Open Math. 2016; 14: 370–383 **DE GRUYTER** OPEN

Open Mathematics **Open Access**

Research Article

Qixiang Dong*, Can Liu, and Zhenbin Fan

Weighted fractional differential equations with infinite delay in Banach spaces

DOI 10.1515/math-2016-0035
Received February 25, 2016; accepted May 23, 206.

Abstract: This paper is devoted to the study of fractional differential equations with Riemann-Liouville fractional derivatives and infinite delay in Banach spaces. The weighted delay is developed to deal with the case of non-zero initial value, which leads to the unboundedness of the solutions. Existence and uniqueness results are obtained based on the theory of measure of non-compactness, Schauder's and Banach's fixed point theorems. As auxiliary results, a fractional Gronwall type inequality is proved, and the comparison property of fractional integral is discussed.

Keywords: Fractional integral, Fractional derivative, Functional differential equation, Infinite delay

MSC: 34A08, 34K37

1 Introduction

In this paper, we study nonlinear functional fractional differential equation with weighted infinite delay in an abstract Banach space X, of the form

$$D^{\alpha} y(t) = f(t, \tilde{y}_t), \quad t \in (0, b], \tag{1}$$

$$\tilde{y}_0 = \phi \in \mathcal{B}, \tag{2}$$

where $0 < \alpha \leq 1$, D^{α} is the Riemann-Liouville fractional derivative, $\tilde{y}(t) = t^{1-\alpha} y(t)$, $f : (0, b] \times \mathcal{B} \to \mathcal{B}$ is a given function satisfying some assumptions, and $\mathcal{B}$ is the phase space that will be specified later. We give the definition of solutions, and investigate the existence and continuous dependence of solutions to such equations in the space $C_{1-\alpha}((a, b]; X)$.

In the past several decades fractional differential equations have attracted a considerable interest in both mathematics and applications, since they have been proved to be valuable tools in modeling many physical phenomena. There has been a significant development in fractional differential equations in the past decades. See, for example, [1–14] and references therein. Fractional differential equations in Banach spaces are also wildly studied [1, 11–14]. Among these works, some authors study functional fractional differential equations [2, 3, 6, 8, 11]. For example, in [2], Benchohra et al. studied fractional order differential equations

$$D^{\alpha} y(t) = f(t, y_t), \quad t \in [0, b], \quad 0 < \alpha < 1,$$

with infinite delay

$$y(t) = \phi(t), \quad t \in (-\infty, 0].$$

However, it is known that the Riemann-Liouville fractional derivative of a function y is unbounded at some neighborhoods of the initial point 0, except that $y(0) = 0$. For this reason, when $y(0) \neq 0$, the solutions to the

***Corresponding Author: Qixiang Dong:** School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, P. R. China, E-mail: qxdongyz@outlook.com, qxdong@yzu.edu.cn
Can Liu: School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, P. R. China, E-mail: cliuyz@163.com
Zhenbin Fan: School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, P. R. China, E-mail: zbfan@yzu.edu.cn

functional fractional differential equations given in the mentioned papers may not be well-defined. An example can be found in [6]. In order to remedy this defect, in [6], the author modified the model and studied the weighted fractional differential equations with infinite delay. Existence and continuous dependence results of solutions are obtained in finite dimensional spaces.

In this paper, we continue the work of [6], to study the weighted fractional differential equations with infinite delay in Banach spaces. The difficulty is that the bounded subsets in Banach spaces are not compact in general. To get a compact subset in the space of continuous functions, we have to suppose some conditions involving the measure of non-compactness on the nonlinear term. The function space $C_{1-\alpha}((0,b] : X)$ we encountered in this paper is a space of unbounded functions, which is different from $C([0,b] : X)$. We need some further discussion on the relevant subsets. As auxiliary results, we also prove a Gronwall type inequality for fractional differential equations, and discuss the comparison property of fractional integral.

2 Preliminaries and lemmas

In this section we collect some definitions and results needed in our further investigations.

Let $(X, \|\cdot\|)$ be a Banach space. Denote by $C([a,b];X)$ the space of all continuous X valued functions defined on $[a,b]$ with the supremum norm, and $L^1_{loc}((a,b);X)$ the space of Bochner integrable functions $u : (a,b) \to X$ with the norm $\|u\| = \int_a^b \|u(t)\|dt$. We also consider the space $C_r((a,b];X)$ consisting of all continuous functions $f : (a,b] \to X$ such that $\lim_{t\to a}(t-a)^r f(t)$ exists, with the norm $\|f\|_{C_r} = \sup\{\|(t-a)^r f(t)\|; t \in (a,b]\}$.

Definition 2.1 ([5]). *Let $\alpha > 0$ be a fixed number. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $h : [a,b] \to X$ is defined by*

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds, \ \ t \in [a,b]$$

provided the right side is pointwisely defined, where $\Gamma(\cdot)$ denotes the well-known gamma function, i.e., $\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \mathrm{d}t$.

Definition 2.2 ([5]). *Let $\alpha > 0$ be fixed and $n = [\alpha] + 1$. The Riemann-Liouville fractional derivative of order α of $h : (a,b] \to X$ at the point t is defined by*

$$D_a^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} h(s)ds, \ \ t \in [a,b]$$

provided the right side is pointwisely defined, where $[\alpha]$ denotes the integer part of the real number α.

When $0 < \alpha < 1$, then

$$D_a^\alpha h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{\alpha-1} h(s)ds.$$

For simplicity, when $a = 0$, we denote D_0^α and I_0^α by D^α and I^α, respectively.

Lemma 2.3 ([5]). *Let $0 < \alpha < 1$. Then the unique solution to the equation $D^\alpha h(t) = 0$ is given by the formula*

$$h(t) = Ct^{\alpha-1},$$

for $t > 0$, where $C \in \mathbf{R}$ is a constant provided $h \in C((0,b]) \cap L^1_{loc}(0,b)$. Further, if $f \in C((0,b]) \cap L^1_{loc}(a,b)$ such that $D^\alpha f \in C((0,b]) \cap L^1_{loc}(0,b)$, then

$$I^\alpha D^\alpha f(t) = f(t) + Ct^{\alpha-1}$$

for $t > 0$ and some constant $C \in \mathbf{R}$.

In the literature devoted to equations with infinite delay, the selection of the state space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [15]. For a detailed discussion on the topic, we refer to the book by Hino et al [16].

