Open Math. 2016; 14: 338–346 DE GRUYTER OPEN

Open Mathematics **Open Access**

Research Article

Qimei Xiao* and Wenjun Liu

On derivations of quantales

DOI 10.1515/math-2016-0030
Received February 29, 2016; accepted May 6, 2016.

Abstract: A quantale is a complete lattice equipped with an associative binary multiplication distributing over arbitrary joins. We define the notions of right (left, two) sided derivation and idempotent derivation and investigate the properties of them. It's well known that quantic nucleus and quantic conucleus play important roles in a quantale. In this paper, the relationships between derivation and quantic nucleus (conucleus) are studied via introducing the concept of pre-derivation.

Keywords: Quantale, Derivation, Quantic nucleus, Pre-derivation, Quantic pre-nucleus

MSC: 06A06, 54A10

1 Introduction

A quantale is a complete lattice equipped with associative binary multiplication distributing over arbitrary joins. Among the numerous examples of quantales are frames, various ideal lattices of rings and C^*-algebras and the power set of a semigroup. The study of such partially ordered algebraic structures goes back to the work of Ward and Dilworth [1–3] on residuated lattices in the late 1930's, motivated by ring-theoretic considerations. The notion of quantale was proposed as a combination of "quantum logic" and "locale" by Mulvey [4] in 1986, with the purpose of studying the foundations of quantum mechanics and the spectra of non-commutative C^*-algebras. In 1990, Yetter [5] revealed the importance of quantales for linear logic, the logical foundation of theoretical computer science, which was proposed by Girard [6]. Since then, the theory of quantales has aroused great interest of many researchers, and a great deal of new ideas and applications of quantales have been discussed [7–10].

Derivation is helpful to the research of structure and property in algebraic system, which was introduced from analytic theory. There are many authors who studied derivations in various algebraic structures, such as rings, lattices, BCI-algebra and subtraction algebras, etc [11–15]. In [16], we introduced the notion of derivation for a quantale, and we discussed some related properties. It is well known that quantic nucleus and quantic conucleus play important roles in quantale theory because they determine the quotients and substructures in the category of quantales [17, 18]. The motivation of this paper is to study the properties of derivation further and the relationships between derivation and quantic nucleus (conucleus) on a quantale.

2 Preliminaries

In this section, we review some elementary notions of quantale theory [10].

***Corresponding Author: Qimei Xiao:** School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, 410004, P.R. China, E-mail: domaintheory@hotmail.com
Wenjun Liu: School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan, 410004, P.R. China, E-mail: liuwjzhlp@126.com

Definition 2.1. *A quantale is a complete lattice Q with an associative binary operation & satisfying:*

$$a\&(\bigvee_i b_i) = \bigvee_i (a\&b_i) \text{ and } (\bigvee_i b_i)\&a = \bigvee_i (b_i\&a)$$

for all $a \in Q$ and $\{b_i\} \subseteq Q$.

Since $a\&-$ and $-\&a$ preserve arbitrary sups, they have right adjoints which we denote by $a \to_r$ and $a \to_l$ respectively. Thus $a\&c \leq b$ if and only if $c \leq a \to_r b$ and $c\&a \leq b$ if and only if $c \leq a \to_l b$.

In this paper, we denote the top element and the bottom element by $\top$ and 0 respectively. It is clear that $0\&x = 0$ and $x\&0 = 0$ for all $x \in Q$.

Definition 2.2. *Let Q be a quantale, $a \in Q$.*
(1) *a is right (left) sided iff $a\&\top \leq a$ ($\top\&a \leq a$).*
(2) *a is strictly right (left) sided iff $a\&\top = a$ ($\top\&a = a$).*
(3) *a is (strictly) two sided iff a is both (strictly) right sided and (strictly) left sided.*
(4) *Q is two sided (right sided, left sided) iff every $a \in Q$ is two sided (right sided, left sided).*
(5) *a is idempotent iff $a\&a = a$.*
(6) *Q is idempotent iff every $a \in Q$ is idempotent.*
(7) *An element $1 \in Q$ is a right (left) unit iff $a\&1 = a$ ($1\&a = a$) for all $a \in Q$.*
(8) *1 is a unit iff 1 is both a right and a left unit.*
(9) *Q is (right, left) unital iff Q has a (right, left) unit.*
A quantale Q is commutative iff $a\&b = b\&a$ holds for all $a, b \in Q$. It is obvious that Q is commutative iff $a \to_r c = a \to_l c$ for all $a, c \in Q$, and we denote by $a \to c$. Q is right commutative iff $(a\&b) \to_r c = (b\&a) \to_r c$ for all $a, b, c \in Q$. It is easy to observe that Q is left commutative iff $a\&b\&c = b\&a\&c$.

