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Research Article

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A class of tridiagonal operators associated to some subshifts

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Abstract: We consider a class of tridiagonal operators induced by not necessary pseudoergodic biinfinite sequences. Using only elementary techniques we prove that the numerical range of such operators is contained in the convex hull of the union of the numerical ranges of the operators corresponding to the constant biinfinite sequences; whilst the other inclusion is shown to hold when the constant sequences belong to the subshift generated by the given biinfinite sequence. Applying recent results by S. N. Chandler-Wilde et al. and R. Hagger, which rely on limit operator techniques, we are able to provide more general results although the closure of the numerical range needs to be taken.

Keywords: Tridiagonal operators, Random operators, Subshifts

MSC: 47B36, 47A12, 37B10

Introduction

Let $b = (b_i)_{i \in \mathbb{Z}}$ be a biinfinite sequence in $\mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A}$ is a finite set, called an alphabet. In this paper we study the operator $A_b \colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ defined as the tridiagonal operator

$$A_b = \begin{pmatrix} \ddots & \ddots & & & & & \\ \ddots & 0 & 1 & & & & \\ & b_{-2} & 0 & 1 & & & \\ & & b_{-1} & \boxed{0} & 1 & & \\ & & & b_0 & 0 & 1 & \\ & & & & b_1 & 0 & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}$$

where the rectangle marks the matrix entry at $(0, 0)$. When the alphabet is the set $\{-1, 1\}$, the corresponding operators are related to the so called “hopping sign model” introduced in [4] and subsequently studied in [1–3] and [5, 6]. We remark that results found in the literature focus on the case when b is a pseudoergodic sequence, that is to say, when every finite sequence of ± 1 appears somewhere in b. In this paper we aim to investigate the case when b is not necessary pseudergodic and although some authors have in fact pointed out that some of their results hold if the pseudoergodic condition is dropped, we try to formalize this approach. For this reason, we begin by thinking that b is an element in a full shift space and as such it is more usual to consider the alphabet $\mathcal{A}$ to be the set $\{0, 1\}$ rather that

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$\{-1, 1\}$. By means of elementary methods, we are able to establish similar results to the ones found in the literature and slightly generalize others. More concretely, we show that the numerical range of A_b has an inclusion-wise maximal, but that such bound is sharp not only when b is pseudoergodic but whenever the subshift generated by b (the closure of the orbit of b) contains the constant biinfinite sequences. With this motivation we employ recent limit operator techniques to extend our results to more general alphabets; however, for these generalizations the closure of the numerical range must be taken.

We feel our approach might be of interest as it employs elementary mathematics to obtain some results that are the motivation to state more general results. We divided this work into three sections. In the first section we review some of the fundamental results needed in the rest of the paper, both operator theoretical concepts and symbolic dynamics notions. In the second section we apply elementary techniques to bound the numerical range of A_b, for any $b \in \{0, 1\}^{\mathbb{Z}}$ and prove that such upper bound is sharp when the subshift generated by b contains the constant sequences. In the last section we generalize our results to more general alphabets by means of recent results which employ limit operator techniques.

1 Background

We review in this section some of the fundamental results and notation to be used throughout in the paper.

1.1 Operators

We are considering bounded linear operators on the Hilbert space $\ell^2(\mathbb{Z})$. Hence, every time we refer to an operator we assume it is a bounded linear operator on $\ell^2(\mathbb{Z})$. The inner product on $\ell^2(\mathbb{Z})$ is denoted by $\langle\ ,\ \rangle$. For an operator A we define the spectrum of A as

$$\sigma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\}.$$

The numerical range of A is defined by

$$W(A) = \left\{\langle Ax, x\rangle : x \in \ell^2(\mathbb{Z}),\ \|x\| = 1\right\}.$$

For easy reference, we state the following well known results (see e.g.[8, Chapter 1]) which are true in general Hilbert spaces.

Theorem 1.1. *Let S and T be operators and the identity operator is denoted by I. Then*

(i) $\sigma(T) \subset \mathbb{C}$, $\sigma(T) \neq \emptyset$ *and* $\sigma(T)$ *is compact set.*
(ii) $W(I) = \{1\}$, *and for* $\alpha, \beta \in \mathbb{C}$, $W(\alpha T + \beta I) = \alpha W(T) + \beta$.
(iii) $W(T + S) \subseteq W(T) + W(S)$.
(iv) $W(T)$ *is a convex subset of* $\mathbb{C}$.
(v) *If* T *is normal, then* $\overline{W(T)}$ *(the closure of the numerical range of* T*) is the convex hull of* $\sigma(T)$.
(vi) $\sigma(T) \subseteq \overline{W(T)}$.
(vii) *If* T *is self-adjoint then* $W(T) \subset \mathbb{R}$ *and* $\sigma(T) \subset \mathbb{R}$.