Definition 2.4 ([16])**.** *A linear topological space of functions from* $(-\infty, 0]$ *into* X*, with seminorm* $\|\cdot\|_{\mathcal{B}}$*, is called an admissible phase space if* $\mathcal{B}$ *has the following properties:*

(A1) *There exist a positive constant* H *and functions* $K(\cdot), M(\cdot) : [0, +\infty) \to [0, +\infty)$*, with* K *continuous and* M *locally bounded, such that for any* $a, b \in \mathbf{R}$ *and* $b > a$*, if* $x : (-\infty, b] \to X, x_a \in \mathcal{B}$*, and* $x(\cdot)$ *is continuous on* $[a, b]$*, then for every* $t \in [a, b]$*, the following conditions hold:*
  - (i) $x_t \in \mathcal{B}$*;*
  - (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$*, for some* $H > 0$*;*
  - (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-a) \sup_{a \leq s \leq t} \|x(s)\| + M(t-a)\|x_a\|_{\mathcal{B}}$.

(A2) *For the function* $x(\cdot)$ *in* (A1)*,* $t \mapsto x_t$ *is a* $\mathcal{B}$ *valued continuous function for* $t \in [a, b]$.

(B) *The space* $\mathcal{B}$ *is complete.*

The theory of the measure of noncompactness was initiated by Kuratowski in 1930s and has played a very important role in nonlinear analysis in recent decades. It is often applied to the theories of differential and integral equations as well as to the operator theory and geometry of Banach spaces [17–22]. One of the most important examples of measure of noncompactness is the *Hausdorff's measure of noncompactness* β_Y, which is defined by

$$\beta_Y(B) = \inf\{r > 0;\ B \text{ can be covered with a finite number of balls of radius equal to } r\}$$

for bounded set B in a Banach space Y.

The following properties of Hausdorff's measure of noncompactness are well known.

Lemma 2.5 ([17])**.** *Let* Y *be a real Banach space and* $B, C \subseteq Y$ *be bounded,the following properties are satisfied :*

(1) B *is pre-compact if and only if* $\beta_Y(B) = 0$ *;*

(2) $\beta_Y(B) = \beta_Y(\overline{B}) = \beta_Y(convB)$ *where* $\overline{B}$ *and* $convB$ *mean the closure and convex hull of* B *respectively;*

(3) $\beta_Y(B) \leq \beta_Y(C)$ *when* $B \subseteq C$*;*

(4) $\beta_Y(B + C) \leq \beta_Y(B) + \beta_Y(C)$ *where* $B + C = \{x + y; x \in B, y \in C\}$*;*

(5) $\beta_Y(B \cup C) \leq max\{\beta_Y(B), \beta_Y(C)\}$*;*

(6) $\beta_Y(\lambda B) = |\lambda|\beta_Y(B)$ *for any* $\lambda \in R$*;*

(7) *If the map* $Q : D(Q) \subseteq Y \to Z$ *is* $Lipschitz$ *continuous with constant* k *then* $\beta_Z(QB) \leq k\beta_Y(B)$ *for any bounded subset* $B \subseteq D(Q)$*, where* Z *be a Banach space;*

(8) $\beta_Y(B) = \inf\{d_Y(B, C); C \subseteq Y\ be\ precompact\} = \inf\{d_Y(B, C); C \subseteq Y\ be\ finite\ valued\}$*, where* $d_Y(B, C)$ *means the nonsymmetric (or symmetric) Hausdorff distance between* B *and* C *in* Y.

(9) *If* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of bounded closed nonempty subsets of* Y *and* $\lim_{n\to+\infty} \beta_Y(W_n) = 0$*, then* $\bigcap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* Y.

In this paper we denote by β the Hausdorff measure of noncompactness of X and by β_c the Hausdorff measure of noncompactness of $C([a, b]; X)$. To discuss the existence we need the following lemmas in this paper.

Lemma 2.6 ([17])**.** *If* $W \subseteq C([a, b]; X)$ *is bounded, then*

$$\beta(W(t)) \leq \beta_c(W)$$

for all $t \in [a, b]$*, where* $W(t) = \{u(t); u \in W\} \subseteq X$*.Furthermore if* W *is equicontinuous on [a,b], then* $\beta(W(t))$ *is continuous on* $[a, b]$ *and*

$$\beta_c(W) = \sup\{\beta(W(t)), t \in [a, b]\}.$$

Lemma 2.7 ([17]). *If $W \subseteq C([a,b];X)$ is bounded and equicontinuous, then $\beta(W(s))$ is continuous and*

$$\beta(\int_a^t W(s)ds) \leq \int_a^t \beta(W(s))ds \tag{3}$$

for all $t \in [a,b]$, where $\int_a^t W(s)ds = \{\int_a^t x(s)ds : x \in W\}$.

3 Auxiliary results

In the qualitative theory of differential and Volterra integral equations, especially in establishing of boundedness and stability, the Gronwall type inequalities play a very important role. See, for example, [23–26]. In 1919, T. H. Gronwall proved a remarkable inequality which has attracted and continues to attract considerable attention in the literature.

Lemma 3.1. *Let u, p and q be real continuous functions defined on $[a,b]$, $q(t) \geq 0$ for $t \in [a,b]$. Suppose that the inequality*

$$u(t) \leq p(t) + \int_a^t q(s)u(s)ds$$

holds for all $t \in [a,b]$. Then we have

$$u(t) \leq p(t) + \int_a^t q(s)p(s)\exp\Big[\int_s^t q(\tau)d\tau\Big]ds$$

for all $t \in [a,b]$.

In 1967, S. C. Chu and F. T. Metcalf proved in [23] a generalized Gronwall inequality for convolution type integral equations as follows.

Lemma 3.2. *Let the functions u and p be continuous on $[0,b]$; let the function K be continuous and nonnegative on the triangle $0 \leq s \leq t \leq b$. If*

$$u(t) \leq p(t) + \int_0^t K(t,s)u(s)ds, \qquad 0 \leq t \leq b,$$

then

$$u(t) \leq p(t) + \int_0^t H(t,s)p(s)ds, \qquad 0 \leq t \leq b,$$

where $H(t,s) = \sum_{i=1}^{\infty} K_i(t,s)$, $0 \leq s \leq t \leq b$, is the resolvent kernel, and the $K_i (i = 1, 2, \cdots)$ are the iterated kernels of K.

The fractional integral is in fact a kind of convolution of functions. However, the kernel of the fractional integral $K(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}$ is unbounded in the triangle $0 \leq s < t \leq b$ when $0 < \alpha < 1$. Observe that Lemma 3.2 holds only for continuous, and hence bounded, kernel $K(t,s)$ on $0 \leq s \leq t \leq b$. For this reason, in 2007, H. Ye *et al.* proved a generalized Grownwall inequality specifically for fractional differential equations ([26], which is the generalization of [25, Lemma 7.1.1].

Lemma 3.3. *Suppose $\alpha > 0$, u and p are nonnegative functions locally integrable on $[0, b]$ and g is a nonnegative, nondecreasing continuous function defined on $[0, b]$, $g(t) \leq M$(constant) on $[0, b]$. If the inequality*

$$u(t) \leq p(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds$$

holds for all $t \in [0, b]$, then we have

$$u(t) \leq p(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}p(s)\Big]ds$$

for all $t \in [0, b]$.

