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Research Article

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New modification of Maheshwari's method with optimal eighth order convergence for solving nonlinear equations

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Abstract: In this paper, we present a family of three-point with eight-order convergence methods for finding the simple roots of nonlinear equations by suitable approximations and weight function based on Maheshwari's method. Per iteration this method requires three evaluations of the function and one evaluation of its first derivative. These class of methods have the efficiency index equal to $8^{\frac{1}{4}} \approx 1.682$. We describe the analysis of the proposed methods along with numerical experiments including comparison with the existing methods. Moreover, the attraction basins of the proposed methods are shown with some comparisons to the other existing methods.

Keywords: Multi-point iterative methods, Maheshwari's method, Kung and Traub's conjecture, Basin of attraction

MSC: 65H05, 37F10

1 Introduction

Finding roots of nonlinear functions $f(x) = 0$ by using iterative methods is a classical problem which has interesting applications in different branches of science, in particular, physics and engineering. Therefore, several numerical methods for approximating simple roots of nonlinear equations have been developed and analyzed by using various techniques based on iterative methods in the recent years. The second order Newton-Raphson's method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ is one of the best-known iterative methods for finding approximate roots and it requires two evaluations for each iteration step, one evaluation of f and one of f' [1, 2].

Kung and Traub [3] conjectured that no multi-point method without memory with n evaluations could have a convergence order larger than 2^{n-1}. A multi-point method with convergence order 2^{n-1} is called optimal. The efficiency index provides a measure of the balance between those quantities, according to the formula $p^{1/n}$, where p is the convergence order of the method and n is the number of function evaluations per iteration.

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Many methods are described of which we note e.g. [2], [4-7]. Using inverse interpolation, Kung and Traub [3] constructed two general optimal classes without memory. Since then, there have been many attempts to construct optimal multi-point methods, utilizing e.g. weight functions [8-16].

In this paper, we construct a new class of optimal eight order of convergence based on Maheshwari's method. This paper is organized as follows: Section 2 is devoted to the construction and convergence analysis of the new class. In Section 3, the new methods are compared with a closest competitor in a series of numerical examples. In addition, comparisons of the basin of attraction with other methods are illustrated in Section 3. Section 4 contains a short conclusion.

2 Description of the method and convergence analysis

2.1 Three-point method of optimal order of convergence

In this section we propose a new optimal three-point method based on Maheshwari's method [6] for solving nonlinear equations. The Maheshwari's method is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n + \frac{1}{f'(x_n)}\left(\frac{f^2(x_n)}{f(y_n)-f(x_n)} - \frac{f^2(y_n)}{f(x_n)}\right), \quad (n = 0, 1, \ldots), \end{cases} \tag{1}$$

where x_0 is an initial approximation of x^*. The convergence order of (1) is four with three functional evaluations per iteration such that this method is optimal. We intend to increase the order of convergence of method (1) by an additional Newton's step. So we have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)}\left(\frac{f^2(x_n)}{f(y_n)-f(x_n)} - \frac{f^2(y_n)}{f(x_n)}\right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \end{cases} \tag{2}$$

Method (2) uses five function evaluations with order of convergence eight. Therefore, this method is not optimal. In order to decrease the number of function evaluations, we approximate $f'(z_n)$ by an expression based on $f(x_n)$, $f(y_n)$, $f(z_n)$ and $f'(x_n)$. Therefore

$$f'(z_n) \approx \frac{f'(x_n)}{F(x_n, y_n, z_n)H(s_n)},$$

where

$$F(x_n, y_n, z_n) = \left(\frac{f^3(y_n)(f(x_n) - 10f(y_n)) + 4f^2(x_n)(f^2(y_n) + f(x_n)f(y_n))}{f(x_n)(2f(x_n) - f(y_n))^2(f(y_n) - f(z_n))}\right), \tag{3}$$

and $H(.)$ is a weight function which will be specified later, and $s_n = \frac{f(z_n)}{f(x_n)}$.
We have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)}\left(\frac{f^2(x_n)}{f(y_n)-f(x_n)} - \frac{f^2(y_n)}{f(x_n)}\right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}F(x_n, y_n, z_n)H(s_n), \end{cases} \tag{4}$$

where $F(x_n, y_n, z_n)$ and s_n are defined as above.

2.2 Convergence analysis

In the following theorem we provide sufficient conditions on the weight function $H(s_n)$, which imply that method (4) has convergence order eight.

