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Review Article

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Some notes to existence and stability of the positive periodic solutions for a delayed nonlinear differential equations

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Abstract: The paper deals with the existence of positive ω-periodic solutions for a class of nonlinear delay differential equations. For example, such equations represent the model for the survival of red blood cells in an animal. The sufficient conditions for the exponential stability of positive ω-periodic solution are also considered.

Keywords: Positive periodic solution, Delay differential equation, Nonlinear, Exponential stability, Red blood cells, Banach space

MSC: 34K13

1 Introduction

In this paper, we consider the existence of positive ω-periodic solutions for the nonlinear delay differential equation of the form

$$\dot{x}(t) = -p(t)x(t) + \sum_{i=1}^{n} q_i(t) f(x(\tau_i(t))), \quad t \geq t_0. \tag{1}$$

With respect to (1) throughout the paper we will assume the following conditions:

(i) $p, q_i \in C([t_0, \infty), (0, \infty)),\ i = 1, \dots, n,\ f \in C^1(R, R),\ f(x) > 0$ for $x > 0$,

(ii) $\tau_i \in C([t_0, \infty), (0, \infty)),\ \tau_i(t) < t$ and $\lim_{t\to\infty} \tau_i(t) = \infty,\ i = 1, \dots, n.$

In the last several years, the problem of the existence of positive periodic solutions for the nonlinear delay differential equations received a considerable attention. It is due to the fact that such equations have found a variety of applications in several fields of natural sciences. They have been proposed as models for physiological, ecological and physical processes, neural interactions [1–3], etc.

One important question is whether these equations can support the existence of positive periodic solutions. Such question has been studied extensively by a number of authors. For example the authors in [2, 4–9] studied the existence, multiplicity and nonexistence of positive periodic solutions for the nonlinear delay differential equations. Periodic properties of solutions of some special types of differential equations are discussed in [10, 11]. Zhang, Wang and Yang [12] and Lin [13] studied the existence and exponential stability of positive periodic solutions.

In this paper, we will obtain existence criteria for the positive ω-periodic solution of (1) and sufficient conditions for the exponential stability of such solution. The existence results in the literature are largely based on the

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assumption that the functions $p(t),\ q_i(t),\ i = 1,\dots,n$ are ω-periodic. It is interesting to know if there is a positive periodic solution of (1) when the periodicity conditions for the functions $p(t),\ q_i(t),\ i = 1,\dots,n$ are not satisfied. This substantially extends and improves the results in [7–9, 14] where the exponential stability for the positive periodic solution is not studied.

The following fixed point theorem will be used to prove the main results in the next section.

Theorem 1.1 (Schauder's Fixed Point Theorem [6, 15])**.** *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $S : \Omega \to \Omega$ be a continuous mapping such that $S(\Omega)$ is a relatively compact subset of X. Then S has at least one fixed point in Ω, that is, there exists an $x \in \Omega$ such that $Sx = x$.*

The remaining of this paper is organized as follows. In Section 2, we consider the existence of positive periodic solutions. In Section 3, the exponential stability of such solution is treated and in Section 4, the obtained results are applied to the model for the survival of red blood cells and illustrated with an example.

2 Existence of positive periodic solutions

In this section, we will study the existence of positive ω-periodic solutions of (1). We choose T sufficiently large so that $\tau_i(t) \geq t_0$ for $t \geq T,\ i = 1,\dots,n$.

Lemma 2.1. *Suppose that there exist functions $k_i \in C([T,\infty),(0,\infty)),\ i = 1,\dots,n$ such that*

$$\int_t^{t+\omega} \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds = 0, \quad t \geq T. \tag{2}$$

Then the function

$$w(t) = \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big), \quad t \geq T,$$

is ω-periodic.