1.2 Symbolic dynamics

For details about symbolic dynamics we refer the reader to [7]. An alphabet $\mathcal{A}$ is nothing but a finite set. We refer to the elements of $\mathcal{A}$ as symbols. The set $\mathcal{A}^{\mathbb{Z}}$ of all biinfinite sequences of symbols from $\mathcal{A}$ is termed the full shift on $\mathcal{A}$. When writing down an element in $\mathcal{A}^{\mathbb{Z}}$ it is customary to distinguish the 0th coordinate with a dot. For example a biinfinite sequence b in $\{0, 1\}^{\mathbb{Z}}$ expressed as

$$b = (\ldots 0\,1\,1\,1\,0\,\dot{1}\,0\,0\,0\,1\,0 \ldots)$$

is such that its coordinates are $b_{-2} = 1, b_{-1} = 0, b_0 = 1, b_1 = 0$ and so on.

Given a biinfinite sequence $b = (b_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$, a block (or word) of b is defined to be a finite subsequence of b. The length of a block of b is the number of symbols it contains. If b belongs to $\mathcal{A}^{\mathbb{Z}}$ and $i < j$, then we will denote the block of coordinates in b from position i to position j by $b_{[i,j]}$. When a finite sequence u of symbols of $\mathcal{A}$ satisfies that $u = b_{[i,j]}$ for some i, j, we will say that u occurs in b. The blocks of the form $b_{[-k,k]}$ are called central blocks of b. Note then an equivalent way to say that $b \in \mathcal{A}$ is pseudoergodic is to require that every block of every size of symbols in $\mathcal{A}$ occur in b.

The full shift $\mathcal{A}^{\mathbb{Z}}$ is actually a metric space with a metric given by

$$\rho(b,c) = \begin{cases} 0 & \text{if } b = c \\ 1 & \text{if } b_0 \neq c_0 \\ 2^{-k} & \text{if } k \text{ is maximal so that } b_{[-k,k]} = c_{[-k,k]}. \end{cases}$$

Hence, we may say that two biinfinite sequences are close to each other when their central blocks agree. We denote the shift map $\varphi\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ as the map which moves the zeroth coordinate one slot to the right, that is $\varphi(b)_i = b_{i+1}$. A subshift X is a subspace of $\mathcal{A}^{\mathbb{Z}}$ which is closed and invariant under φ. The orbit of a point $b \in \mathcal{A}^{\mathbb{Z}}$ is the set of iterates $\{\varphi^n(b)\}_{n\in\mathbb{Z}}$ and we will denote it as $\operatorname{orb}(b)$. Given $b \in \mathcal{A}^{\mathbb{Z}}$, the subshift generated by b is defined to be $\overline{\operatorname{orb}(b)}$ and denoted by X_b. Notice that when b is pseudoergodic, it follows that $X_b = \mathcal{A}^{\mathbb{Z}}$; indeed, if c belongs to $\mathcal{A}^{\mathbb{Z}}$, then for any k the central block $c_{[-k,k]}$ occurs in b, so there is j_0 such that $c_{[-k,k]} = b_{[j_0,j_0+2k]}$. But $b_{[j_0,j_0+2k]} = \varphi^{j_0+k}(b)_{[-k,k]}$ so that there is always an element in $\operatorname{orb}(b)$ as close as desired to c, proving that c belongs to X_b, as wanted.

For $\mathfrak{a} \in \mathcal{A}$, we denote its corresponding biinfinite sequence in $\mathcal{A}^{\mathbb{Z}}$ with boldface $\boldsymbol{\mathfrak{a}} = (\cdots \mathfrak{a}\,\mathfrak{a}\,\dot{\mathfrak{a}}\,\mathfrak{a} \cdots)$. The tridiagonal operators $A_{\boldsymbol{\mathfrak{a}}}$ corresponding to the constant sequences $\boldsymbol{\mathfrak{a}}$ are known as Laurent operators.