Theorem 2.1. *Assume that function $f : D \to \mathbb{R}$ is eight times continuously differentiable on an interval $D \subset \mathbb{R}$ and has a simple zero $x^* \in D$. Moreover, H is one time continuously differentiable. If the initial approximation x_0 is sufficiently close to x^* then the class defined by* (4) *converges to x^* and the order of convergence is eight under the conditions*

$$H(0) = 1, \quad H^{'}(0) = 2,$$

with the error term

$$e_{n+1} = \left(\frac{1}{2}c_2^2(4c_2^2 - c_3)(c_2^3 - 8c_2c_3 + 2c_4)\right) e_n^8 + O(e_n^9),$$

where $e_n := x_n - x^*$ for $n \in \mathbb{N}$ and $c_k := \frac{f^{(k)}(x^*)}{k!f^{'}(x^*)}$ for $k = 2, 3, \cdots$.

Proof. Let $e_{n,y} := y_n - x^*$, $e_{n,z} := z_n - x^*$ for $n \in \mathbb{N}$. Using the fact that $f(x^*) = 0$, Taylor expansion of f at x^* yields

$$f(x_n) = f^{'}(x^*)\left(e_n + c_2e_n^2 + c_3e_n^3 + \cdots + c_8e_n^8\right) + O(e_n^9), \tag{5}$$

and

$$f^{'}(x_n) = f^{'}(x^*)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \cdots + 9c_9e_n^8\right) + O(e_n^9). \tag{6}$$

Therefore,

$$\begin{aligned}\frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + \left(2c_2^2 - 2c_3\right)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &+ (8c_2^4 - 20c_2^2c_3 + 6c_3^2 + 10c_2c_4 - 4c_5)e_n^5 \\ &+ (-16c_2^5 + 52c_2^3c_3 - 28c_2^2c_4 + 17c_3c_4 - c_2(33c_3^2 - 13c_5))e_n^6 + O(e_n^7),\end{aligned}$$

and hence

$$\begin{aligned}e_{n,y} = y_n - x^* &= c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &+ (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 \\ &+ (16c_2^5 - 52c_2^3c_3 + 28c_2^2c_4 - 17c_3c_4 + c_2(33c_3^2 - 13c_5))e_n^6 + O(e_n^7).\end{aligned}$$

For $f(y_n)$ we also have

$$f(y_n) = f^{'}(x^*)\left(e_{n,y} + c_2e_{n,y}^2 + c_3e_{n,y}^3 + \cdots + c_8e_{n,y}^8\right) + O(e_{n,y}^9). \tag{7}$$

Therefore, by substituting (5), (6) and (7) into (2), we get

$$\begin{aligned}e_{n,z} = z_n - x^* &= (4c_2^3 - c_2c_3)e_n^4 + (-27c_2^4 + 26c_2^2c_3 - 2c_3^2 - 2c_2c_4)e_n^5 \\ &+ \left(120c_2^5 - 196c_2^3c_3 + 39c_2^2c_4 - 7c_3c_4 + c_2(54c_3^2 - 3c_5)\right)e_n^6 + O(e_n^7).\end{aligned}$$

For $f(z_n)$ we also get

$$f(z_n) = f^{'}(x^*)\left(e_{n,z} + c_2e_{n,z}^2 + c_3e_{n,z}^3 + \cdots + c_8e_{n,z}^8\right) + O(e_{n,z}^9). \tag{8}$$

From (5), (7) and (8) we obtain

$$\begin{aligned}F(x_n, y_n, z_n) &= 1 + 2c_2e_n + 3c_3e_n^2 + (-8c_2^3 + 2c_2c_3 + 4c_4)e_n^3 + (\frac{83}{2}c_2^4 - 45c_2^2c_3 + 4c_3^2 + 3c_2c_4 \\ &+ 5c_5)e_n^4 + (\frac{7167}{8}c_2^6 + 1731c_2^4c_3 - 56c_3^3 + 429c_2^3c_4 - 245c_2c_3c_4 + 9c_4^2 + c_2^2(815c_3^2 \\ &- 84c_5) + 16c_3c_5)e_n^6 + O(e_n^7).\end{aligned} \tag{9}$$