Remark 2.3. *Let $a, b, c \in Q$ with Q a quantale. Then*
(1) *$a\&(a \to_r b) \leq b$;*
(2) *$(a \to_l b)\&a \leq b$.*

Definition 2.4. *Let P be an ordered set. A map $f : P \to P$ is called a closure (coclosure) operator on P if, for all $a, b \in P$,*
(1) *$a \leq f(a)$ ($a \geq f(a)$);*
(2) *$a \leq b \Rightarrow f(a) \leq f(b)$;*
(3) *$ff(a) = f(a)$.*

Definition 2.5. *Let Q be a quantale. A quantic nucleus (conucleus) on Q is a closure (coclosure) operator j such that $j(a)\&j(b) \leq j(a\&b)$ for all $a, b \in Q$. We say that a quantic nucleus (conucleus) j is strict if it satisfies $j(a)\&j(b) = j(a\&b)$.*

Picado introduced the notion of quantic pre-nucleus in [9], which generalizes Banaschewski's definition of a (localic) pre-nucleus [19, 20].

Definition 2.6 (Picado [9])**.** *Let Q be a quantale. An order preserving mapping $j_o : Q \to Q$ is called a quantic pre-nucleus iff it satisfies*
(1) *$a \leq j_o(a)$ for all $a \in Q$*
(2) *$a\&j_o(b) \leq j_o(a\&b)$ and $j_o(a)\&b \leq j_o(a\&b)$ for all $a, b \in Q$.*

It is easy to prove that $Q_{j_o} = \{a \in Q | j_o(a) = a\}$ is a closure system and the associated closure operator is given by

$$j(a) = \bigwedge\{b \in Q_{j_o} | a \leq b\}.$$

Theorem 2.7 (Picado [9]). *Let j_o be a quantic pre-nucleus on a quantale Q. If $j(a) = \bigwedge\{b \in Q_{j_o} | a \le b\}$ for all $a \in Q$, then j is a quantic nucleus on Q.*

Definition 2.8. *Let Q be a quantale. A nonempty subset $I \subseteq Q$. I is an ideal of Q if it satisfies the following two conditions:*
(1) *if $\{a_i\} \subseteq Q$, then $\bigvee a_i \in I$;*
(2) *for all $x \in Q$ and $a \in I$, we have $a\&x \in I$ and $x\&a \in I$.*

3 Derivation on quantales

In [16], we defined the notion of derivation on quantales as follows:

Let d be a mapping on a quantale Q, then d is a *derivation* on Q, if it satisfies the following conditions

$$d(\bigvee_i b_i) = \bigvee_i d(b_i) \text{ and } d(a\&b) = (a\&d(b)) \bigvee (d(a)\&b).$$

for all $a, b \in Q$ and $\{b_i\} \subseteq Q$.

Remark 3.1.
(1) *Let Q be a quantale. We define a mapping d by $d(a) = 0$ for all $a \in Q$. It's clear that $d(\bigvee_i b_i) = \bigvee_i d(b_i) = 0$ for all $\{b_i\} \subseteq Q$. Since $a\&0 = 0\&b = 0$, we have $(d(a)\&b) \vee (a\&d(b)) = (0\&b) \vee (a\&0) = 0 = d(a\&b)$ for all $a, b \in Q$. Then d is a derivation on Q, which is called a zero derivation.*
(2) *Let d be an identity mapping on a quantale Q. Then d is a derivation on Q, which is called an identity derivation.*

According to the above definition, we obtain the following basic properties:

Proposition 3.2. *Let d be a derivation on a quantale Q and $a, b, c \in Q$, we have:*
(1) *$b \le d(a) \to_r d(a\&b)$, $a \le d(b) \to_l d(a\&b)$.*
(2) *$a \to_r b \le d(a) \to_r d(b)$, $d(a \to_r b) \le a \to_r d(b)$.*
(3) *$a \to_l b \le d(a) \to_l d(b)$, $d(a \to_l b) \le a \to_l d(b)$.*
(4) *If $a \le b$, then $d(a) \le d(b)$.*
(5) *If $a\&b = b$ (or $b\&a = b$) and $d(a) \ge a$, then $d(b) \ge b$.*
(6) *If $c \le a \to_r b$, then $c \le d(a) \to_r d(b)$ and $a \le d(c) \to_l d(b)$.*
(7) *If $a\&a = a$, then $a \le d(a) \to_r d(a)$ and $a \le d(a) \to_l d(a)$.*
(8) *If Q is right (left) unital, then $a\&d(\top) \le d(a)$ ($d(\top)\&a \le d(a)$).*
(9) *If Q is right (left) unital and $d(1) = 1$, then $a \le d(a)$ and $d(\top) = \top$.*
(10) *If Q is right (left) unital and $d(1) = 1$, then $d(a\&\top) = d(a)\&\top$ ($d(\top\&a) = \top\&d(a)$).*

The following corollary follows from Proposition 3.2 (5) .

Corollary 3.3. *Let d be a derivation on a right (left) unital quantale Q. If $d(1) \ge 1$, then $d(a) \ge a$ for all $a \in Q$.*

Proposition 3.4. *Let d be a derivation on a quantale Q.*
(1) *$d(0) = 0$.*
(2) *$d(\top) = \top$ if Q is right (left) unital and $d(1) \ge 1$.*

Proof. (1) By the definition of derivation and $0\&x = 0$ and $x\&0 = 0$ for all $x \in Q$, we have $d(0) = d(0\&0) = (d(0)\&0) \vee (0\&d(0)) = 0 \vee 0 = 0$.