2 Tridiagonal operators for biinfinite sequences of zeroes and ones

In this section we discuss tridiagonal operators A_b where b is a biinfinite sequence of symbols in the alphabet $\mathcal{A} = \{0, 1\}$. Similar arguments can be given for $\mathcal{A} = \{-1, 1\}$ and we will comment about this at the end of this section. We begin this section by computing the numerical range of the Laurent operators $A_{\mathbf{0}}$ and $A_{\mathbf{1}}$; although it is a well known result, we provide a sketch of the proof since we believe it sheds some light in the understanding of the computation of $W(A_b)$, for more general b, which is one of the main results of this section.

Proposition 2.1. $W(A_{\mathbf{0}}) = \mathbb{D}$ *and* $W(A_{\mathbf{1}}) = (-2, 2)$.

Proof. For the proof of the inclusions $W(A_{\mathbf{0}}) \supset \mathbb{D}$ and $W(A_{\mathbf{1}}) \supset (-2, 2)$, we will show that for each $\lambda \in \mathbb{D}$ there is x in the unit ball of $\ell^2(\mathbb{Z})$ such that $\langle A_{\mathbf{0}}x, x\rangle = \lambda$ and $\langle A_{\mathbf{1}}x, x\rangle = 2\mathrm{Re}(\lambda)$. Indeed, we define

$$x_k = \begin{cases} \sqrt{1-|\lambda|^2}\,\lambda^k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

It follows that

$$\|x\|^2 = \sum_{k\in\mathbb{Z}} |x_k|^2 = (1-|\lambda|^2)\sum_{k\geq 0}(|\lambda|^2)^k = 1.$$

Furthermore

$$\langle A_{\mathbf{0}}x, x\rangle = \sum_{k\in\mathbb{Z}} \overline{x_k}x_{k+1} = \lambda\,(1-|\lambda|^2|)\sum_{k\geq 0}|\lambda|^{2k} = \lambda$$

and

$$\langle A_{\mathbf{1}}x, x\rangle = \sum_{k\in\mathbb{Z}} (\overline{x_k}x_{k+1} + \overline{x_k}x_{k-1}) = \langle A_{\mathbf{0}}x, x\rangle + \overline{\langle A_{\mathbf{0}}x, x\rangle} = 2\mathrm{Re}(\lambda),$$

as wanted.

To prove the other inclusions, first notice that since $A_{\mathbf{1}}$ is self-adjoint then $W(A_{\mathbf{1}}) \subset \mathbb{R}$ by Theorem 1.1. Let $x \in \ell^2(\mathbb{Z})$ such that $\|x\| = 1$. Since x and $A_{\mathbf{0}}x$ are linearly independent, Cauchy-Schwarz inequality gives $|\langle A_{\mathbf{0}}x, x\rangle| < \|A_{\mathbf{0}}x\| \cdot \|x\| = 1$ and so $|\langle A_{\mathbf{1}}x, x\rangle| = |2\mathrm{Re}(\lambda)| < 2$ so that $W(A_{\mathbf{0}}) \subset \mathbb{D}$ and $W(A_{\mathbf{1}}) \subset (-2, 2)$, as was to be proved. □

Lemma 2.2. *Let $x = (x_j)_{j\in\mathbb{Z}}$ be an element in the unit ball of $\ell^2(\mathbb{Z})$ and let $b = (b_j)_{j\in\mathbb{Z}}$ be in $\{0,1\}^{\mathbb{Z}}$. Then*

$$\langle A_b x, x\rangle = \langle A_0 x, x\rangle + \sum_{k\in\mathbb{Z}} b_k x_k \overline{x_{k+1}} = 2\mathrm{Re}(\langle A_0 x, x\rangle) - \sum_{k\in\mathbb{Z}} (1-b_k) x_k \overline{x_{k+1}}.$$

Proof. A direct computation shows

$$\langle A_b x, x\rangle = \sum_{k\in\mathbb{Z}} (b_{k-1}x_{k-1} + x_{k+1})\overline{x_k} = \langle A_{\mathbf{0}} x, x\rangle + \sum_{k\in\mathbb{Z}} b_k x_k \overline{x_{k+1}} \tag{1}$$

On the other hand we have

$$\langle A_{\mathbf{0}} x, x\rangle = \sum_{k\in\mathbb{Z}} \overline{x_k} x_{k+1} = \sum_{k\in\mathbb{Z}} (1-b_k)\overline{x_k} x_{k+1} + \sum_{k\in\mathbb{Z}} b_k \overline{x_k} x_{k+1}$$

and so

$$\sum_{k\in\mathbb{Z}} b_k x_k \overline{x_{k+1}} = \overline{\langle A_{\mathbf{0}} x, x\rangle} - \sum_{k\in\mathbb{Z}} (1-b_k)\, x_k \overline{x_{k+1}}.$$