From (5) and (8) we have

$$\begin{aligned}s_n &= (4c_2^3 - c_2c_3)e_n^3 + (-31c_2^4 + 27c_2^2c_3 - 2c_3^2 - 2c_2c_4)e_n^4 + (151c_2^5 \\ &- 227c_2^3c_3 + 41c_2^2c_4 - 7c_3c_4 + c_2(57c_3^2 - 3c_5))e_n^5 + (592c_2^6 + 1266c_2^4c_3 \\ &+ 38c_3^3 - 325c_2^3c_4 + 170c_2c_3c_4 - 6c_4^2 - 10c_3c_5 + c_2^2(-608c_3^2 + 55c_5))e_n^6 + O(e_n^7).\end{aligned} \tag{10}$$

Expanding H at 0 yields

$$H(s_n) = H(0) + H'(0)s_n + O(s_n^2). \tag{11}$$

Substituting (5)-(11) into (4) we obtain

$$e_{n+1} = x_{n+1} - x^* = R_4e_n^4 + R_5e_n^5 + R_6e_n^6 + R_7e_n^7 + R_8e_n^8 + O(e_n^9),$$

where

$$\begin{aligned} R_4 &= -c_2(4c_2^2 - c_3)(-1 + H(0)), \\ R_5 &= 0, \\ R_6 &= 0, \\ R_7 &= -c_2^2(-4c_2^2 + c_3)^2(-2 + H'(0)). \end{aligned}$$

By setting $R_4 = R_7 = 0$ and $R_8 \neq 0$ the convergence order becomes eight. Obviously,

$$\begin{aligned} H(0) = 1 \quad &\Rightarrow \quad R_4 = 0, \\ H'(0) = 2 \quad &\Rightarrow \quad R_7 = 0, \end{aligned}$$

consequently, the error term becomes

$$e_{n+1} = \left(\frac{1}{2}c_2^2(4c_2^2 - c_3)(c_2^3 - 8c_2c_3 + 2c_4)\right)e_n^8 + O(e_n^9),$$

which completes the proof of the theorem. □

In what follows we give some concrete explicit representations of (4) by choosing different weight functions satisfying the provided condition for the weight function $H(s_n)$ in Theorem 2.1.

Method 1. Choose the weight function $H(s_n)$ as:

$$H(s_n) = 1 + 2s_n, \tag{12}$$

where $s_n = \frac{f(z_n)}{f(x_n)}$.

The function $H(s_n)$ in (12) satisfies the assumptions of Theorem 2.1 and we get

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)}\left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)}\right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left(\frac{f(x_n) + 2f(z_n)}{f(x_n)}\right)F(x_n, y_n, z_n), \end{cases} \tag{13}$$

where $F(x_n, y_n, z_n)$ is evaluated by (3).

Method 2. Choose the weight function $H(s_n)$ as:

$$H(s_n) = \frac{1 + 4s_n}{1 + 2s_n}, \tag{14}$$

where $s_n = \frac{f(z_n)}{f(x_n)}$.

The function $H(s_n)$ in (14) satisfies the assumptions of Theorem 2.1 and we obtain

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)}\left(\frac{f^2(x_n)}{f(y_n) - f(x_n)} - \frac{f^2(y_n)}{f(x_n)}\right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left(\frac{f(x_n) + 4f(z_n)}{f(x_n) + 2f(z_n)}\right)F(x_n, y_n, z_n), \end{cases} \tag{15}$$

where $F(x_n, y_n, z_n)$ is evaluated by (3).

Method 3. Choose the weight function $H(s_n)$ as:

$$H(s_n) = \frac{1}{1 - 2s_n}, \tag{16}$$

where $s_n = \frac{f(z_n)}{f(x_n)}$.
The function H in (16) satisfies the assumptions of Theorem 2.1 and we get

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = x_n + \frac{1}{f'(x_n)} \left(\frac{f^2(x_n)}{f(y_n)-f(x_n)} - \frac{f^2(y_n)}{f(x_n)} \right), \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} (\frac{f(x_n)}{f(x_n)-2f(z_n)}) F(x_n, y_n, z_n), \end{cases} \tag{17}$$

where $F(x_n, y_n, z_n)$ is evaluated by (3).

We apply the new methods (13), (15) and (17) to several benchmark examples and compare them with the existing three-point methods which have the same convergence order $r = 8$ and the same computational efficiency index equal to $\sqrt[\theta]{r} = 1.682$, which is optimal for $\theta = 4$ function evaluations per iteration [1, 2].