Proof. For $t \geq T$, we obtain

$$\begin{aligned} w(t+\omega) &= \exp\Big(\int_T^{t+\omega} \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big) \\ &= \exp\Big(\int_T^{t} \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big) \cdot \exp\Big(\int_t^{t+\omega} \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big) = w(t). \end{aligned}$$

Thus, the function $w(t)$ is ω-periodic. □

Theorem 2.2. *Suppose that there exist functions $k_i \in C([T,\infty),(0,\infty)),\ i = 1,\dots,n$ such that* (2) *holds and*

$$\begin{aligned} & f\Big(\exp\Big(\int_T^{\tau_j(t)} \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big)\Big) \\ & \times \exp\Big(\int_T^{t} \Big[p(s) - \sum_{i=1}^n q_i(s)k_i(s)\Big]\, ds\Big) = k_j(t),\ \tau_j(t) \geq T,\ j = 1,\dots,n. \end{aligned} \tag{3}$$

Then, (1) *has a positive ω-periodic solution.*

Proof. Let $X = \{x \in C([t_0, \infty), R)\}$ be the Banach space with the norm $||x|| = \sup_{t \geq t_0} |x(t)|$. With regard to Lemma 2.1 for the function

$$w(t) = \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\,ds\Big), \quad t \geq T,$$

we get $0 < m \leq w(t) \leq M,\ t \geq T$, where

$$m = \min_{t\in[T,\infty)} \Big\{\exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\,ds\Big)\Big\},$$

$$M = \max_{t\in[T,\infty)} \Big\{\exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\,ds\Big)\Big\}. \tag{4}$$

We now define a closed, bounded and convex subset Ω of X as follows:

$$\begin{aligned}\Omega = \{x \in X : {} & x(t+\omega) = x(t), \quad t \geq T,\\ & m \leq x(t) \leq M, \quad t \geq T,\\ & k_i(t)x(t) = f(x(\tau_i(t))), \quad i = 1,\ldots,n,\ t \geq T,\\ & x(t) = 1, \quad t_0 \leq t \leq T\}.\end{aligned}$$

Define the operator $S : \Omega \to X$ as follows:

$$(Sx)(t) = \begin{cases} \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)\dfrac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big), & t \geq T,\\ 1, \quad t_0 \leq t \leq T. \end{cases}$$

We will show that for any $x \in \Omega$, we have $Sx \in \Omega$. For every $x \in \Omega$ and $t \geq T$, we get

$$(Sx)(t) = \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big) = \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\,ds\Big) \leq M.$$

Furthermore, for $t \geq T$ and $x \in \Omega$, we obtain

$$(Sx)(t) = \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)k_i(s)\Big]\,ds\Big) \geq m.$$

For $t \in [t_0, T]$ we have $(Sx)(t) = 1$. By hypothesis (3) for every $x \in \Omega$ and $\tau_j(t) \geq T,\ j = 1,\ldots,n$, we get

$$\begin{aligned} f\big((Sx)(\tau_j(t))\big) &= f\Big(\exp\Big(\int_T^{\tau_j(t)} \Big[-p(s) + \sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big)\Big)\\ &= f\Big(\exp\Big(\int_T^{\tau_j(t)} \Big[-p(s) + \sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big)\\ &\quad\times \exp\Big(\int_T^t \Big[p(s) - \sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big)\\ &\quad\times \exp\Big(\int_T^t \Big[-p(s) + \sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]\,ds\Big)\end{aligned}$$

$$= f\Big(\exp\Big(\int\limits_T^{\tau_j(t)} \Big[-p(s)+\sum_{i=1}^n q_i(s)k_i(s)\Big]ds\Big)\Big)$$
$$\times \exp\Big(\int\limits_T^{t} \Big[p(s)-\sum_{i=1}^n q_i(s)k_i(s)\Big]ds\Big)(Sx)(t)$$
$$= k_j(t)(Sx)(t), \quad j=1,\dots,n.$$