Substituting this last equality in (1) we complete the proof. □

Proposition 2.3. *Let $x = (x_j)_{j\in\mathbb{Z}}$ be in the unit ball of $\ell^2(\mathbb{Z})$ and let $b \in \{0,1\}^{\mathbb{Z}}$. Then the complex number $\langle A_b x, x\rangle$ is contained in the interior of the ellipse with major axis of length 1 and focal points $\langle A_{\mathbf{0}} x, x\rangle$ and* $2\mathrm{Re}(\langle A_{\mathbf{0}} x, x\rangle)$.

Proof. Since x is in the unit ball of $\ell^2(\mathbb{Z})$ then the element y in $\ell^2(\mathbb{Z})$ defined by $y_k = |x_k|$ is also in the unit ball. Therefore, by Proposition 2.1, we obtain $\langle A_{\mathbf{0}} y, y\rangle < 1$. This together with Lemma 2.2 give us

$$\begin{aligned}
|\langle A_b x, x\rangle - \langle A_{\mathbf{0}} x, x\rangle| &+ |\,\langle A_b x, x\rangle - 2\mathrm{Re}\,\langle A_{\mathbf{0}} x, x\rangle\,| \\
&= \left|\sum_{k\in\mathbb{Z}} b_k x_k \overline{x_{k+1}}\right| + \left|-\sum_{k\in\mathbb{Z}} (1-b_k) x_k \overline{x_{k+1}}\right| \\
&\le \sum_{k\in\mathbb{Z},\, b_k=1} |\overline{x_k} x_{k+1}| + \sum_{k\in\mathbb{Z},\, b_k=0} |x_k \overline{x_{k+1}}| \\
&= \sum_{k\in\mathbb{Z}} |\overline{x_k} x_{k+1}| \\
&= \langle A_{\mathbf{0}} y, y\rangle < 1.
\end{aligned}$$

as was to be proved. □

Definition 2.4. *Let $x = (x_j)_{j\in\mathbb{Z}}$ be in the unit ball of $\ell^2(\mathbb{Z})$. We will denote by E_x the open convex set limited by the ellipse with focal points at $\langle A_{\mathbf{0}} x, x\rangle$ and $2\mathrm{Re}\big(\langle A_{\mathbf{0}} x, x\rangle\big)$ and major axis of length 1.*

Theorem 2.5. *Let $b \in \{0,1\}^{\mathbb{Z}}$, then*

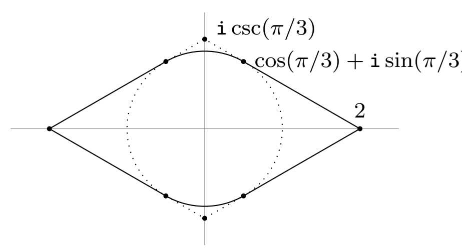
$$W(A_b) \subseteq \mathrm{conv}\big(W(A_{\mathbf{0}}) \cup W(A_{\mathbf{1}})\big).$$

Proof. For convenience, let us denote by Γ the right hand side of the inclusion in the theorem, see Figure 1. Using Proposition 2.3, it will suffice to show that, for each $x = (x_j)_{j\in\mathbb{Z}}$ in the unit ball of $\ell^2(\mathbb{Z})$, the corresponding ellipse E_x is contained in Γ. This is achieved by showing that the boundary of Γ lies outside of each E_x. By symmetry, we may only prove the case when $\langle A_0 x, x\rangle$ lies in the first quadrant. Since the case $\langle A_0 x, x\rangle = 0$ implies E_x is the open unit ball, satisfying the desired property, we may assume that $\lambda = \langle A_0 x, x\rangle \neq 0$, say $\lambda = re^{\mathrm{i}\theta}$ with $0 < r < 1$ and $0 \leq \theta \leq \pi/2$. Since the distance from $2\mathrm{Re}(\lambda)$ to the closest boundary line is $\frac{1}{2}\left(2 - 2\mathrm{Re}(\lambda)\right) = 1 - r\cos\theta$, by *Heron's Problem*, the minimum sum of the distances from the boundary line of Γ to the points λ and $2\mathrm{Re}(\lambda)$ is

$$\begin{aligned}\left|\left(2\mathrm{Re}(\lambda) + (1 - r\cos\theta)\left(1 + \mathrm{i}\sqrt{3}\right)\right) - \lambda\right| &= \left|1 + \mathrm{i}\left(\sqrt{3} - r\left(\sqrt{3}\cos\theta + \sin\theta\right)\right)\right| \\ &= \left|1 + \mathrm{i}\left(\sqrt{3} - 2r\cos(\theta - \pi/6)\right)\right| \\ &= \sqrt{4 + 4(r\cos(\theta - \pi/6))^2 - 4\sqrt{3}(r\cos(\theta - \pi/6))}.\end{aligned}$$