3 Numerical performance

In this section we test and compare our proposed methods with some existing methods. We compare methods (13), (15) and (17) with the following related three-point methods.

Bi, Ren and Wu's method. The method by Bi et al. [8] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2) f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)} H(t_n), \end{cases} \tag{18}$$

where $\beta = -\frac{1}{2}$ and weight function

$$H(t_n) = \frac{1}{(1-\alpha t_n)^2}, \quad \alpha = 1, \tag{19}$$

where $t_n = \frac{f(z_n)}{f(x_n)}$.

If $x_0, x_1, \ldots, x_n$ are points of D, the divided difference of order 1 can be expressed by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

and in general, the divided difference of order n is obtained by

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

In addition, for $x_0 = x_1 = \ldots = x_n = x$, we write

$$f[x, x, \ldots, x] = \frac{f^{(n+1)}(x)}{(n+1)!}.$$

Chun and Lee's method. The method by Chun and Lee [9] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{1}{\left(1 - \frac{f(y_n)}{f(x_n)}\right)^2}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \cdot \frac{1}{(1 - H(t_n) - J(s_n) - P(u_n))^2}, \end{cases} \tag{20}$$

with weight functions

$$H(t_n) = -\beta - \gamma + t_n + \frac{t_n^2}{2} - \frac{t_n^3}{2}, \quad J(s_n) = \beta + \frac{s_n}{2}, \quad P(u_n) = \gamma + \frac{u_n}{2}, \tag{21}$$

where $t_n = \frac{f(y_n)}{f(x_n)}$, $s_n = \frac{f(z_n)}{f(x_n)}$, $u_n = \frac{f(z_n)}{f(y_n)}$ and $\beta, \gamma \in \mathbb{R}$.

Sharma and Sharma's method. The Sharma and Sharma method [16] is given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} = z_n - \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]} W(t_n), \end{cases} \tag{22}$$

with weight function

$$W(t_n) = 1 + \frac{t_n}{1+\alpha t_n}, \quad \alpha = 1, \tag{23}$$

where $t_n = \frac{f(z_n)}{f(x_n)}$.

The three point method (4) is tested on a number of nonlinear equations. To obtain a high accuracy and avoid the loss of significant digits, we employed multi-precision arithmetic with 7000 significant decimal digits in the programming package of Mathematica 8 [17].

In order to test our proposed methods (13), (15) and (17), and also to compare them with the methods (18), (20) and (22), we compute the error and the approximated computational order of convergence (ACOC) that was introduced by Cordero et al. [18]

$$\mathrm{ACOC} \approx \frac{\ln|(x_{n+1}-x_n)/(x_n-x_{n-1})|}{\ln|(x_n-x_{n-1})/(x_{n-1}-x_{n-2})|}.$$

In Tables 1, 2, 3 and 4, the proposed methods (13), (15) and (17) with the methods (18), (20) and (22) have been tested on different nonlinear equations. It is clear that these methods are in accordance with the developed theory.

Table 1. Comparison for $f(x) = \ln(1+x^2) + e^{x^2-3x}\sin(x), x^* = 0, x_0 = 0.35$, for different methods (M) and weight functions (W-F).

M	W-F	$\lvert x_1 - x^*\rvert$	$\lvert x_2 - x^*\rvert$	$\lvert x_3 - x^*\rvert$	$\lvert x_4 - x^*\rvert$	ACOC
(13)	**(12)**	**0.657e−4**	**0.466e−30**	**0.299e−239**	**0.865e−1913**	**8.0000**
(15)	**(14)**	**0.568e−4**	**0.145e−30**	**0.259e−243**	**0.272e−1945**	**8.0000**
(17)	**(16)**	**0.755e−4**	**0.141e−29**	**0.206e−235**	**0.423e−1882**	**8.0000**
(18)	**(19)**	**0.720e−4**	**0.584e−30**	**0.110e−238**	**0.175e−1908**	**8.0000**
(20)	**(21)**	**0.721e−4**	**0.230e−30**	**0.252e−242**	**0.528e−1938**	**7.9999**
(22)	**(23)**	**0.753e−4**	**0.619e−31**	**0.128e−247**	**0.453e−1981**	**8.0000**

Table 2. Comparison for $f(x) = \ln(1-x+x^2) + 4\sin(1-x), x^* = 1, x_0 = 1.1$, for different methods (M) and weight functions (W-F).