(2) Since Q is right (left) unital and $d(1) \ge 1$, by Corollary 3.3, we have $d(\top) \ge \top$. Since $\top$ is the top element, we get $d(\top) = \top$. □

Proposition 3.5. *Let d be a derivation on a quantale Q. If $a \in Q$ is right (left, two) sided, then $d(a)$ is right (left, two) sided.*

Proof. Since a is right sided, then $a\&\top \leq a$. By the Proposition 3.2 (4), we have $d(a) \geq d(a\&\top) = (d(a)\&\top) \vee (a\&d(\top)) \geq d(a)\&\top$, so $d(a)$ is right sided. □

Proposition 3.6. *Let d be a derivation on a right unital quantale Q and $d(1) = 1$. If $a \in Q$ is right (left,two) sided, then $d(a)$ is strictly right (left,two) sided.*

Proof. Since a is right sided, then $a\&\top \leq a$. By the Proposition 3.2 (4), we have $d(a) \geq d(a\&\top) = (d(a)\&\top) \vee (a\&d(\top)) \geq d(a)\&\top$ and $d(a) = d(a\&1) \leq d(a\&\top)$. That $d(a\&\top) = d(a)\&\top$ follows from Proposition 3.2 (10). From the above, we get $d(a) = d(a)\&\top$ which implies that $d(a)$ is strictly right sided. □

In the following, we give some definitions about derivation which are similar to those on quantic nucleus.

Definition 3.7. *Let d be a derivation on a quantale Q, $a \in Q$.*
(1) *d is right sided iff $d(\top\&a) \leq d(a)$.*
(2) *d is left sided iff $d(a\&\top) \leq d(a)$.*
(3) *d is two sided iff d is both right and left sided.*
(4) *d is idempotent iff $d(a\&a) = d(a)$.*

Example 3.8. *Let Q be the complete lattice shown in Fig. 1 and the operations & on Q is shown in Table 1. It is straightforward to verify that $(Q, \&)$ is a quantale. We define $d_1 : Q \to Q$ by $d_1(0) = 0, d_1(a) = a, d_1(\top) = \top$ and $d_2 : Q \to Q$ by $d_2(0) = 0, d_2(a) = d_2(\top) = \top$. It is easy to prove that d_1 and d_2 are derivations on $(Q, \&)$. It is clear that d_1 is left sided and d_2 is two sided. Obviously, any derivation on $(Q, \&)$ is idempotent.*

Fig. 1

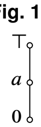

Table 1

$\&_1$	0	a	⊤
0	0	0	0
a	0	a	a
⊤	0	⊤	⊤

Proposition 3.9. *Let d be a derivation on a quantale Q. If d is idempotent, right (left) sided and $d(a) \geq a$ for all $a \in Q$, then $d(a)$ is an idempotent element of Q.*

Proof. Since d is a right-sided derivation, we have $d(a\&\top) = (d(a)\&\top) \vee (a\&d(\top)) \leq d(a)$ for all $a \in Q$. So $d(a)\&\top \leq d(a)$, and then $d(a)\&d(a) \leq d(a)$. Making use of the idempotence of d, $d(a) = d(a\&a) = (d(a)\&a) \vee (a\&d(a)) \leq d(a)\&d(a)$. Therefore, $d(a)\&d(a) = d(a)$ for all $a \in Q$. □

Proposition 3.10. *Let d be a derivation on a quantale Q. If $d(a) \geq a$ for all $a \in Q$, then d is right (left) sided iff $d(a)$ is right (left) sided.*

Proof. If $d(a)$ is right sided for all $a \in Q$, then $d(a)\&\top \leq d(a)$. So $a\&d(\top) \leq a\&\top \leq d(a)\&\top \leq da$. Thus $d(a\&\top) = (d(a)\&\top) \vee (a\&d(\top)) \leq d(a)$.

Conversely, if d is right sided, we have $d(a\&\top) = (d(a)\&\top) \vee (a\&d(\top)) \leq d(a)$, then $d(a)\&\top \leq d(a)$. □

Let $(Q, \&)$ be a quantale, we denote the collection of all derivations on Q by $D(Q)$. Define the operation "$\bigvee$" and "$\circ$" on $D(Q)$ by $(d_1 \bigvee d_2)() = d_1() \bigvee d_2()$ and $(d_1 \circ d_2)() = d_1()\&d_2()$ for $d_1, d_2 \in D(Q)$.

Theorem 3.11. *Let Q be a quantale. Then,*
(1) *$(D(Q), \circ)$ is a complete lattice;*
(2) *if Q is two sided and right commutative, then $(D(Q), \circ)$ is a quantale.*

Proof. (1) We prove that $D(Q)$ is a complete lattice under the pointwise order.