Furthermore, the above expression, as function of $r\cos(\theta - \pi/6)$, reaches its minimum value 1 when $r\cos(\theta - \pi/6) = \frac{\sqrt{3}}{2}$. This proves that the sum of the distances from the boundary line to the points λ and $2\mathrm{Re}(\lambda)$ has minimal value 1. Hence it lies outside E_x, as desired.

Fig. 1. Boundary of $\Gamma = \mathrm{conv}\,(W(A_0) \cup W(A_1))$

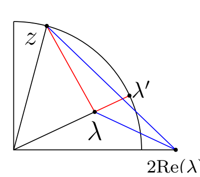


It remains to prove that the sum of the distances from a point in the curved part of the boundary of Γ to λ and $2\mathrm{Re}(\lambda)$ is also at least 1. To this purpose, let $z = e^{\mathrm{i}\varphi}$ with $\pi/3 < \varphi \leq \pi/2$ and let $\lambda' = \frac{1}{r}\lambda$. We consider two cases, namely, either $0 \leq \theta \leq \varphi$ or $\varphi < \theta \leq \pi/2$. In case $0 \leq \theta \leq \varphi$, we look at the triangle with vertices z, λ,and λ' and the triangle with vertices z, λ, and $2\mathrm{Re}(\lambda)$, see Figure 2. Since λ' is, by its definition, the closest point to λ in the unit circle, we obtain $|z - \lambda| \geq |\lambda - \lambda'|$. On the other hand, by letting $z = a + \mathrm{i}b$ and $\lambda = c + \mathrm{i}d$ and using the assumption on φ and θ we have $0 \leq a \leq c$ and $0 \leq d \leq b$. A straightforward computation then shows that $|z - 2\mathrm{Re}(\lambda)| \geq 1 \geq |\lambda| = |\lambda - 2\mathrm{Re}(\lambda)|$. We obtain

$$|z - \lambda| + |z - 2\mathrm{Re}(\lambda)| \geq |\lambda - \lambda'| + |\lambda| = 1,$$

as wanted.

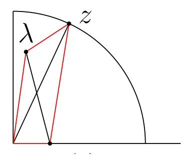
Fig. 2. Case $0 \leq \theta \leq \varphi$



To complete the proof we now assume $\varphi < \theta \leq \pi/2$. We use *Ptolemy's inequality* (see Figure 3) on the quadrilateral with vertices λ, z, $2\mathrm{Re}(\lambda)$ and the origin to obtain

$$|z - \lambda|\,|2\mathrm{Re}(\lambda)| + |\lambda|\,|z - 2\mathrm{Re}(\lambda)| \geq |\lambda - 2\mathrm{Re}(\lambda)|.$$

Fig. 3. Case $\varphi < \theta \leq \pi/2$



Using $|2\mathrm{Re}(\lambda)| = |2r\cos\theta| \leq |2r\cos(\pi/3)| = r = |\lambda|$ and again that $|\lambda - 2\mathrm{Re}(\lambda)| = |\lambda|$, we get

$$|z-\lambda|\,|\lambda| + |\lambda|\,|z - 2\mathrm{Re}(\lambda)| \geq |\lambda - z|\,|2\mathrm{Re}(\lambda)| + |\lambda|\,|z-2\mathrm{Re}(\lambda)| \geq |\lambda|,$$

and so

$$|z-\lambda| + |z - 2\mathrm{Re}(\lambda)| \geq 1,$$

as was to be proved. □

As mentioned in the introduction, the following result might be derived from limit operator techniques when the closure of the numerical range is considered, see e.g. [5]. However, our approach might be of interest since it is elementary and provides the construction of the unitary elements in $\ell^2(\mathbb{Z})$ corresponding to a given element in the numerical range.

Proposition 2.6. *Let* $b \in \{0,1\}^{\mathbb{Z}}$.
(i) *If* $\mathbf{0} \in X_b$ *then* $W(A_{\mathbf{0}}) = \mathbb{D} \subset W(A_b)$.
(ii) *If* $\mathbf{1} \in X_b$ *then* $W(A_{\mathbf{1}}) = (-2,2) \subset W(A_b)$.