M	W-F	$\lvert x_1 - x^*\rvert$	$\lvert x_2 - x^*\rvert$	$\lvert x_3 - x^*\rvert$	$\lvert x_4 - x^*\rvert$	ACOC
(13)	**(12)**	**0.444e−11**	**0.399e−94**	**0.170e−758**	**0.189e−6073**	**8.0000**
(15)	**(14)**	**0.445e−11**	**0.404e−94**	**0.187e−758**	**0.394e−6073**	**8.0000**
(17)	**(16)**	**0.443e−11**	**0.395e−94**	**0.155e−758**	**0.902e−6077**	**8.0000**
(18)	**(19)**	**0.423e−12**	**0.134e−114**	**0.445e−1037**	**0.211e−9339**	**8.0000**
(20)	**(21)**	**0.211e−11**	**0.264e−97**	**0.250e−782**	**0.181e−6264**	**7.9999**
(22)	**(23)**	**0.172e−11**	**0.581e−98**	**0.984e−790**	**0.663e−6324**	**8.0000**

Table 3. Comparison for $f(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5, x^* = \sqrt{2}, x_0 = 1.5$, for different methods (M) and weight functions (W-F).

M	W-F	$\lvert x_1 - x^* \rvert$	$\lvert x_2 - x^* \rvert$	$\lvert x_3 - x^* \rvert$	$\lvert x_4 - x^* \rvert$	ACOC
(13)	(12)	0.783e−8	0.648e−64	0.142e−512	0.765e−4102	8.0000
(15)	(14)	0.749e−8	0.656e−64	0.855e−514	0.132e−4111	8.0000
(17)	(16)	0.816e−8	0.908e−64	0.212e−511	0.187e−4092	8.0000
(18)	(19)	0.673e−8	0.113e−64	0.726e−519	0.208e−4152	8.0000
(20)	(21)	0.433e−8	0.134e−66	0.116e−534	0.373e−4279	7.9999
(22)	(23)	0.642e−10	0.101e−81	0.389e−656	0.184e−5251	8.0000

Table 4. Comparison for $f(x) = (x-2)(x^{10} + x + 1)e^{-x-1}, x^* = 2, x_0 = 2.1$, for different methods (M) and weight functions (W-F).

M	W-F	$\lvert x_1 - x^* \rvert$	$\lvert x_2 - x^* \rvert$	$\lvert x_3 - x^* \rvert$	$\lvert x_4 - x^* \rvert$	ACOC
(13)	(12)	0.119e−3	0.253e−26	0.106e−207	0.992e−1659	8.0000
(15)	(14)	0.143e−3	0.109e−25	0.124e−202	0.353e−1618	8.0000
(17)	(16)	0.916e−4	0.307e−27	0.493e−215	0.221e−1717	8.0000
(18)	(19)	0.183e−4	0.319e−33	0.278e−263	0.920e−2104	8.0000
(20)	(21)	0.344e−5	0.134e−38	0.702e−306	0.398e−2444	8.0000
(22)	(23)	0.239e−4	0.138e−32	0.170e−258	0.938e−2066	8.0000

3.1 Graphical comparison by means of attraction basins

We have already observed that all methods converge if the initial guess is chosen suitably. From the numerical point of view, the dynamical behavior of the rational function associated with an iterative method gives us important information about convergence and stability. Therefore, we now investigate the stability region. In other words, we numerically approximate the domain of attraction of the zeros as a qualitative measure of stability. To answer the important question on the dynamical behavior of the algorithms, we investigate the dynamics of the new methods and compare them with common and well-performing methods from the literature. In the following, we recall some basic concepts such as basin of attraction. For more details one can consult [19-22].