Let $\{d_i\} \subseteq D(Q)$ and $\{a_j\} \subseteq Q$, we have $\bigvee_i d_i(\bigvee_j a_j) = \bigvee_i(d_i(\bigvee_j a_j)) = \bigvee_i(\bigvee_j(d_i(a_j))) = \bigvee_j(\bigvee_i d_i(a_j))$. Let $a, b \in Q$, we have $\bigvee_i d_i(a\&b) = \bigvee_i((d_i(a)\&b) \vee (a\&d_i(b))) = (\bigvee_i(d_i(a)\&b)) \vee (\bigvee_i(a\&d_i(b))) = ((\bigvee_i d_i(a))\&b) \vee (a\&(\bigvee_i d_i(b)))$. So $\bigvee_i d_i \in D(Q)$.

Since $d : Q \to Q$ defined by $d(a) = 0$ for all $a \in Q$ is a derivation, then $D(Q)$ has a bottom element. Therefore, $D(Q)$ is a complete lattice.

(2) We need to prove "$\circ$" is a binary operation on $D(Q)$.

Let $d_1, d_2 \in D(Q)$, $a, b \in Q$, then $(d_1 \circ d_2)(a\&b) = d_1(a\&b)\&d_2(a\&b) = ((d_1(a)\& b) \vee (a\&d_1(b)))\&((d_2(a)\&b) \vee (a\&d_2(b)))=((d_1(a)\&b)\&(d_2(a)\&b)) \vee ((a\&d_1(b))\& (d_2(a)\&b)) \vee ((d_1(a)\&b)\&(a\&d_2(b))) \vee ((a\&d_1(b))\&(a\&d_2(b)))$.

Since Q is right sided, we have $((d_1(a)\&b)\&(d_2(a)\&b)) \vee ((a\&d_1(b))\&(d_2(a)\&b)) = ((d_1(a)\&b) \vee (a\&d_1(b)))\&(d_2(a)\&b) = d_1(a\&b)\&(d_2(a)\&b) \leq d_1(a\&\top)\&(d_2(a)\&b) \leq (d_1(a)\& d_2(a))\&b = (d_1 \circ d_2)(a)\&b$.

Since Q is left sided and right commutative, then $((d_1(a)\&b)\&(a\&d_2(b))) \vee ((a\&d_1(b))\&(a\& d_2(b))) = ((d_1(a)\&b) \vee (a\&d_1(b)))\&(a\&d_2(b)) = d_1(a\&b)\&(a\&d_2(b)) \leq d_1(\top\&b)\&(a\&d_2(b)) \leq d_1(b)\&a\&d_2(b) = a\&d_1(b)\&d_2(b) = a\&(d_1 \circ d_2)(b)$.

From the above, we have $(d_1 \circ d_2)(a\&b) = ((d_1 \circ d_2)(a)\&b) \vee (a\&(d_1 \circ d_2)(b))$. So $d_1 \circ d_2 \in D(Q)$.

It is clear that the operation $\circ$ is an associative binary operation satisfying the distribution over arbitrary joins, so $(D(Q), \circ)$ is a quantale. □

Given a derivation d on a quantale $(Q, \&)$, we denote the fixed set $\{a \in Q | d(a) = a\}$ by Q_d.

Theorem 3.12. *Let d be a derivation on a quantale $(Q, \&)$, then $(Q_d, \&)$ is a quantale.*

Proof. We first prove that Q_d is a complete lattice under the pointwise order of Q. Let $\{b_i\} \subseteq Q_d$, we have $d(\bigvee_i b_i) = \bigvee_i d(b_i) = \bigvee_i b_i$, then $\bigvee_i b_i \in Q_d$. By Proposition 3.4, we have $d(0) = 0$, then $0 \in Q_d$.

Then we prove that & is a binary operation on Q_d. Let $a, b \in Q_d$, we have $d(a\&b) = (d(a)\&b) \vee (a\&d(b)) = a\&b$, so $a\&b \in Q_d$.

From the above, we have $(Q_d, \&)$ is a quantale. □

Example 3.13. *Let $Q = \{0, \top\}$ and define a binary operation & on Q by $0\&0 = 0$, $0\&\top = 0$, $\top\&0 = 0$, $\top\&\top = \top$. The quantale $(Q, \&)$ is two sided, idempotent, unital and commutative. There are only zero derivation and identity derivation on Q, denoted by d_1 and d_2. It is clear that $D(Q) = \{d_1, d_2\}$, $Q_{d_1} = \{0\}$ and $Q_{d_2} = \{0, \top\}$ are quantales.*

Theorem 3.14. *Let d be a derivation on a quantale Q. If d is two-sided, idempotent and $d(a) \geq a$ for all $a \in Q$, then d is a quantale homomorphism from Q onto Q_d.*

Proof. By Proposition 3.9 and Proposition 3.10, we have $d(a)$ is idempotent and two-sided for all $a \in Q$. Therefore, $d(d(a)) = d(d(a)\&d(a)) = (d(d(a))\&d(a)) \vee (d(a)\&d(d(a))) \leq (\top\&d(a)) \vee (d(a)\&\top) \leq d(a)$. On the other hand, $d(d(a)) \geq d(a)$. Thus $d(d(a)) = d(a)$ which implies that $d(a) \in Q_d$ for all $a \in Q$. So d is a mapping from Q onto Q_d.