Proof. To prove the proposition, it will suffice to show that for each $\lambda \in \mathbb{D}$ we can provide elements x and y in the unit ball of $\ell^2(\mathbb{Z})$ such that $\langle A_b x, x\rangle = \lambda$ and $\langle A_b y, y\rangle = 2\mathrm{Re}(\lambda)$.

We first observe that the case $\lambda = 0$ is taken care of by choosing $x \in \ell^2(\mathbb{Z})$ with just one component equal to 1 and all others equal to zero, since then $\langle A_b x, x\rangle = 0$. So we may assume in what follows that $\lambda \neq 0$. Let us assume we have $\lambda \in \mathbb{D}$, say $\lambda = re^{\mathrm{i}\theta}$. Since $0 < r < 1$, then $0 < \frac{r+1}{2} < 1$. Let k_0 be an integer such that

$$\frac{1-r}{2} > \left(\frac{r+1}{2}\right)^{2k_0+1}. \tag{2}$$

Consider the polynomial function $f(t) = t - t^{2k_0+1} - r$. Note that (2) implies $f(r) = -r^{2k_0+1} < 0$ and $f(\frac{r+1}{2}) = \frac{1-r}{2} - \left(\frac{r+1}{2}\right)^{2k_0+1} > 0$. The Intermediate Value Theorem gives us then a $r < t_0 < \frac{r+1}{2} < 1$ such that $f(t_0) = 0$, i.e.

$$\begin{aligned} t_0 - t_0^{2k_0+1} - r &= 0 \\ 1 - t_0^{2k_0} &= \frac{r}{t_0}. \end{aligned} \tag{3}$$

Now, since the constant sequence $\mathbf{0}$ (resp. $\mathbf{1}$) is a limit point of the orbit of b, there exists an integer j_0 (resp. l_0) such that $b_{[j_0+1, j_0+k_0]}$ (resp. $b_{[l_0+1, l_0+k_0]}$) is a sub-word of zeros (resp. of ones) of b with length k_0. We define $x = (x_k)_{k\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z})$ in the following way. For each $k \in \mathbb{Z}$

$$x_k = \begin{cases} \sqrt{1-t_0^2}\; t_0^{k-j_0-1}\; e^{\mathrm{i}\theta(k-j_0)} & \text{if } j_0 < k < j_0 + k_0 + 2 \\ t_0^{k_0+1} & \text{if } k = j_0 + k_0 + 3 \\ 0 & \text{otherwise.} \end{cases}$$

and define y_k similarly by replacing j_0 with l_0 in the definition of x_k. It follows that

$$\|x\|^2 = \sum_{k\in\mathbb{Z}} |x_k|^2$$

$$
\begin{aligned}
&= t_0^{2(k_0+1)} + \sum_{k=0}^{k_0} (1-t_0^2)t_0^{2k} \\
&= t_0^{2(k_0+1)} + (1-t_0^2)\frac{1-t_0^{2(k_0+1)}}{1-t_0^2} \\
&= 1
\end{aligned}
$$

and similarly $\|y\| = 1$. Thus x and y are elements in the unit ball of $\ell^2(\mathbb{Z})$. Using $b_{j_0+1} = \cdots = b_{j_0+k_0} = 0$ and (3) we obtain

$$
\begin{aligned}
\langle A_b x, x\rangle &= \sum_{k\in\mathbb{Z}} b_k x_k \overline{x_{k+1}} + \overline{x_k} x_{k+1} = \sum_{k=j_0+1}^{j_0+k_0} \overline{x_k} x_{k+1} \\
&= \sum_{k=j_0+1}^{j_0+k_0} (1-t_0^2)t_0^{2(k-j_0)} t_0^{-1} e^{\mathrm{i}\theta} = t_0(1-t_0^{2k_0})e^{\mathrm{i}\theta} = re^{\mathrm{i}\theta} = \lambda
\end{aligned}
$$

Similarly

$$
\langle A_b y, y\rangle = \overline{\langle A_b x, x\rangle} + \langle A_b x, x\rangle = \overline{\lambda} + \lambda = 2\mathrm{Re}(\lambda)
$$

as was to be proved. □

Part (i) of the following corollary is analogous to [1, Lemma 3.1] for b pseudoergodic and the alphabet $\mathcal{A} = \{-1, 1\}$. Part (ii) may also be derived from [5, Theorem 16]. However, notice we do not require the operator A_b to be pseudoergodic and, thus far, we do not have employed limit operator techniques. In case we replace the numerical range with its closure, our result is analogous to [5, Corollary 14], where the operator is taken to be pseudoergodic and the union is taken over all periodic operators.