Now we suppose that

$$u(t) \leq p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)u(s)ds \tag{4}$$

for $0 \leq t \leq b$. Let $k(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)$. Define

$$k_1(t,s) = k(t,s), \quad k_i(t,s) = \int_s^t k(t,\tau)k_{i-1}(\tau,s)d\tau, \quad i = 2, 3, \cdots. \tag{5}$$

Then we have

Lemma 3.4. *Let the functions u, p and q be continuous and nonnegative on $[0, b]$. If the inequality* (4) *holds, then*

$$u(t) \leq p(t) + \int_0^t R(t,s)p(s)ds, \qquad 0 \leq t \leq b, \tag{6}$$

where $R(t,s) = \sum_{i=1}^{\infty} k_i(t,s)$, $0 \leq s < t \leq b$, is the resolvent kernel, and the $k_i (i = 1, 2, \cdots)$ are the iterated kernels defined by (5).

Proof. From (4), we have

$$\begin{aligned} u(t) &\leq p(t) + \int_0^t k(t,s)p(s)ds + \int_0^t k(t,s)ds\int_0^s k(s,\tau)u(\tau)d\tau \\ &= p(t) + \int_0^t k_1(t,s)p(s)ds + \int_0^t u(s)ds\int_s^t k(t,\tau)k_1(\tau,s)d\tau \\ &= p(t) + \int_0^t k_1(t,s)p(s)ds + \int_0^t k_2(t,s)u(s)ds \end{aligned}$$

for all $t \in [0, b]$. By induction, we have for all $n \in \mathbf{N}$,

$$u(t) \leq p(t) + \int_0^t \sum_{i=1}^{n} k_i(t,s)p(s)ds + \int_0^t k_{n+1}(t,s)u(s)ds \tag{7}$$

for all $t \in [0, b]$. Now we prove that

$$|k_n(t,s)| \leq \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}\|q\|_{\infty}^n \tag{8}$$

for all $n \in \mathbf{N}$ and $0 \le s < t \le b$. In fact,

$$|k_1(t,s)| = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|q(s)| \le \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\|q\|_\infty$$

for $0 \le s < t \le b$. Suppose that (8) holds for n. Then for $n+1$,

$$|k_{n+1}(t,s)| = |\int_s^t k(t,\tau)k_n(\tau,s)d\tau| \le \frac{\|q\|_\infty}{\Gamma(\alpha)}\frac{\|q\|_\infty^n}{\Gamma(n\alpha)}\int_s^t (t-\tau)^{\alpha-1}(\tau-s)^{n\alpha-1}d\tau$$
$$= \frac{\|q\|_\infty^{n+1}}{\Gamma(\alpha)\Gamma(n\alpha)}(t-s)^{n\alpha}\frac{\Gamma(\alpha)\Gamma(n\alpha)}{\Gamma((n+1)\alpha)} = \frac{(t-s)^{n\alpha}}{\Gamma((n+1)\alpha)}\|q\|_\infty^{n+1}$$

for all $0 \le s < t \le b$. By mathematical induction, (8) holds for all $n \in \mathbf{N}$.

We now show that $\int_0^t R(t,s)p(s)ds = \int_0^t \sum_{i=1}^{\infty} k_i(t,s)p(s)ds$ uniformly for $t \in [0,b]$ and $\lim_{i\to\infty} k_i(t,s) = 0$ uniformly for $0 \le s < t \le b$. Write

$$R(t,s) = \sum_{i=1}^{[1/\alpha]} k_i(t,s) + \sum_{i=[1/\alpha]+1}^{\infty} k_i(t,s).$$

For $1 \le i \le [1/\alpha]$,

$$\Big|\int_0^t k_i(t,s)p(s)ds\Big| \le \frac{\|p\|_\infty^i}{\Gamma(i\alpha)}\int_0^t (t-s)^{i\alpha-1}ds = \frac{\|p\|_\infty^i}{\Gamma(i\alpha+1)}t^{i\alpha}. \tag{9}$$

While

$$\sum_{i=[1/\alpha]+1}^{\infty}\Big|k_i(t,s)\Big| \le \sum_{i=[1/\alpha]+1}^{\infty}\frac{\|p\|_\infty^i}{\Gamma(i\alpha)}(t-s)^{i\alpha-1} \le \sum_{i=[1/\alpha]+1}^{\infty}\frac{\|p\|_\infty^i b^{i\alpha-1}}{\Gamma(i\alpha)} = \frac{1}{b}\sum_{i=[1/\alpha]+1}^{\infty}\frac{\left(\|p\|_\infty b^\alpha\right)^i}{\Gamma(i\alpha)}.$$

Using the ratio test it is easily seen that $\sum_{i=[1/\alpha]+1}^{\infty}\frac{\left(\|p\|_\infty b^\alpha\right)^i}{\Gamma(i\alpha)}$ is convergent. Hence $\sum_{i=[1/\alpha]+1}^{\infty} k_i(t,s)$ converges absolutely and uniformly for $0 \le s < t \le b$. This, combined with (9), implies that $\int_0^t R(t,s)p(s)ds = \int_0^t \sum_{i=1}^{\infty} k_i(t,s)p(s)ds$ uniformly for $t \in [0,b]$. The convergence of $\sum_{i=[1/\alpha]+1}^{\infty} k_i(t,s)$ in turn implies that $\lim_{i\to\infty} k_i(t,s) = 0$ uniformly for $0 \le s < t \le b$.

At last, letting $n \to \infty$ in (7), we get the desired inequality (6). □

Remark 3.5. *The proof of inequality* (8) *and the convergence of series can be found in [5, Page 145-146]. For the convenience of the readers and the completeness of the paper we repeat the proof here.*

Remark 3.6. *From the proof of Lemma 3.4 we can see that, if the functions p and q are locally integrable and bounded, then the result in Lemma 3.4 still holds. Further, by a slightly modification of the proof, we can get the following result, which is the generalization of [26, Theorem 1].*

Lemma 3.7. *Let $\alpha > 0$, the functions p and q be nonnegative, locally integrable and bounded on $[0,b]$, g be nonnegative and nondecreasing continuous on $[0,b]$. Suppose that u is nonnegative and continuous on $[0,b)$ satisfying*

$$u(t) \le p(t) + g(t)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}q(s)u(s)ds$$

for all $t \in [0,b)$, then

$$u(t) \le p(t) + \int_0^t \widetilde{R}(t,s)p(s)ds, \qquad 0 \le t \le b,$$

where $\widetilde{R}(t,s) = \sum_{i=1}^{\infty} \widetilde{k_i}(t,s)$, $0 \le s < t \le b$, *and* $\widetilde{k_i}$ $(i = 1, 2, \cdots)$ *are defined by*

$$\widetilde{k_1}(t,s) = \widetilde{k}(t,s) = \frac{(t-s)^{\alpha-1} g(t)}{\Gamma(\alpha)} q(s),$$

$$\widetilde{k_i}(t,s) = \int_s^t \widetilde{k}(t,\tau) \widetilde{k}_{i-1}(\tau,s) d\tau, \quad i = 2, 3, \cdots .$$