We need to show that $d(a\&b) = d(a)\&d(b)$ for all $a, b \in Q$. Since d is idempotent, we have $d(a\&b) = d(d(a\&b)) = d((d(a)\&b) \vee (a\&d(b))) = d(d(a)\&b) \vee d(a\&d(b)) \geq d(d(a)\&b) = (d(d(a))\&b) \vee (d(a)\&d(b)) \geq d(a)\&d(b)$. On the other hand, $d(a\&b) = (d(a)\&b) \vee (a\&d(b)) \leq (d(a)\&d(b)) \vee (d(a)\&d(b)) = d(a)\&d(b)$. So $d(a\&b) = d(a)\&d(b)$. Since d preserves arbitrary sups, we know that d is a quantale homomorphism from Q onto Q_d. □

By the proof of Theorem 3.14, we have the following corollary.

Corollary 3.15. *Let d be a derivation on a quantale Q. If d is two-sided, idempotent and $d(a) \geq a$ for all $a \in Q$, then $Q_p = \{a \in Q | d(a) \geq p, p\&p = p\}$ is a subquantale of Q.*

In the following, we shall give a description for a derivation on general quantales.

Lemma 3.16 (Paseka and Kruml [8]). *Let $(Q, \&)$ be a quantale and $Q[e] = \{a \vee k : a \in Q, k \in \{0, e\}\}$, where e is an arbitrary element such that $e \notin Q$. We define the supremum on $Q[e]$: $e \vee 0 = e$ and*

$$\bigvee_i^{Q[e]} (a_i \vee k_i) = \begin{cases} (\bigvee_i a_i) \vee e, & \text{if } \exists k_i = e, \\ \bigvee_i a_i, & \text{otherwise}. \end{cases}$$

The multiplication $\&'$ on $Q[e]$ is as follows:

$$(a \vee k')\&'(b \vee k'') = \begin{cases} a\&b, & \text{if } k' = k'' = 0, \\ (a\&b) \vee b, & \text{if } k' = e, k'' = 0, \\ (a\&b) \vee a, & \text{if } k' = 0, k'' = e, \\ ((a\&b) \vee a \vee b) \vee e, & \text{if } k' = k'' = e. \end{cases}$$

Then $(Q[e], \&')$ is a unital quantale with the unit e.

Let Q be a quantale and $d : Q \to Q$ a map on Q. We define the map $\bar{d} : Q[e] \to Q[e]$ such that

$$\bar{d}(a) = \begin{cases} d(a), & \text{if } a \in Q, \\ d(a'), & \text{if } a = a' \vee e, a' \in Q. \end{cases}$$

It is clear that $d = \bar{d}|_Q$.

Theorem 3.17. *Let Q be a quantale and $d : Q \to Q$ a map on Q. Then d is a derivation on Q if and only if $\bar{d}$ is a derivation on $Q[e]$.*

Proof. Clearly, if $\bar{d}$ is a derivation on $Q[e]$, then d is a derivation on Q.

Conversely, we assume that d is a derivation on Q. Let $\{b_i \vee k_i\}_i \subseteq Q[e]$ and $b_i \in Q, k_i \in \{0, e\}$. We have

$$\bar{d}(\bigvee_i^{Q[e]} (b_i \vee k_i)) = \begin{cases} \bar{d}((\bigvee_i b_i) \vee e), & \text{if } \exists k_i = e, \\ \bar{d}(\bigvee_i b_i), & \text{otherwise}. \end{cases}$$

$$= d(\bigvee_i b_i) = \bigvee_i d(b_i) = \bigvee_i \bar{d}(b_i \vee k_i)$$

Let $a \vee k', b \vee k'' \in Q[e]$ and $a, b \in Q, k_1', k'' \in \{0, e\}$. Then

$$\bar{d}((a \vee k')\&'(b \vee k'')) = \begin{cases} d(a\&b), & \text{if } k' = k'' = 0, \\ d((a\&b) \vee b), & \text{if } k' = e, k'' = 0, \\ d((a\&b) \vee a), & \text{if } k' = 0, k'' = e, \\ d((a\&b) \vee a \vee b), & \text{if } k' = k'' = e. \end{cases}$$

Case 1: $k' = k'' = 0$, $[(\bar{d}(a \vee k'))\&'(b \vee k'')] \vee [(a \vee k')\&'(\bar{d}(b \vee k''))] = [d(a)\&'(b \vee k'')] \vee [(a \vee k')\&' d(b)] = (d(a)\&b) \vee (a\&d(b)) = d(a\&b)$;