Corollary 2.7. *Let $b \in \{0, 1\}^{\mathbb{Z}}$. If $\mathbf{0}, \mathbf{1} \in X_b$ then*
(i) $W(A_b) = \mathrm{conv}\,(W(A_{\mathbf{0}}) \cup W(A_{\mathbf{1}}))$.
(ii) $\overline{W(A_b)} = \mathrm{conv}\,(\sigma(A_{\mathbf{0}}) \cup \sigma(A_{\mathbf{1}}))$.

Proof. To prove (i), observe that since $W(A_b)$ is a convex set, it follows from Proposition 2.6 that $W(A_{\mathbf{0}}) \cup W(A_{\mathbf{1}}) \subseteq W(A_b)$. The other inclusion is an application of Theorem 2.5.

For equality (ii), since $A_{\mathbf{0}}$ and $A_{\mathbf{1}}$ are normal operators, Theorem 1.1 gives us that $\mathrm{conv}\,(\sigma(A_{\mathbf{0}})) = \overline{W(A_{\mathbf{0}})}$ and $\mathrm{conv}\,(\sigma(A_{\mathbf{1}})) = \overline{W(A_{\mathbf{1}})}$. Furthermore, since the numerical range of an operator is a convex set, by applying the closure to both sides of the equality in (i), a straightforward computation gives the desired result. □

Notice that in the proof of Theorem 2.6, by replacing 0 with -1, it is easy to see that actually $\langle A_{-\mathbf{1}} x, x\rangle = \overline{\lambda}$. Hence, it might be possible to use our techniques to recover some known results for the hopping sign model operators.

3 Tridiagonal operators associated to subshifts

For this section, $\mathcal{A}$ denotes a finite alphabet.

Proposition 3.1. *Let $b \in \mathcal{A}^{\mathbb{Z}}$. If $d \in \mathrm{orb}\,(b)$ then the operators A_b and A_d are unitarily equivalent.*

Proof. Since $d \in \mathrm{orb}\,(b)$, then exist $k_0 \in \mathbb{Z}$ such that $d = \varphi^k(b)$. We define $S = A_{\mathbf{0}}^k$. Note S is an unitary operator. For $j \in \mathbb{Z}$ it follows that

$$
\begin{aligned}
((SA_b)\,x)_j &= (A_b x)_{j+k} \\
&= x_{j+k+1} + b_{j+k-1} x_{j+k-1}
\end{aligned}
$$

$$
\begin{aligned}
&= (Sx)_{j+1} + \left(\varphi^k(b)\right)_{j-1}(Sx)_{j-1} \\
&= (Sx)_{j+1} + d_{j-1}(Sx)_{j-1} \\
&= ((A_d S)x)_j .
\end{aligned}
$$

Thus $S^{-1}A_d S = A_b$. □

Lemma 3.2. *If A and B are unitarily equivalent, then they have the same numerical range and spectrum, furthermore, they have the same eigenvalues.*

Proof. Since A and B are unitarily equivalent there exists a unitary operator U such that $A = U^{-1}BU$. Let $\lambda \in \mathbb{C}$, it follows that $A - \lambda I = U^{-1}(B - \lambda I)U$. Therefore $A - \lambda I$ is an invertible operator with bounded inverse and dense range if and only if $B - \lambda I$ is. Hence $\sigma(A) = \sigma(B)$ and A and B have the same eigenvalues.

Finally, since $U^{-1} = U^*$, it follows that $\langle Ax, x\rangle = \langle BUx, Ux\rangle$ and thus since U is an isometry we conclude $W(A) = W(B)$. □

Theorem 3.3. *Let $b \in \mathcal{A}^{\mathbb{Z}}$, then for every $d \in \operatorname{orb}(b)$ we have $W(A_d) = W(A_b)$ and $\sigma(A_d) = \sigma(A_b)$.*

Proof. Follows from Lemma 3.2 and Proposition 3.1. □

The following may be derived from [5, Lemma 12]; however, we provide a simplified proof here for the sake of completeness.