Finally, we will show that for $x\in\Omega,\ t\geq T$ the function $(Sx)(t)$ is ω-periodic. For $x\in\Omega,\ t\geq T$ and according to (2), we obtain

$$(Sx)(t+\omega) = \exp\Big(\int\limits_T^{t+\omega} \Big[-p(s)+\sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]ds\Big)$$
$$= \exp\Big(\int\limits_T^{t} \Big[-p(s)+\sum_{i=1}^n q_i(s)k_i(s)\Big]ds\Big)$$
$$\times \exp\Big(\int\limits_t^{t+\omega} \Big[-p(s)+\sum_{i=1}^n q_i(s)k_i(s)\Big]ds\Big) = (Sx)(t).$$

Thus $(Sx)(t)$ is ω-periodic on $[T,\infty)$. Therefore we have proved that $Sx\in\Omega$ for any $x\in\Omega$.

We now show that S is completely continuous. At first we will show that S is continuous. Let $x_k = x_k(t)\in\Omega$ be such that $x_k(t)\to x(t)\in\Omega$ as $k\to\infty$. For $t\geq T$, we get

$$|(Sx_k)(t)-(Sx)(t)| = \Big|\exp\Big(\int\limits_T^{t} \Big[-p(s)+\sum_{i=1}^n q_i(s)\frac{f(x_k(\tau_i(s)))}{x_k(s)}\Big]ds\Big)$$
$$- \exp\Big(\int\limits_T^{t} \Big[-p(s)+\sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]ds\Big)\Big|.$$

Since $f(x_k(\tau_i(t)))/x_k(t) \to f(x(\tau_i(t)))/x(t)$ as $k\to\infty$ for $i=1,2,\dots,n$, by applying the Lebesgue dominated convergence theorem, we obtain that (cf. [13, p.66], [16, p.95])

$$\lim_{k\to\infty} \|(Sx_k)(t)-(Sx)(t)\| = 0.$$

For $t\in[t_0,T]$ the relation above is also valid. This means that S is continuous.

Now, we will show that $S(\Omega)$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{Sx : x\in\Omega\}$ is uniformly bounded and equicontinuous on every finite subinterval of $[t_0,\infty)$. The uniform boundedness follows from the definition of Ω. It remains to prove the equicontinuity. Using (4), we get for $t\geq T$ and $x\in\Omega$:

$$\Big|\frac{d}{dt}(Sx)(t)\Big| = \Big|-p(t)+\sum_{i=1}^n q_i(t)\frac{f(x(\tau_i(t)))}{x(t)}\Big|\exp\Big(\int\limits_T^{t} \Big[-p(s)+\sum_{i=1}^n q_i(s)\frac{f(x(\tau_i(s)))}{x(s)}\Big]ds\Big)$$
$$= \Big|-p(t)+\sum_{i=1}^n q_i(t)k_i(t)\Big|\exp\Big(\int\limits_T^{t} \Big[-p(s)+\sum_{i=1}^n q_i(s)k_i(s)\Big]ds\Big) \leq M_1,\ M_1>0.$$

For $t\in[t_0,T]$ and $x\in\Omega$, we obtain:

$$\Big|\frac{d}{dt}(Sx)(t)\Big| = 0.$$

This shows the equicontinuity of the family $S(\Omega)$ and, therefore, S is completely continuous (cf. [6, p.265]). Hence $S(\Omega)$ is relatively compact. By Theorem 1.1, there is an $x_0\in\Omega$ such that $Sx_0 = x_0$. Therefore, by the definition of S, we have that $x_0(t)$ is a positive ω-periodic solution of (1). The proof is complete. □

3 Stability of positive periodic solution

In this section, we consider the exponential stability of the positive periodic solution of (1). Let $r = \min_{1 \le i \le n}\{\inf_{t \ge T} \tau_i(t)\}$. We denote $x(t;T,\varphi),\ t \ge r,\ \varphi \in C([r,T],(0,\infty))$ for a solution of (1) satisfying the initial condition $x(t;T,\varphi) = \varphi(t),\ t \in [r,T]$, where T is the initial point. Let $x(t) = x(t;T,\varphi),\ x_1(t) = x(t;T,\varphi_1)$ and $y(t) = x(t) - x_1(t),\ t \in [r,\infty)$. By (1), we get

$$\dot{y}(t) = -p(t)y(t) + \sum_{i=1}^{n} q_i(t)[f(x(\tau_i(t))) - f(x_1(\tau_i(t)))], \quad t \ge T.$$