Case 2: $k' = e, k'' = 0$, $[(\bar{d}(a \vee k'))\&'(b \vee k'')] \vee [(a \vee k')\&'(\bar{d}(b \vee k''))] = [d(a)\&'(b \vee k'')] \vee [(a \vee k')\&' d(b)] = (d(a)\&b) \vee (a\&d(b)) \vee d(b) = d((a\&b) \vee b)$;

Case 3: $k' = 0, k'' = e$, $[(\bar{d}(a \vee k'))\&'(b \vee k'')] \vee [(a \vee k')\&'(\bar{d}(b \vee k''))] = [d(a)\&'(b \vee k'')] \vee [(a \vee k')\&' d(b)] = (d(a)\&b) \vee da \vee (a\&d(b)) = d((a\&b) \vee b)$;

Case 4: $k' = e, k'' = e$, $[(\bar{d}(a \vee k'))\&'(b \vee k'')] \vee [(a \vee k')\&'(\bar{d}(b \vee k''))] = [d(a)\&'(b \vee k'')] \vee [(a \vee k')\&' d(b)] = (d(a)\&b) \vee da \vee (a\&d(b)) \vee d(b) = d((a\&b) \vee a \vee b)$.

Therefore, $\bar{d}$ is a derivation on $Q[e]$. □

4 The relation between derivation and quantic nucleus

In the following, we introduce the concept of pre-derivation which is the generalization of derivation.

Definition 4.1. *Let Q be a quantale. A mapping d_o on Q is a pre-derivation if d_o satisfies:*

$$d_o(\bigvee_i b_i) \geq \bigvee_i d_o(b_i) \text{ and } d(a\&b) \geq (a\&d_o(b))\bigvee(d_o(a)\&b)$$

for all $a, b \in Q$ and $\{b_i\} \subseteq Q$.

Remark 4.2.

(1) *If d_o is a pre-derivation on a right (left) unital quantale Q, then $d_o(1) \geq 1$ if and only if $d_o(a) \geq a$ for all $a \in Q$.*

(2) *If d_o is a pre-derivation on a quantale Q and $d_o(a) \geq a$ for all $a \in Q$, then d_o is a quantic pre-nucleus.*

(3) *If j_o is a quantic pre-nucleus on a quantale Q, then j_o is a pre-derivation on Q.*

Proposition 4.3. *Let Q be a commutative quantale and $s \in Q$, then $s \rightarrow$ is a pre-derivation on Q.*

Proof. Let $a, b \in Q$, by Remark 2.3, we have $s\&((s \rightarrow_r a)\&b) = (s\&(s \rightarrow_r a))\&b \leq a\&b$, so $(s \rightarrow_r a)\&b \leq s \rightarrow_r (a\&b)$. Similarly, $a\&(s \rightarrow_l b) \leq s \rightarrow_l (a\&b)$. Since Q is commutative, we have $a \rightarrow_r c = a \rightarrow_l c$ for all $a, c \in Q$. So $s \rightarrow (a\&b) \geq ((s \rightarrow a)\&b) \vee (a\&(s \rightarrow b))$. Since $s \rightarrow$ is order preserving, $s \rightarrow$ is a pre-derivation on Q. □

Corollary 4.4. *Let Q be a commutative quantale and $s \in Q$.*

(1) *If Q is unital and $s \leq 1$, then $s \rightarrow$ is a quantic pre-nucleus on Q.*

(2) *If Q is right (left) sided, then $s \rightarrow$ is a quantic pre-nucleus on Q.*

Proof. (1) Since $s \leq 1$, we have $1\&s = s\&1 = s \leq 1 \Rightarrow 1 \leq s \rightarrow 1$. By Remark 4.2 and Proposition 4.3, we get the conclusion.

(2) Since Q is right (left) sided, we have $s\&a = a\&s \leq a\&T \leq a$ for all $a \in Q$, then $a \leq s \rightarrow a$. By Remark 4.2 and Proposition 4.3, we get the conclusion. □

Proposition 4.5. *Let d_o be a pre-derivation on a quantale Q. If $d_o(a) \geq a$ and $j(a) = \bigwedge\{x \in Q_{d_o} | a \leq x\}$ for all $a \in Q$, then j is a quantic nucleus on Q.*

Proof. By Remark 4.2 and Theorem 2.7, we can prove it immediately. □

Theorem 4.6. *Let d be a derivation on a quantale Q. If d is idempotent, two-sided and $d(a) \geq a$ for all $a \in Q$, then d is a strict quantic nucleus of Q.*

Proof. Let $a, b \in Q$. From the proof of Theorem 3.14, we have $d(a) = d(d(a))$ and $d(a\&b) = d(a)\&d(b)$. Since $d(a) \geq a$ and d is order preserving, we have d is a strict quantic nucleus. □

A quantic nucleus j on Q is called localic iff $j(a\&b) = j(a) \wedge j(b)$ for all $a, b \in Q$.