Lemma 3.4. *Let $b, d \in \mathcal{A}^{\mathbb{Z}}$. If $\varphi^{n_k}(b) \to d$ for some sequence of integers $\{n_k\}_{k>0}$ then $A_{\varphi^{n_k}(b)} \to A_d$ in the weak operator topology.*

Proof. Let $m > 0$. Since $\varphi^{n_k}(b) \to d$, there is $M > 0$ such that $d_{[-m,m]} = \varphi^{n_k}(b)_{[-m,m]} = b_{[-m+n_k, m+n_k]}$ for all $k > M$. Since m is arbitrary, this yields $\left\langle A_{\varphi^{n_k}(b)}x, y\right\rangle$ to be as close as wanted to $\langle A_d x, y\rangle$ for k sufficiently large, as desired. □

Proposition 3.5. *Let $b \in \mathcal{A}^{\mathbb{Z}}$. If $d \in X_b$ then*

$$W(A_d) \subset \overline{W(A_b)} \qquad \textit{and} \qquad \sigma(A_d) \subset \overline{W(A_b)}.$$

Proof. Since $d \in \overline{\operatorname{orb}(b)}$, there exists a sequence of integers $\{n_k\}_{k\in\mathbb{Z}}$ such that $\varphi^{n_k}(b) \to d$ as $k \to \infty$. Using Lemma 3.4, we also have $A_{\varphi^{n_k}(b)} \to A_d$ in the weak operator topology. To prove the first inclusion, we let $z \in W(A_d)$. Then there is $x \in \ell^2(\mathbb{Z})$ with $\|x\| = 1$ such that $z = \langle A_d x, x\rangle$. Hence $\left\langle A_{\varphi^{n_k}(b)}x, x\right\rangle - z = \left\langle (A_{\varphi^{n_k}(b)} - A_d)x, x\right\rangle \to 0$ as $k \to \infty$ and so $\left\langle A_{\varphi^{n_k}(b)}x, x\right\rangle \to z$. Using Theorem 3.3, we obtain that $\left\langle A_{\varphi^{n_k}(b)}x, x\right\rangle$ belongs to $W\left(A_{\varphi^{n_k}(b)}\right) = W(A_b)$, this proves $z \in \overline{W(A_b)}$, as wanted. The second inclusion is immediate from Theorem 1.1. □

Corollary 3.6. *Let $b \in \mathcal{A}^{\mathbb{Z}}$. Then*

$$\bigcup_{d\in X_b} \overline{W(A_d)} = \overline{W(A_b)}.$$

Proof. The nontrivial inclusion is an application of Proposition 3.5 □

Part (i) of the following corollary is analogous to [5, Theorem 16], in fact, we rely on its proof. This will allow us to argue Part (ii) which is somewhat more general than a particular case of [5, Corollary 17], as it applies to not necessarily pseudoergodic sequences.

Corollary 3.7 (Hagger)**.** *Let $b \in \mathcal{A}^{\mathbb{Z}}$. If for all $\mathfrak{a} \in \mathcal{A}$ the constant sequence $\mathbf{a}$ belongs to X_b then*

(i) $\overline{W(A_b)} = \operatorname{conv}\left(\bigcup_{\mathfrak{a}\in\mathcal{A}} \overline{W(A_{\mathbf{a}})}\right) = \operatorname{conv}\left(\bigcup_{\mathfrak{a}\in\mathcal{A}} \sigma(A_{\mathbf{a}})\right).$

(ii) $\overline{W(A_b)} = \operatorname{conv}(\sigma(A_b))$.

Proof. The equality in the right hand side of (i) follows from Theorem 1.1 as the Laurent operator $A_{\mathfrak{a}}$ is normal. For the left hand side equality in (i), the inclusion "$\supset$" follows from an application of Proposition 3.5 by taking closures and using that the closure of a convex set is convex. The inclusion "$\subset$" follows from the proof of [5, Theorem 16] by taking $U_{-1} = \mathcal{A}$, $U_0 = \{0\}$ and $U_1 = \{1\}$ and noticing that the hypothesis on $A = A_b$ to be pseudoergodic is not required in that part of the proof.

For equality (ii) we use Theorem 1.1 to conclude the "$\supset$" inclusion. For the other inclusion we observe that the hypothesis ensures that Laurent operators $A_{\mathfrak{a}}$ are limit operators of A_b (see e.g. [2, Section 2] for an introduction of this notion). We may apply now [2, Theorem 2.1] to obtain $\bigcup_{\mathfrak{a}\in\mathcal{A}} \sigma(A_{\mathfrak{a}}) \subset \sigma(A_b)$. This together with an application of (i) completes the proof. □

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