Next we discuss the comparison property of the fractional integral. It is well-known that if an integrable function f is nonnegative on $[0, b]$, then $\int_{t_1}^{t_2} f(s)ds \ge 0$ for any $t_1, t_2 \in [0, b]$ with $t_1 < t_2$. Equivalently, the function $\Phi(t) = \int_0^t f(s)ds$ is nondecreasing on $[0, b]$. For fractional case, a natural generalization of this property might be $I_0^\alpha f(t_2) - I_0^\alpha f(t_1) \ge 0$ for $t_1, t_2 \in [0, b]$ with $t_1 < t_2$. Or, equivalently, the function $F(t) = I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$ is nondecreasing. However, it is not always the case.

Example 3.8. *Consider* $f(t) = t^{\alpha-1}$ *for* $0 < \alpha < 1$ *and* $t > 0$. *Then* $f(t) > 0$ *for all* $t > 0$.

$$F(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^{\alpha-1} ds = \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} t^{2\alpha-1}.$$

So for $0 < \alpha < \frac{1}{2}$, $2\alpha - 1 < 0$, *and the function* $F(\cdot)$ *is strictly decreasing for* $t > 0$.

Fortunately, we have the following relatively weak result.

Lemma 3.9. *Suppose that* $\alpha > 0$ *and* $f \in C[0, b]$ *is nonnegative and nondecreasing. Then* $F(t) = I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds$ *is nondecreasing on* $[0, b]$.

Proof. We first suppose that $f \in C^1[0, b]$ is nonnegative and nondecreasing. Then $f'(t) \ge 0$ for all $t \in [0, b]$. It follows that

$$F'(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f'(s)ds + f(0)t^{\alpha-1} \ge 0$$

for all $t \in [0, b]$ and hence $F(\cdot)$ is nondecreasing on $[0, b]$.

For the general case that $f \in C[0, b]$, we choose a sequence of nonnegative and nondecreasing functions $\{f_n\} \in C^1[0, b]$ such that $\lim_n f_n = f$ in $C[0, b]$. Then each $F_n = I_0^\alpha f_n$ is nondecreasing, and the limit $F = \lim_n F_n$ is therefore nondecreasing. □

4 Existence results

In this section, we study the existence of solutions to the weighted functional differential equations (1)-(2). We begin with the definition of solutions to these equations.

Definition 4.1. *A function* $y : (-\infty, b] \to X$ *is said to be a solution to* (1)-(2), *if* $y|_{(0,b]} \in C((0, b]) \cap L_{loc}^1(0, b)$, $\tilde{y}_0 = \phi$ *and satisfies* (1).

For the existence results on the problem (1)-(2), we need to transform the fractional differential equation into an integral equation. From Lemma 2.3 we can obtain that, if $0 < \alpha < 1$ and $h \in C((0, b]) \cap L_{loc}^1(0, b)$ such that $I^{1-\alpha}h$ is absolutely continuous on $[0, b]$, then the function y solves the fractional differential equation

$$D^\alpha y(t) = h(t), \quad t \in (0, b]$$

if and only if y satisfies

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds + Ct^{\alpha-1}, \quad t \in (0,b]$$

for some constant C. See also [5, Theorem 2.23].

For the forthcoming analysis, we need the following hypothesis.

(H1) $f : (0,b] \times \mathcal{B} \to X$ is continuous.

(H2) There exist a nonnegative $\eta \in L^p(0,b]$ with $p > 1/\alpha$ and a continuously non-decreasing function $\Omega : [0,+\infty) \to [0,+\infty)$ such that

$$\|f(t,u)\| \leq \eta(t)\Omega(\|u\|_{\mathcal{B}}) \tag{10}$$

for $t \in (0,b]$ and every $u \in \mathcal{B}$, and

$$\beta(f(t,D)) \leq \eta(t) \sup_{-\infty<\theta\leq 0} \beta(D(\theta)) \tag{11}$$

for $t \in (0,b]$ and every bounded subset $D \subset \mathcal{B}$, where $D(\theta) = \{u(\theta) : u \in D\}$.

Theorem 4.2. *Assume the hypotheses (H1) and (H2) hold. If*

$$\lim_{r\to+\infty} \sup \frac{\Omega(r)}{r} < \frac{\Gamma(1+\alpha)}{K_b b^2 \|\eta\|_p}, \tag{12}$$

then there exists at least a solution to (1)-(2) *on* $(-\infty, b]$.

Proof. From the comment above, we know that y is a solution to (1) if and only if y satisfies

$$y(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\widetilde{y_s}) ds + \phi(0)t^{\alpha-1}, & t \in (0,b], \\ \phi(t), & t \in (-\infty,0]. \end{cases}$$

For given $\phi : (-\infty,0]$ which belongs to $\mathcal{B}$, let $\widetilde{\phi}$ be a function defined by

$$\widetilde{\phi}(t) = \begin{cases} 0, & t \in (0,b], \\ \phi(t), & t \in (-\infty,0]. \end{cases}$$

Then we have $\widetilde{\phi}_0 = \phi$. For $z \in C_{1-\alpha}((0,b],X)$,where $C_{1-\alpha}((0,b],X)$ is the Banach space consisting of all continuous functions $f : (0,b] \to X$ such that $\lim_{t\to 0} t^{1-\alpha} f(t)$ exists, endowed with the norm $\|f\|_{C_{1-\alpha}} = \sup |t^{1-\alpha} f(t)|; t \in (0,b]$, we extend $\widetilde{z}$ to $(-\infty,b]$, also denoted by $\widetilde{z}$, defined by

$$\widetilde{z}(t) = \begin{cases} t^{1-\alpha} z(t), & t \in (0,b], \\ 0, & t \in (-\infty,0]. \end{cases}$$

It is easily seen that if $y(.)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\widetilde{y_s}) ds + \phi(0)t^{\alpha-1}, \qquad t > 0,$$

we can decompose $y(.)$ as $y(t) = \phi(t) + z(t)$, which implies that $\widetilde{y}_t = \widetilde{\phi_t} + \widetilde{z}_t$ for $t \in (0,b]$, and the function $z(.)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\widetilde{\phi}_s + \widetilde{z_s}) ds + \phi(0)t^{\alpha-1}, \qquad t > 0. \tag{13}$$

Set $W = \{z : (-\infty, b] \to X; z|_{(0,b]} \in C_{1-\alpha}((0, b]; X), z_0 = 0\}$. For $z \in W$, define $\|z\|_W = \|z_0\|_{\mathcal{B}} + \|z\|_{C_{1-\alpha}}$. Then $(W, \|z\|_W)$ becomes a Banach space. Define an operator $P : W \to W$ by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi_s} + \widetilde{z_s}) ds + \phi(0) t^{\alpha-1}, \qquad t > 0.$$

We will prove by the Schauder's fixed point theorem that P has at least a fixed point z, and hence $z + \widetilde{\phi}$ is a solution to (1)-(2).