Lemma 4.7. *Let d be a derivation on a quantale Q. If $d(a) \geq a$ for all $a \in Q$, then the following conditions are equivalent:*

(1) *d is two sided and idempotent;*

(2) *$d(a\&b) = d(a) \wedge d(b)$ for all $a, b \in Q$.*

Proof. (1) $\Rightarrow$ (2): Since d is two sided, we have $d(a\&b) \leq d(a\&\top) \leq d(a)$ and $d(a\&b) \leq d(\top\&b) \leq d(b)$. So $d(a\&b) \leq d(a) \wedge d(b)$. By Theorem 4.6, we know that d is a strict quantic nucleus, then $d(a\&b) = d(d(a\&b)) = d(d(a)\&d(b)) \geq d((d(a) \wedge d(b))\&(d(a) \wedge d(b))) = d(d(a) \wedge d(b)) \geq d(a) \wedge d(b)$. Therefore, $d(a\&b) = d(a) \wedge d(b)$.

(2) $\Rightarrow$ (1): $d(a\&\top) = d(a) \wedge d(\top) \leq d(a)$ and $d(\top\&a) = d(\top) \wedge d(a) \leq d(a)$, so d is two-sided. And $d(a\&a) = d(a) \wedge d(a) = d(a)$, then d is idempotent. □

Theorem 4.8. *Let d be a derivation on a quantale Q. If d is two sided, idempotent and $d(a) \geq a$ for all $a \in Q$, then d is localic.*

Proof. By Theorem 4.6 and Lemma 4.7, we have that d is localic. □

Theorem 4.9. *Let d_o be a pre-derivation on a quantale Q. If $d_o(a) \leq a$ and $g(a) = \vee\{x \in Q_{d_o} | x \leq a\}$ for all $a \in Q$, then g is a quantic conucleus.*

Proof. For all $a, b \in Q$, we can prove that

(i) $g(a) \leq a$.

(ii) If $a \leq b$, we have $\{x \in Q_{d_o} | x \leq a\} \subseteq \{x \in Q_{d_o} | x \leq b\}$, so $g(a) \leq g(b)$.

(iii) It is clear that $g(a) = a$ for $a \in Q_{d_o}$. So $g(g(a)) = g(\bigvee x | x \in Q_{d_o}, x \leq a) \geq \bigvee\{g(x) | x \in Q_{d_o}, x \leq a\} = \bigvee\{x | x \in Q_{d_o}, x \leq a\} = g(a)$. On the other hand, we have $g(g(a)) \leq g(a)$ by (i). Therefore, $g(g(a)) = g(a)$.

(iv) Since $g(a) = a$ for $a \in Q_{d_o}$ and d_0 is order preserving, we have $g(a) = \vee\{x \in Q_{d_o} | x \leq a\} = \vee\{d_o(x) \in Q_{d_o} | x \leq a\} \leq d_o(a)$. So $d_o(a \& g(b)) \geq a \& d_o(g(b)) \geq a \& g(g(b)) = a \& g(b)$. On the other hand, $a \& g(b) \geq d_o(a \& g(b))$. Therefore, $a \& g(b) \in Q_{d_o}$. Then $g(a \& g(b)) = \vee\{x \in Q_{d_o} | x \leq a \& g(b)\} = a \& g(b)$. So $g(a) \& g(b) = g(g(a) \& g(b)) \leq g(a \& b)$. □

Proposition 4.10. *Let Q be a quantale. If I is an ideal of Q, then $I = Q_{d_o}$ for some pre-derivation d_o.*

Proof. Define $d_o : Q \to Q$ by $d_o(a) = \bigvee\{i \in I | i \leq a\}$. It is immediate that $I = Q_{d_o}$ and $d_o(\bigvee b_i) \geq \bigvee d_o(b_i)$ for $\{b_i\} \subseteq Q$. Let $a, b \in Q$. Since I is an ideal of Q, we have $a \& d_o(b) = a \& \bigvee\{i \in I | i \leq b\} = \bigvee\{a \& i \in I | i \leq b\} \leq \bigvee\{a \& i \in I | a \& i \leq a \& b\} \leq \bigvee\{i \in I | i \leq a \& b\} = d_o(a \& b)$. Similarly, $d_o(a) \& b \leq d_o(a \& b)$. So $d_o(a \& b) \geq (a \& d_o(b)) \bigvee (d_o(a) \& b)$. □

Remark 4.11. *From Proposition 4.10, we know that every ideal of Q can be represented by the fixed set of a pre-derivation of Q. Conversely, a fixed set of a pre-derivation of Q is not an ideal in general. In Example 3.8, we know that d_2 is a derivation of Q but $Q_{d_2} = \{0, \top\}$ is not an ideal of Q for $a \& \top = a \notin Q_{d_2}$.*

Corollary 4.12. *Let I be an ideal of a quantale Q. If $g(a) = \vee\{x \in I | x \leq a\}$ for all $a \in Q$, then g is a quantic conucleus on Q.*

Proof. By Proposition 4.10, there exists a pre-derivation $d_o(a) = \bigvee\{i \in I | i \leq a\}$ such that $I = Q_{d_o}$ and $d_o(a) \leq a$. By Theorem 4.9, we have g is a quantic conucleus. □

Let a be an element of a commutative quantale Q. A mapping $d_a : Q \to Q$ defined by $d_a(b) = a \& b$ is a derivation on Q, we call d_a is a simple derivation [16]. It is obvious that zero derivation and identity derivation are special simple derivations.

