) \right]
\tag{38}
$$

$$
\times \left\{ \frac{1}{\Gamma(q)} \left[\int_a^{t_i} (t_i - s)^{q-1} f ds + \int_{t_i}^{t} (t-s)^{q-1} f ds - \int_a^{t} (t-s)^{q-1} f ds \right] \right.
$$

$$+\frac{(t-t_i)}{\Gamma(q-1)}\int_a^{t_i}(t_i-s)^{q-2}f ds+\frac{(t-t_i)^2}{2!\Gamma(q-2)}\int_a^{t_i}(t_i-s)^{q-3}f ds\Bigg\}$$

$$for\ t\in J_k, k=0,1,2,...,K.$$

provided that the integral in (38) exists.

4 Example

Example 4.1. *The analytical solution of system* (1) *is difficult to obtain when* f *is a nonlinear function in* (1). *So, let us consider a linear impulsive system.*

$$\begin{cases} {}_0D_t^{9/4}x(t)=t,\ \ t\in[0,2]\backslash\{1\}, \\ x(1^+)=x(1^-)+I\left(x(1^-)\right), \\ x'(1^+)=x'(1^-)+\bar{I}\left(x(1^-)\right), \\ x''(1^+)=x''(1^-)+\hat{I}\left(x(1^-)\right), \\ x(0)=x_0,\ x'(0)=\bar{x}_0,\ x''(0)=\hat{x}_0. \end{cases} \tag{39}$$

Next, we give the general solution by

$$x(t)=\begin{cases} x_0+\bar{x}_0t+\dfrac{\hat{x}_0}{2!}t^2+\dfrac{1}{\Gamma(\frac{9}{4})}\dfrac{16}{9\times13}t^{\frac{13}{4}}\ \ for\ t\in[0,1], \\[4mm] x_0+\bar{x}_0t+\dfrac{\hat{x}_0}{2!}t^2+I\left(x(1^-)\right)+\bar{I}\left(x(1^-)\right)(t-1)+\dfrac{1}{2!}\hat{I}\left(x(1^-)\right)(t-1)^2+\dfrac{1}{\Gamma(\frac{9}{4})}\dfrac{16}{9\times13}t^{\frac{13}{4}}\bigg|_{t\geq0} \\[4mm] +\left(\xi_0I(x(1^-))+\xi_1\bar{I}(x(1^-))+\xi_2\hat{I}(x(1^-))\right)\left[\dfrac{1}{\Gamma(\frac{9}{4})}\dfrac{16}{9\times13}+\dfrac{1}{\Gamma(\frac{9}{4})}\dfrac{4}{9\times13}(t-1)^{\frac{9}{4}}(4t+9)\bigg|_{t\geq1}\right. \\[4mm] \left.-\dfrac{1}{\Gamma(\frac{9}{4})}\dfrac{16}{9\times13}t^{\frac{13}{4}}\bigg|_{t\geq0}+\dfrac{(t-1)}{\Gamma(\frac{9}{4}+1)}\bigg|_{t\geq1}+\dfrac{(t-1)^2}{2!\Gamma(\frac{9}{4})}\bigg|_{t\geq1}\right]\ \ for\ t\in(1,2]. \end{cases} \tag{40}$$

where ξ_0, ξ_1 *and* ξ_2 *are arbitrary constants.*

    Next, for Eq. (40), *we have*

$$_0D_t^{\frac{9}{4}}x(t)={}_0D_t^{\frac{9}{4}}\left\{x_0+\bar{x}_0t+\frac{\hat{x}_0}{2!}t^2+\frac{1}{\Gamma(\frac{9}{4})}\frac{16}{9\times13}t^{\frac{13}{4}}\right\}=t\ \ for\ t\in[0,1],$$

and

$$_0D_t^{\frac{9}{4}}x(t)={}_0D_t^{\frac{9}{4}}\Bigg\{x_0+\bar{x}_0t+\frac{\hat{x}_0}{2!}t^2+I\left(x(1^-)\right)+\bar{I}\left(x(1^-)\right)(t-1)$$

$$+\frac{1}{2!}\hat{I}\left(x(1^-)\right)(t-1)^2+\frac{1}{\Gamma(\frac{9}{4})}\frac{16}{9\times13}t^{\frac{13}{4}}\bigg|_{t\geq0}$$

$$+\left[\xi_0I(x(1^-))+\xi_1\bar{I}(x(1^-))+\xi_2\hat{I}(x(1^-))\right]\left[\frac{1}{\Gamma(\frac{9}{4})}\frac{16}{9\times13}+\frac{1}{\Gamma(\frac{9}{4})}\frac{4}{9\times13}(t-1)^{\frac{9}{4}}(4t+9)\bigg|_{t\geq1}\right.$$

$$\left.-\frac{1}{\Gamma(\frac{9}{4})}\frac{16}{9\times13}t^{\frac{13}{4}}\bigg|_{t\geq0}+\frac{(t-1)}{\Gamma(\frac{9}{4}+1)}\bigg|_{t\geq1}+\frac{(t-1)^2}{2!\Gamma(\frac{9}{4})}\bigg|_{t\geq1}\right]\Bigg\}_{t\in(1,2]}$$

$$
\begin{aligned}
= {}_0 D_t^{\frac{9}{4}} &\left\{ \frac{1}{\Gamma(\frac{9}{4})} \frac{16}{9 \times 13} t^{\frac{13}{4}} \Big|_{t \geq 0} \right. \\
&+ \frac{\xi_0 I(x(1^-)) + \xi_1 \bar{I}(x(1^-)) + \xi_2 \hat{I}(x(1^-))}{\Gamma(\frac{9}{4})} \frac{4}{9 \times 13} \left[(t-1)^{\frac{9}{4}}(4t+9) \Big|_{t \geq 1} - 4t^{\frac{13}{4}} \Big|_{t \geq 0} \right] \Big\}_{t \in (1,2]} \\
= {}& t_{t \in [0,2]} \big|_{t \in (1,2]} + \left[\xi_0 I(x(1^-)) + \xi_1 \bar{I}(x(1^-)) + \xi_2 \hat{I}(x(1^-)) \right] \left[t_{t \in (1,2]} - t_{t \in [0,2]} \right]_{t \in (1,2]} \\
= {}& t, t \in (1,2].
\end{aligned}
$$

So, Eq. (40) *satisfies Caputo fractional derivative condition in* (39).

Secondly, we can verify that Eq. (40) *satisfies*

$$
x(1^+) = x(1^-) + I\left(x(1^-)\right), \ x'(1^+) = x'(1^-) + \bar{I}\left(x(1^-)\right) \ and \ x''(1^+) = x''(1^-) + \hat{I}\left(x(1^-)\right).
$$

Moreover, we can verify that Eq. (40) *satisfies the corresponding conditions (i)-(vii) of* (39). *Therefore, Eq.* (40) *is the general solution of* (39).

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