First note that the continuity of P can be derived by $(H2)$ and the Lebesgue dominated convergence theorem. We now show that P maps bounded subsets in W into bounded subsets. Let $B_r = \{z \in W; \|z\|_W \le r\}$. Then, for any $z \in B_r$ and $t \in (0, b]$, we have

$$\begin{aligned}\|t^{1-\alpha}(Pz)(t)\| \le & \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, \widetilde{\phi_s} + \widetilde{z_s})\| ds + \|\phi(0)\| \\ \le & \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \Omega(\|\widetilde{\phi_s} + \widetilde{z_s}\|_{\mathcal{B}}) ds + \|\phi(0)\|.\end{aligned}$$

Since

$$\|\widetilde{z}_s + \widetilde{\phi}_s\|_{\mathcal{B}} \le K(s) \sup_{0 \le \tau \le s} \|\widetilde{z}(\tau)\| + M(s)\|\widetilde{z}_0\|_{\mathcal{B}} + K(s) \sup \|\widetilde{\phi}(\tau)\| + M(s)\|\widetilde{\phi_0}\|_{\mathcal{B}} \le K_b r + M_b \|\phi\|_{\mathcal{B}},$$

where $M_b = \sup_{0 \le s \le b} M(s)$. It follows from (H2) and Holder's inequality that

$$\begin{aligned}\|t^{1-\alpha}(Pz)(t)\| \le & \frac{b^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) ds \Omega(K_b r + M_b \|\phi\|_{\mathcal{B}}) + \|\phi(0)\| \\ \le & \frac{b^{1-\alpha}}{\Gamma(\alpha)} \Omega(K_b r + M_b \|\phi\|_{\mathcal{B}}) (\int_0^t (t-s)^{(\alpha-1)q} ds)^{1/q} \|\eta\|_p + \|\phi(0)\|. \\ \le & \Omega(K_b r + M_b \|\phi\|_{\mathcal{B}}) \frac{b^2 \|\eta\|_p}{\Gamma(1+\alpha)} + \|\phi(0)\| := l,\end{aligned}$$

where $\|\eta\|_p = (\int_0^b |\eta(s)|^p ds)^{1/p}$ and $1/p + 1/q = 1, (\alpha - 1)q > -1$. Therefore, $\|Pz\|_W \le l$ for every $z \in B_r$, which implies that P maps bounded subsets into bounded subsets in W.

Next we prove that $\widetilde{PB}$ is equicontinuous for every bounded subsets $B \subset W$. Let $z \in B_r$ and $t_1, t_2 \in (0, b]$ with $t_1 < t_2$, then we have

$$\begin{aligned}&\|t_2^{1-\alpha}(Pz)(t_2) - t_1^{1-\alpha}(Pz)(t_1)\| \\ \le & \frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} \|f(s, \widetilde{\phi}_s + \widetilde{z}_s)\| ds + \frac{t_1^{1-\alpha}}{\Gamma(\alpha)} [\int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) \|f(s, \widetilde{\phi}_s + \widetilde{z}_s)\| ds \\ & + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|f(s, \widetilde{\phi}_s + \widetilde{z}_s)\| ds] \\ \le & \frac{t_2^{1-\alpha} - t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} \eta(s) \Omega(\|\widetilde{\phi}_s + \widetilde{z}_s\|_{\mathcal{B}}) ds \\ & + \frac{t_1^{1-\alpha}}{\Gamma(\alpha)} [\int_0^{t_1} ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}) \eta(s) \Omega(\|\widetilde{\phi}_s + \widetilde{z}_s\|_{\mathcal{B}}) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \eta(s) \Omega(\|\widetilde{\phi}_s + \widetilde{z}_s\|_{\mathcal{B}}) ds]\end{aligned}$$

$$\leq \frac{(t_z^{1-\alpha} - t_1^{1-\alpha})\Omega(r_0)}{\Gamma(\alpha)} (\int_0^{t_2} (t_2 - s)^{(\alpha-1)q} ds)^{1/q} (\int_0^{t_2} \eta^p(s) ds)^{1/p}$$
$$+ \frac{t_1^{1-\alpha}\Omega(r_0)}{\Gamma(\alpha)} [(\int_0^{t_1} ((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1})^q ds)^{1/q} (\int_0^{t_1} \eta^p(s) ds)^{1/p}$$
$$+ (\int_{t_1}^{t_2} (t_2 - s)^{(\alpha-1)q} ds)^{1/q} (\int_0^{t_1} \eta^p(s) ds)^{1/p}]$$
$$\leq \frac{\|\eta\|_p \Omega(r_0)}{r_1 \Gamma(\alpha)} (b^{r_2}(t_z^{1-\alpha} - t_1^{1-\alpha}) + b^{1-\alpha}(2(t_2 - t_1)^{r_2} + (t_2^{r_2} - t_1^{r_2}))),$$

where $r_0 = K_b r + M_b \|\phi\|_{\mathcal{B}}$, $r_1 = ((\alpha-1)q+1)^{1/q}$ and $r_2 = [(\alpha-1)q+1]/q > 0$. It follows that $\|t_2^{1-\alpha}(Pz)(t_2) - t_1^{1-\alpha}(Pz)(t_1)\| \to 0$ as $t_2 - t_1 \to 0$, and the convergence is independent of $z \in B_r$, which implies that the set $\widetilde{PB_r}$ is equicontinuous.

Now we have to verify that there exists a closed convex bounded subset B_{r_0}, such that $PB_{r_0} \subset B_{r_0}$. We derive from inequlity (12), i.e., $\lim_{r\to+\infty} \frac{\Omega(r)}{r} < \frac{\Gamma(1+\alpha)}{K_b b^2 \|\eta\|_p}$, that there exists a constant $r_0 > 0$ such that

$$\frac{b^2\|\eta\|_p}{\Gamma(1+\alpha)} \Omega(K_b r_0 + M_b\|\phi\|_{\mathcal{B}}) + \|\phi(0)\| < r_0.$$

Define $B_{r_0} = \{z \in W; \|z\|_W \leq r_0\}$. Then B_{r_0} is closed, convex and bounded in W. Then, for every $z \in B_{r_0}$ and $t \in (0, b]$, similar to the proof of P maps bounded subsets in W into bounded subset, we have

$$\begin{aligned} \|t^{1-\alpha}(Pz)(t)\| \leq & \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, \widetilde{\phi}_s + \widetilde{z}_s)\| ds + \|\phi(0)\| \\ \leq & \frac{b^{1-\alpha}}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \eta(s) \Omega(\|\widetilde{\phi}_s + \widetilde{z}_s\|_{\mathcal{B}}) ds + \|\phi(0)\| \\ \leq & \Omega(K_b r_0 + M_b\|\phi\|_{\mathcal{B}}) \frac{b^2\|\eta\|_p}{\Gamma(1+\alpha)} + \|\phi(0)\| \leq r_0. \end{aligned}$$

It then follows that $\|Pz\|_W \leq r_0$ for all $z \in Br_0$, and hence $PB_{r_0} \subset B_{r_0}$.