Theorem 4.13 (Xiao and Li [16]). *Let a be an idempotent element of a commutative quantale Q. If Q is unital, then:*

(1) *d_a is a strict quantic nucleus on Q if $a > 1$,*

(2) *d_a is a strict quantic conucleus on Q if $a < 1$,*

(3) *d_a is a strict quantic nucleus and a quantic conucleus on Q if $a = 1$.*

Theorem 4.14. *Let a be an idempotent element of a commutative quantale Q. If Q is left (right) sided, then d_a is a quantic conucleus on Q.*

Proof. Let $b, c \in Q$, then

(i) Since Q is left sided, we have $d_a(b) = a \& b \leq \top \& b \leq b$.

(ii) $b \leq c$, we have $d_a(b) \leq d_a(c)$.

(iii) Since a is idempotent, then $d_a(d_a(b)) = a \& (d_a(b)) = a \& (a \& b) = (a \& a) \& b = a \& b = d_a(b)$.

(iv) By (i) and (ii), we have $d_a(b)\&d_a(c) = a\&b\&(a\&c) = a\&((a\&b)\&(a\&c)) = d_a(d_a(b)\&d_a(c)) \leq d_a(b\&c)$.

From the above, we can obtain that d_a is a quantic conucleus on Q. □

5 Conclusion

In this paper, we first introduced the notions of right (left, two) sided derivation and idempotent derivation for a quantale, and discussed some interesting structural properties about these derivations on quantales. We got an important result that the collection of derivations in a quantale Q is a complete lattice, furthermore, it is a quantale if the quantale Q is right sided and right commutative. Also, the fixed set Q_d of a quantale Q is a quantale and d is a quantale homomorphism from Q onto Q_d if d is two sided, idempotent and expansive. Then, we introduced pre-derivation which is an extended notion of derivation and studied the relationship between derivation and quantic nucleus. We believe that these results will be useful in theoretical computer science. We will study the generalized derivations on quantales in future work.

Acknowledgement: This work was supported by the National Science Foundation of China (No.11401052), Hunan Provincial Education Department Foundation (No.15B004) and the Science and Technology Program of Hunan of China (no. 2015RS4049).

References

[1] Dilworth R.P., Noncommutative residuated lattices, Transactions of the American Mathematical Society, 1939, 46, 426-444.
[2] Ward M., Structure residuation, Annals of Mathematics, 1938, 39, 558-569.
[3] Ward M., Dilworth R.P., Residuated lattice, Transactions of the American Mathematical Society, 1938, 45, 335-354.
[4] Mulvey C.J., &, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie, 1986, 12, 99-104.
[5] Yetter D.N., Quantales and (noncommutative) linear logic, Journal of Symbolic Logic, 1990, 55, 41-64.
[6] Girard J.Y., Linear logic, Theoretical Computer Science, 1987, 50, 1-102.
[7] Kruml D., Spatial quantales, Applied Categorical Structures, 2002, 10, 49-62.
[8] Paseka J., Kruml D., Embeddings of quantales into simple quantales, Journal of Pure and Applied Algebra, 2000, 148, 209-216.
[9] Picado J., The Quantale of Galois connections, Algebra Universalis, 2004, 52, 527-540.
[10] Rosenthal K.I., Quantales and their applications, Longman Scientific and Technical, New York,1990.
[11] Albas E., Argac N., De Filippis V. and Demir C., Generalized skew derivations on multilinear polynomials in right ideals of prime rings, Hacettepe Journal of Mathematics and Statistic, 2014, 43, 69–83.
[12] Bell H.E., Mason G., On derivations in near-rings, North-Holland Mathatic Studies, 1987, 137, 31-35.
[13] Jun Y.B., Xin X.L., On derivations of BCI-algebras, Information Sciences, 2004, 159,167-176.
[14] Xin X.L., Li T.Y., Lu J.H., On derivations of lattices, Information Sciences, 2008, 178, 307-316.
[15] Yon Y.H., Kim K.H., On derivations of subtraction algebras, Hacettepe Journal of Mathematics and Statistic, 2012, 41, 157–168.
[16] Xiao Q., Li Q., Derivation of quantales, Journal of Hunan University (in Chinese), 2012, 8, 87-89.
[17] Han S., Zhao B., Nuclei and conuclei on reiduated lattices, Fuzzy Sets and Systems, 2011, 172, 51-70.
[18] Han S., Zhao B., The quantic conuclei on quantales, Algebra Universalis, 2009, 61, 97-114.
[19] Banaschewski B., Another look at the localic Tychonoff theorem, Commentationes Mathematicae Universitatis Carolinae, 1988, 29, 647-656.
[20] Banaschewski B., On certain localic nuclei, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 1994, 35, 227-237.