By the mean value theorem, we obtain

$$\begin{aligned} \dot{y}(t) &= -p(t)y(t) + \sum_{i=1}^{n} q_i(t) f'(x_i^*)[x(\tau_i(t)) - x_1(\tau_i(t))], \quad f'(x) = \frac{df(x)}{dx}, \\ \dot{y}(t) &= -p(t)y(t) + \sum_{i=1}^{n} q_i(t) f'(x_i^*) y(\tau_i(t)), \quad t \ge T. \end{aligned} \tag{5}$$

Lemma 3.1. *Assume that* $|f'(x)| \le a,\ x \in (0,\infty),\ t - \tau_i(t) \le b,\ t \ge T,\ i = 1,\dots,n$ *and*

$$\sup_{t \ge T}\Big\{-p(t) + a\sum_{i=1}^{n} q_i(t)\Big\} < 0.$$

Then there exists $\lambda \in (0,1]$ *such that*

$$-p(t) + \lambda + a\,e^{\lambda b}\sum_{i=1}^{n} q_i(t) < 0 \quad \textit{for} \quad t \ge T.$$

Proof. Define a continuous function $H(u)$ by

$$H(u) = \sup_{t \ge T}\Big\{-p(t) + u + a\,e^{ub}\sum_{i=1}^{n} q_i(t)\Big\}, \quad u \in [0,1].$$

By hypothesis, we get

$$H(0) = \sup_{t \ge T}\Big\{-p(t) + a\sum_{i=1}^{n} q_i(t)\Big\} < 0.$$

According to the continuity of $H(u)$ and $H(0) < 0$, there exists $\lambda \in (0,1]$ such that $H(\lambda) < 0$, that is

$$-p(t) + \lambda + a\,e^{\lambda b}\sum_{i=1}^{n} q_i(t) < 0 \quad \text{for } t \ge T.$$

We have achieved the desired result. □

Next we will assume that the function

$$F(t,x,x_1,\dots,x_n) = -p(t)x(t) + \sum_{i=1}^{n} q_i(t) f(x_i(t)), \quad t \ge r,$$

satisfies Lipschitz-type condition with respect to $x, x_i > 0,\ i = 1,\dots,n$.

Definition 3.2. *Let* $x_1(t)$ *be a positive solution of* (1). *If there exist constants* $T_{\varphi,x_1},\ K_{\varphi,x_1}$ *and* $\lambda > 0$ *such that for every solution* $x(t;T,\varphi)$ *of* (1)

$$|x(t;T,\varphi) - x_1(t)| < K_{\varphi,x_1} e^{-\lambda t} \quad \textit{for all } t > T_{\varphi,x_1}.$$

Then $x_1(t)$ *is said to be exponentially stable.*

In the next lemma, we establish sufficient conditions for the exponential stability of the positive solution $x_1(t) = x(t; T, \varphi_1)$ of (1).

Lemma 3.3. *Suppose that* $|f'(x)| \le a,\ x \in (0,\infty),\ t - \tau_i(t) \le b,\ t \ge T,\ i = 1,\dots,n$ *and*

$$\sup_{t \ge T} \Big\{ -p(t) + a \sum_{i=1}^{n} q_i(t) \Big\} < 0.$$

Then there exists $\lambda \in (0,1]$ *such that*

$$|x(t;T,\varphi) - x(t;T,\varphi_1)| < K_{\varphi,x_1} e^{-\lambda t}, \quad t > T,$$