Now we prove that there exists a compact subset $M \subset B_{r_0}$ such that $PM \subset M$. We first construct a series of sets $\{M_n\} \subset B_{r_0}$ by

$$M_0 = B_{r_0},\ M_1 = \overline{conv}PM_0, \quad M_{n+1} = \overline{conv}PM_n,\ \ n = 1, 2, \cdots.$$

From the above proof it is easy to see that $M_{n+1} \subset M_n$ for $n = 1, 2, \cdots$ and each $\widetilde{M}_n$ is equi-continuous. Further, from Lemma 2.5, 2.6 and 2.7 we can derive that

$$\begin{aligned} \beta(\widetilde{M}_{n+1}(t)) = & \beta(t^{1-\alpha} M_{n+1}(t)) = \beta(t^{1-\alpha} PM_n(t)) \\ = & \beta[\tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{M}_{ns}) ds + \phi(0)] \\ \leq & \beta[\tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{M}_{ns}) ds] \\ \leq & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \beta(f(s, \widetilde{\phi}_s + \widetilde{M}_{ns})) ds] \\ \leq & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{-\infty<\theta\leq 0} \beta(\widetilde{\phi}_s(\theta) + \widetilde{M}_{ns}(\theta)) ds \\ \leq & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{-\infty<\theta\leq 0} [\beta(\widetilde{\phi}_s(\theta)) + \beta(\widetilde{M}_{ns}(\theta))] ds \\ \leq & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{-\infty<\theta\leq 0} \beta(\widetilde{M}_{ns}(\theta)) ds \\ = & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{-\infty<\theta\leq 0} \beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha} M_n(\theta+s)) ds \\ = & \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \textstyle\int_0^t (t-s)^{\alpha-1} \eta(s) \sup_{-s\leq\theta\leq 0} \beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha} M_n(\theta+s)) ds \end{aligned} \tag{14}$$

From the proof above we know that $\{(\theta+s)^{1-\alpha}M_n(\theta+s)\}$ is equicontinuous at θ and $(\theta+s)^{1-\alpha}M_n(\theta+s)\mid_{\theta=-s}=\{\phi(0)\}$. So from Lemma 2.5 we know that the function $\theta\mapsto\beta((\theta+s)^{1-\alpha}M_n(\theta+s))$ is continuous, and $\beta((\theta+s)^{1-\alpha}M_n(\theta+s))\mid_{\theta=-s}=0$. Therefore, if the supermum $\sup_{-s\le\theta\le 0}\beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha}M_n(\theta+s))$ is taken at $\theta=-s$, then $\sup_{-s\le\theta\le 0}\beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha}M_n(\theta+s))\equiv 0$. Otherwise, there exist $\delta_n>0$, such that

$$\begin{aligned}
&\sup_{-s\le\theta\le 0}\beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha}M_n(\theta+s))\\
&=\sup_{-s+\delta_n\le\theta\le 0}\beta(s^{1-\alpha}(\theta+s)^{\alpha-1}(\theta+s)^{1-\alpha}M_n(\theta+s))\\
&=\sup_{\delta_n\le\tau\le s}\beta(s^{1-\alpha}\tau^{\alpha-1}\tau^{1-\alpha}M_n(\tau))\\
&\le\sup_{\delta_n\le\tau\le s}\beta(s^{1-\alpha}\delta_n^{\alpha-1}\tau^{1-\alpha}M_n(\tau))\\
&=s^{1-\alpha}\delta_n^{\alpha-1}\sup_{0\le\tau\le s}\beta(\widetilde{M}_n(\tau)).
\end{aligned}\tag{15}$$

Substituting (15) into (14) we obtain that

$$\beta(\widetilde{M}_{n+1}(t))\le\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)s^{1-\alpha}\delta_n^{\alpha-1}\sup_{0\le\tau\le s}\beta(\widetilde{M}_n(\tau))ds\tag{16}$$

for $n=1,2,\cdots$. Define the functions $f_n(t)=\sup_{\tau\in[0,t]}\beta(\widetilde{M}_n(\tau))$ for $n=1,2,\cdots$ and take supermum on both sides of (16). We get due to Lemma 3.9 that

$$f_{n+1}(t)\le\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)s^{1-\alpha}\delta_n^{\alpha-1}f_n(s)ds\tag{17}$$

for $n=1,2,\cdots$. The fact that $M_{n+1}\subset M_n$ implies that $f_{n+1}(t)\le f_n(t)$ for all $t\in[0,b]$ and $n=1,2,\cdots$, and each f_n is continuous on $[0,b]$. Therefore, the limit $\lim_n f_n(t)=f(t)$ exists for $t\in[0,b]$. We claim that $f(t)=0$ for all $t\in[0,b]$. In fact, take $\inf_{n\le 1}\delta_n=\delta$. If $\delta>0$, then inequality (17) implies that

$$f_{n+1}(t)\le\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)s^{1-\alpha}\delta^{\alpha-1}f_n(s)ds\tag{18}$$

for all $t\in[0,b]$ and $n=1,2,\cdots$. Taking limit as $n\to\infty$ in (18) we get

$$f(t)\le\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\eta(s)s^{1-\alpha}\delta^{\alpha-1}f(s)ds$$

for all $t\in[0,b]$. An application of Lemma 3.7 yields $f(t)=0$ for all $t\in[0,b]$.

If $\delta=0$, then there are two cases. In case when there exists an n_0 such that $\delta_{n_0}=0$, $f_{n_0}(t)=0$ for all $t\in[0,b]$ according to the definition of δ_n, the inequality (17) implies that $f_n(t)=0$ for all $n>n_0$ and $t\in[0,b]$, and hence $f(t)=\lim_n f_n(t)=0$ for all $t\in[0,b]$. In case when $\delta_n\ne 0$ for all $n\ge 1$, $\lim_n\delta_n=0$, the definition of f_n and δ_n imply that $\lim_n f_n(t)=0=f(t)$ for all $t\in[0,b]$. So we have $f(t)=0$ for all $t\in[0,b]$ in each case, as claimed. Therefore, $\cap_{n=1}^{\infty}M_n=M$ is nonempty and compact in W due to Lemma 2.5, and $PM\subset M$ by the definition of M_n.