Let $G : \mathbb{C} \to \mathbb{C}$ be a rational map on the complex plane. The orbit of a point $z \in \mathbb{C}$ is defined as

$$orb(z) = \{z,\ G(z),\ G^2(z), \dots\}.$$

A point $z_0 \in \mathbb{C}$ is called a periodic point with minimal period m if $G^m(z_0) = z_0$, where m is the smallest integer with this property. A periodic point with minimal period 1 is called a fixed point. Moreover, a point z_0 is called attracting if $|G'(z_0)| < 1$, repelling if $|G'(z_0)| > 1$, and neutral otherwise. The Julia set of a nonlinear map $G(z)$, denoted by $J(G)$, is the closure of the set of its repelling periodic points. The complement of $J(G)$ is the Fatou set $F(G)$, where the basin of attraction of the different roots lie [23]. For the dynamical point of view, in fact, we take a 256×256 grid of the square $[-3, 3] \times [-3, 3] \in \mathbb{C}$ and assign a color to each point $z_0 \in D$ according to the simple root to which the corresponding orbit of the iterative method starting from z_0 converges, and we mark the point as black if the orbit does not converge to a root, in the sense that after at most 100 iterations it has a distance to any of the roots, which is larger than 10^{-3}. In this way, we distinguish the attraction basins by their color for different methods.

We use the basins of attraction for comparing the iteration algorithms. The basin of attraction is a method to visually comprehend how an algorithm behaves as a function of the various starting points. In the following figures, the roots of each functions are drawn with a different color. In the basin of attractions, the number of iteration needed to achieve the solution is showed in darker or brighter colors. Black color denotes lack of convergences to any of the roots or convergence to the infinity.

We have tested several different examples, and the results on the performance of the tested methods were similar. Therefore we report the general observation here for test problems $p_1(z) = z^2 - 1$ with roots $-1, 1$ and $p_2(z) = z(z^2 + 1)$ with roots $0, i, -i$.

In Figures 1 and 2, basins of attractions of methods (13), (18), (20) and (22) with two test problems $p_1(z)$ and $p_2(z)$ are illustrated from left to right respectively. As a result, in Figure 1 the basin of attraction of method (13) is similar to other methods, however in Figure 2, first two figures seem to produce larger basins of attraction than the last two figures.

Fig. 1. Comparison of basin of attraction of methods (13), (18), (20) and (22) for test problem $p_1(z) = z^2 - 1$

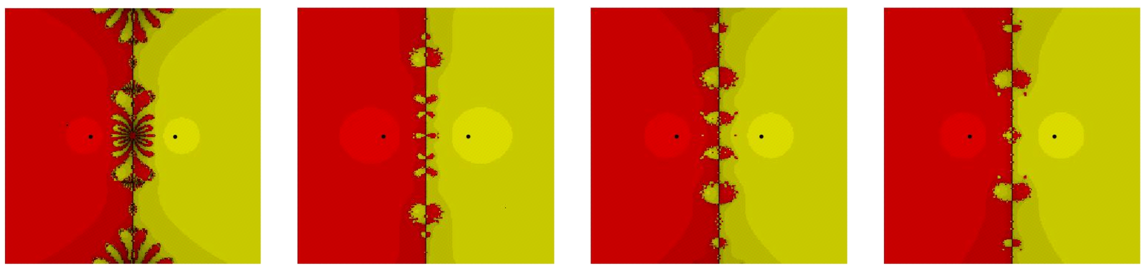
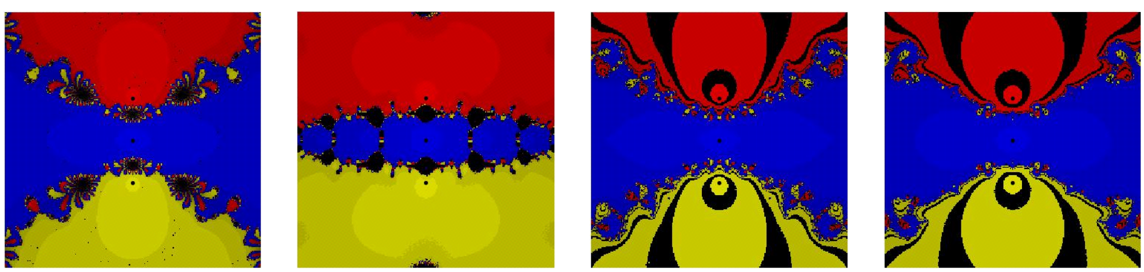


Fig. 2. Comparison of basin of attraction of methods (13), (18), (20) and (22) for test problem $p_2(z) = z^3 + z$



4 Conclusion

We presented a new optimal class of three-point methods without memory for approximating a simple root of a given nonlinear equation. Our proposed methods use five function evaluations for each iteration. Therefore they support Kung and Traub's conjecture. Numerical examples show that our methods work and can compete with other methods in the same class, as well as we used the basin of attraction for comparing the iteration algorithms.

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