where $K_{\varphi,x_1} = \max_{t\in[r,T]} e^{\lambda T}|y(t)| + 1.$

Proof. We consider the Lyapunov function

$$L(t) = |y(t)|e^{\lambda t}, \quad t \ge r,\ \lambda \in (0,1].$$

We claim that $L(t) < K_{\varphi,x_1}$ for $t > T$. In order to prove it, suppose that there exists $t_* > T$ such that $L(t_*) = K_{\varphi,x_1}$ and $L(t) < K_{\varphi,x_1}$ for $t \in [r, t_*)$. Calculating the upper left derivative of $L(t)$ along the solution $y(t)$ of (5), we obtain

$$D^-(L(t)) \le -p(t)|y(t)|e^{\lambda t} + e^{\lambda t} \sum_{i=1}^{n} q_i(t) f'(x_i^*)|y(\tau_i(t))| + \lambda |y(t)|e^{\lambda t},\ t \ge T.$$

For $t = t_*$ and applying Lemma 3.1, we get

$$\begin{aligned}
0 \le D^-(L(t_*)) &\le [\lambda - p(t_*)]|y(t_*)|e^{\lambda t_*} + a\, e^{\lambda t_*} \sum_{i=1}^{n} q_i(t_*)|y(\tau_i(t_*))| \\
&= [\lambda - p(t_*)]|y(t_*)|e^{\lambda t_*} + a \sum_{i=1}^{n} q_i(t_*)|y(\tau_i(t_*))|e^{\lambda \tau_i(t_*)} e^{\lambda(t_* - \tau_i(t_*))} \\
&= [\lambda - p(t_*)]K_{\varphi,x_1} + a \sum_{i=1}^{n} q_i(t_*) L(\tau_i(t_*)) e^{\lambda(t_* - \tau_i(t_*))} \\
&< \Big[\lambda - p(t_*) + a\, e^{\lambda b} \sum_{i=1}^{n} q_i(t_*)\Big] K_{\varphi,x_1} < 0,
\end{aligned}$$

which is a contradiction. Therefore we obtain

$$L(t) = |y(t)|e^{\lambda t} < K_{\varphi,x_1} \text{ for } t > T \text{ and for some } \lambda \in (0,1].$$

The proof is complete. □

Theorem 3.4. *Suppose that* $|f'(x)| \le a,\ x \in (0,\infty),\ t - \tau_i(t) \le b,\ t \ge T,\ i = 1,\dots,n,$

$$\sup_{t \ge T} \Big\{ -p(t) + a \sum_{i=1}^{n} q_i(t) \Big\} < 0$$

and there exist functions $k_i \in C([T,\infty),(0,\infty)),\ i = 1,\dots,n$ *such that* (2)*,* (3) *hold. Then* (1) *has a positive* ω*-periodic solution which is exponentially stable.*

Proof. The proof follows from the Theorem 2.2 and Lemma 3.3. □

4 Model for the survival of red blood cells

In this section, we consider the existence of positive ω-periodic solutions for the nonlinear delay differential equation of the form

$$\dot{x}(t) = -p(t)x(t) + q(t)e^{-\gamma\, x(\tau(t))}, \quad t \geq t_0, \tag{6}$$

which is a special case of (1), where $q_1(t) = q(t)$, $q_i(t) = 0$, $i = 2, \dots, n$, and $f(x(\tau_1(t))) = \exp(-\gamma x(\tau(t)))$, $\gamma > 0$. We will also establish the sufficient conditions for the exponential stability of the positive periodic solution.

The autonomous case of (6) is given by:

$$\dot{x}(t) = -p\, x(t) + q\, e^{-\gamma\, x(t-\tau)}, \quad t \geq t_0,$$

and it has been used by Wazewska-Czyzewska and Lasota in [17] as a model for the survival of red blood cells in an animal. The function $x(t)$ denotes the number of red blood cells at time t. The positive constants p, q and γ are related to the production of red blood cells per unit of time and τ is the time required to produce red blood cells.

Rewriting the Theorem 3.4 to the equation (6) we obtain the next result.