Up to now we have verified that there exists a nonempty bounded convex and compact subset $M\subset W$ such that $PM\subset M$. An employment of Schauder's fixed point theorem shows that there exists at least a fixed point z of P in M. Then $y=z+\widetilde{\phi}$ is the solution to (1)-(2) on $[0,b]$, which completes the proof. □

Below we consider the existence result which is based on the Lipschitz condition. We need the following hypothesis.

(H3) There exists a constant $L>0$ such that

$$\|f(t,u)-f(t,u)\|\le L\|u-v\|_{\mathcal{B}}$$

for $t\in(0,b]$ and every $u,v\in\mathcal{B}$.

Theorem 4.3. *Assume that (H1) and (H3) hold. Then there exists a unique solution to* (1)-(2) *on* $(-\infty, b]$.

Proof. As in the proof of Theorem 4.2, we define the operator $P : W \to W$ by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \phi(0) t^{\alpha-1}, \qquad t > 0. \tag{19}$$

If $z \in W$ is a fixed point of P, then $y = z + \widetilde{\phi}$ is a solution of (1)-(2).

Let $K(b) = \sup\{K(t); t \in (0, b]\}$, where $K(.)$ is the function that appeared in Definition 2.4. Let $N = [b(2LK_b b^{1-\alpha}/\Gamma(1+\alpha))^{1/\alpha}]$, and $h_i = ib/N$. Then $0 < h_1 < h_2 < ... < h_N = b$ and

$$\frac{LK_b b^{1-\alpha}(h_{i+1} - h_i)^\alpha}{\Gamma(1+\alpha)} < \frac{1}{2} \tag{20}$$

for $i = 1, 2, ..., N$.

We first focus on the interval $(0, h_1]$. Let $W_1 = \{z : (-\infty, h_1] \to X; z|_{(0,h_1]} \in C_{1-\alpha}((0, h_1]; X), z|_0 = 0\}$ and define $\|z\|_{W_1} = \|z_0\|_{\mathcal{B}} + \sup\{|t^{1-\alpha} z(t)|; 0 < t < h_1\} = \sup\{|t^{1-\alpha} z(t)|; 0 < t < h_1\}$ for $z \in W_1$.Then $(W_1, \|z\|_{W_1})$ is a Banach space. Define the operator $P_1 : W_1 \to W_1$ by

$$(P_1 z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z_s}) ds + \phi(0) t^{\alpha-1}. \qquad t \in (0, h_1]. \tag{21}$$

For $z, z^* \in W_1$ and $t \in (0, h_1]$, we have

$$\begin{aligned}
\|t^{1-\alpha}(P_1 z)(t) - t^{1-\alpha}(P_1 z^*)(t)\| \leq& \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, \widetilde{\phi}_s + \widetilde{z}_s) - f(s, \widetilde{\phi}_s + \widetilde{z}^*_s)\| ds \\
\leq& \frac{Lt^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\widetilde{z_s} - \widetilde{z}^*_s\|_{\mathcal{B}} ds
\end{aligned}$$

Since

$$\begin{aligned}
\|\widetilde{z}_s - \widetilde{z}^*_s\|_{\mathcal{B}} \leq& K(s) \sup_{0 \leq \tau \leq s} \{\|\widetilde{z}(\tau) - \widetilde{z}^*(\tau)\|\} + M(s) \|\widetilde{z}_0 - \widetilde{z}^*_0\|_{\mathcal{B}} \\
\leq& K_b \sup_{0 \leq \tau \leq s} \{\|\tau^{1-\alpha} z(\tau) - \tau^{1-\alpha} z^*(\tau)\|\} \leq K_b \|z - z^*\|_{W_1},
\end{aligned}$$

we have

$$\|t^{1-\alpha}(P_1 z)(t) - t^{1-\alpha}(P_1 z^*)(t)\| \leq \frac{LK_b t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \|z - z^*\|_{W_1} \leq \frac{LK_b b^{1-\alpha} h_1^\alpha}{\Gamma(1+\alpha)} \|z - z^*\|_{W_1},$$

and hence

$$\|P_1 z - P_1 z^*\|_{W_1} \leq \frac{LK_b b^{1-\alpha} h_1^\alpha}{\Gamma(1+\alpha)} \|z - z^*\|_{W_1}. \tag{22}$$

From (20) and the Banach contraction principle we know that there exists a unique $z \in W_1$ satisfying

$$z(t) = \frac{1}{\alpha} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \phi(0) t^{\alpha-1} \tag{23}$$

for $t \in (0, h_1]$, which is the unique solution to the integral equation (13) on the interval $(0, h_1]$.

Next we consider the interval $(h_1, h_2]$. Restrict the function $z \in W$ on the interval $(h_1, h_2]$ to construct W_2 and define $\|z\|_{W_2} = \|z_0\|_{\mathcal{B}} + \sup\{|t^{1-\alpha} z(t)|; h_1 < t \le h_2\} = \sup\{|t^{1-\alpha} z(t)|; h_1 < t \le h_2\}$ for $z \in W_2$. Then $(W_2, \|z\|_{W_2})$ is Banach space . For $t \in (h_1, h_2]$, rewrite equation (13) as

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_{h_1}^{t} (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{h_1} (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \phi(0) t^{\alpha-1} \tag{24}$$

Since the function z is uniquely defined on $(0, h_1]$, the second integral can be considered as a known function. Using the same arguments as above, we can obtain that there exists a unique function $z \in W_2$ satisfying

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_{h_1}^{t} (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{h_1} (t-s)^{\alpha-1} f(s, \widetilde{\phi}_s + \widetilde{z}_s) ds + \phi(0) t^{\alpha-1} \tag{25}$$

for $t \in (h_1, h_2]$, which is the unique solution to the integral equation (13) on the interval $(h_1, h_2]$. Taking the next interval $(h_2, h_3]$, repeating this process, we conclude that there exists a unique solution to the integral equation (13) on the interval $(0, h_N] = (0, b]$. Set $y = z + \widetilde{\phi}$, then y is the unique solution to the fractional differential equation (1)-(2). □

Remark 4.4. *In the most of similar results, there is a restriction on the Lipschitz constant. Here we do not have any restriction on the Lipschitz constant L.*

5 An example

In this section, we discuss an example to illustrate our results. For any real constant $\gamma > 0$ we set

$$C_\gamma = \{\varphi \in C((-\infty, 0]; \mathbf{R}) : \lim_{\theta \to -\infty} e^{\gamma\theta} \varphi(\theta) \text{ exists in } \mathbf{R}\},$$

and set

$$|\varphi|_\gamma = \sup\{e^{\gamma\theta} |\varphi(\theta)| : -\infty < \theta \le 0\}$$

for $\varphi \in C_\gamma$. Then by [16, Theorem 3.7], C_γ satisfies (A1), (A2) and (B) in Definition 2.4 with $H = 1$, $K(t) = \max\{1, e^{-\gamma t}\}$ and $M(t) = e^{-\gamma t}$, C_γ is a phase space.