Theorem 4.1. *Suppose that $\gamma > 0$, $t - \tau(t) \leq b$, $t \geq T$,*

$$\sup_{t \geq T} \{-p(t) + \gamma\, q(t)\} < 0 \tag{7}$$

and there exists function $k \in C([T, \infty), (0, \infty))$ such that

$$\int_t^{t+\omega} [-p(s) + q(s)k(s)]\, ds = 0, \quad t \geq T, \tag{8}$$

$$\ln k(t) = \int_T^t [p(s) - q(s)k(s)]\, ds - \gamma \exp\Big(\int_T^{\tau(t)} [-p(s) + q(s)k(s)]\, ds \Big), \ \tau(t) \geq T. \tag{9}$$

Then (6) has a positive ω-periodic solution which is exponentially stable.

Example 4.2. *Consider the nonlinear delay differential equation*

$$\dot{x}(t) = -p(t)x(t) + q(t)e^{-\gamma\, x(\tau(t))}, \quad t \geq t_0, \tag{10}$$

where $\gamma > 0$, $\tau(t) = t - \pi$,

$$p(t) = \frac{1}{4}(4 + e^{-t} + \sin t),$$
$$q(t) = \frac{1}{4}(4 + e^{-t}) \exp\Big(\frac{1}{4}(\cos t - \cos T)\Big) \exp\Big(\gamma e^{-0.25(\cos t + \cos T)}\Big), \ T > 0.$$

We choose

$$k(t) = \exp\Big(-\frac{1}{4}(\cos t - \cos T)\Big) \exp\Big(-\gamma e^{-0.25(\cos t + \cos T)}\Big).$$

Then for conditions (8), (9) and $\omega = 2\pi$, we get

$$\int_t^{t+\omega} [-p(s) + q(s)k(s)]\, ds = -\frac{1}{4} \int_t^{t+2\pi} \sin s\, ds = 0.$$

Therefore:

$$\int_T^t [p(s) - q(s)k(s)]\, ds - \gamma \exp\Big(\int_T^{t-\pi} [-p(s) + q(s)k(s)]\, ds \Big)$$

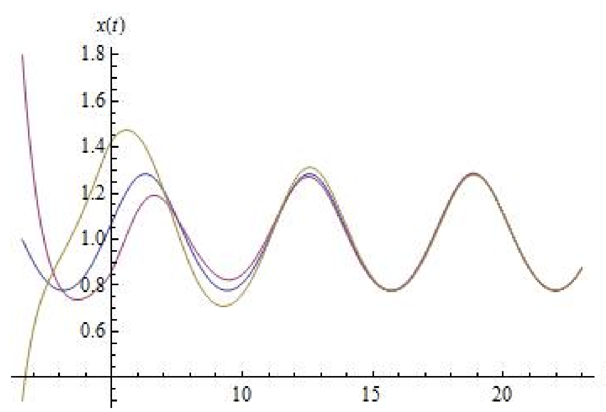
$$= \frac{1}{4}\int_T^t \sin s\, ds - \gamma \exp\Big(-\frac{1}{4}\int_T^{t-\pi} \sin s\, ds\Big)$$
$$= -\frac{1}{4}(\cos t - \cos T) - \gamma \exp\Big(-\frac{1}{4}(\cos t + \cos T)\Big) = \ln k(t),\ t \ge T + \pi.$$

The conditions (8), (9) *of Theorem 4.1 are satisfied and* (10) *has a positive* $\omega = 2\pi$ *periodic solution*

$$x(t) = \exp\Big(\int_T^t [-p(s) + q(s)k(s)]\, ds\Big) = \exp\Big(-\frac{1}{4}\int_T^t \sin s\, ds\Big)$$
$$= \exp\Big(\frac{1}{4}(\cos t - \cos T)\Big), \quad t \ge T.$$

If we put $\gamma = 0.4,\ T = \frac{\pi}{2}$, *then also the condition* (7) *is satisfied and solution* $x(t)$ *is exponentially stable. The numerical simulation in Figure 1 supports the conclusion.*

Fig. 1. Numerical simulation of exponential stability



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