Let us consider the weighted fractional differential equation with infinite delay

$$D^\alpha y(t) = \frac{c e^{-\gamma t + t} |\tilde{y}_t|_\gamma}{(e^t + e^{-t})(1 + |\tilde{y}_t|_\gamma)}, \quad t \in (0, b],\ \alpha \in (0, 1), \tag{26}$$

$$y(t) = \varphi(t), \quad t \in (-\infty, 0], \tag{27}$$

where $\varphi \in C_\gamma$ and $c > 0$ can be any constant. Set

$$f(t, x) = \frac{c e^{-\gamma t + t} x}{(e^t + e^{-t})(1 + x)}, \quad (t, x) \in (0, b] \times \mathbf{R}^+.$$

Then for any $x, y \in \mathbf{R}^+$, one has

$$|f(t, x) - f(t, y)| = \frac{c e^{-\gamma t + t} |x - y|}{(e^t + e^{-t})(1 + x)(1 + y)} \le \frac{c e^t |x - y|}{(e^t + e^{-t})} \le c|x - y|,$$

i.e., f is Lipschitz with respect to the second variable with Lipschitz constant c. Then by Theorem 4.3 the problem (26)-(27) has a unique solution on $(-\infty, b]$.

In [2], the authors discussed an example similar to the problem (26)-(27) without weight. They have to suppose that $\varphi(0) = 0$ since otherwise the solution y may be unbounded at the right neighbourhood of 0. Here we do not need this restriction. Furthermore, we don't have any restriction on the constant c.

Acknowledgement: This research was supported by the National Natural Science Foundation of China (11271316 and 11571300) and TAPP of Jiangsu Higher Education Institutions(No. PPZY2015B109). Zhenbin Fan's work was also supported by Qing Lan Project of Jiangsu Province of China and High-Level Personal Support Project of Yangzhou University.

References

[1] Agarwal R.P., Lupulescu V., O'Regan D., ur RahmanG., Fractional calculus and fractional differential equations in nonreflexive Banach spaces, Commun. Nonlinear Sci. Numer. Simulat., 2015, 20, 59-73.
[2] Benchohra M., Henderson J., Ntouyas S.K., Ouahab A., Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 2008, 338, 1340-1350.
[3] Darwish M.A., Ntouyas S.K., On initial and boundary value problem for fractional order mixed type functional differential inclusions, Comput. Math. Appl., 2010, 59, 1253-1265.
[4] Delbosco D., Rodino L., Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 1996, 204, 609-625.
[5] Diethelm K., The analysis of fractional differential equations, Lecture Notes in Mathmatics 2004, Springer-Verlag, Berlin, 2010.
[6] Dong Q., Existence and continuous dependence for weighted fractional differential equations with infinite delay, Adv. Differ. Equ., 2014, 2014, No. 190, 1-10.
[7] Dong Q., Existence and viability for fractional differential equations with initial conditions at inner points, J. Nonlinear Sci. Appl., 2016, 9, 2590-2603.
[8] Henderson J., Ouahab A., Fractional functional differential inclusions with finite delay, Nonlinear Anal. 2009, 70, 2091-2105.
[9] Kilbas A.A., Srivastava H.M., Trujillo J.J., Theory and applications of fractional differential equations, North-Holland Math. Stud., vol 204, Elsevier Science B.V., Amsterdam, 2006.
[10] Peng S., Wang J., Existence and Ulam-Hyers stability of ODEs involving two Caputo fractional derivatives, Electron. J. Qual. Theory Differ. Equ., 2015, 2015, No. 52, 1-16.
[11] Kamaljeet, Bahuguna D., Monotone iterative technique for nonlocal fractional differential equations with finite delay in a Banach space, Electron. J. Qual. Theory Differ. Equ., 2015, 2015, No. 9, 1-16.
[12] Lakshmikantham V., Devi J.V., Theory of fractional differential equations in Banach space, European J. Pure Appl. Math., 2008, 1(1), 38-45.
[13] Mei Z., Peng J., Gao J., Existence and uniqueness of solutions for nonlinear general fractional differential equations in Banach spaces, Indagationes Mathematicae, 2015, 26, 669-678.
[14] Wang J., Ibrahim A.G., Feckan M., Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces, Appl. Math. Comput., 2015, 257, 103-118.
[15] Hale J.K., Kato J., Phace space for retarded equations with infinite delay, Funkcial. Ekvac. 1978, 21, 11-41.
[16] Hino Y., Murakami S., Naito T., Functional differential equations with infinite delay, Lecture Notes in Mathmatics 1473, Springer-Verlag, Berlin, 1991.
[17] Banas J., Goebel K., Measure of Noncompactness in Banach spaces, Lecture Notes in Pure and Applied Math., Vol 60, Dekker, New York, 1980.
[18] Dong Q., Li G., Existence of Solutions for Semilinear Differential Equations with Nonlocal Conditions in Banach Spaces, Electronic J. Quali. Theory Diff. Equati., 2009, 2009, No. 47, 1-13.
[19] Dong Q., Li G., Measure of noncompactness and semilinear nonlocal functional differential equations in Banach spaces, Acta Mathematica Sinica, English Series, 2015, 31, No. 1, 140-150
[20] Dong Q., Li G., Zhang J., Quasilinear Nonlocal intergrodifferential equations in Banach spaces, Electronic J. Diff. Equati., 2008, 2008, No. 19, 1-8.
[21] Heinz H.P., On the behavior of measure of noncompactness with respect to differentiation and integration of vector-valued functions, Nonlinear Anal. TMA., 1983, 7, 1351-1371.
[22] Kamenskii M., Obukhovskii V., Zecca P., Condensing multivalued maps and semilinear differential inclusions in Banach spaces, De Gruyter Ser. Nonlinear Anal. Appl., Vol.7, de Gruyter, Berlin, 2001.
[23] Chu S.C., Metcalf F.T., On Gronwall's inequality, Proc. Amer. Math. Soc., 1967, 18, 439-440.
[24] Feckan M., Pospisil M., Note on fractional difference Gronwall inequalities, Electron. J. Qual. Theory Differ. Equ., 2014, 2014, No. 44, 1-18.
[25] Henry D., Geometric theory of semilinear parabolic equations, Lecture Notes in Math., vol. 840, Springer-Verlag, New York/Berlin, 1981.
[26] Ye H., Gao J., Ding Y., A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl., 2007, 328, 1075-1081.