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Research Article

Yu Xi Zou, Xiao Long Xin*, and Peng Fei He

On annihilators in BL-algebras

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Abstract: In the paper, we introduce the notion of annihilators in BL-algebras and investigate some related properties of them. We get that the ideal lattice $(I(L), \subseteq)$ is pseudo-complemented, and for any ideal I, its pseudo-complement is the annihilator $I^{\perp}$ of I. Also, we define the $An(L)$ to be the set of all annihilators of L, then we have that $(An(L); \cap, \vee_{An(L)}, \perp, \{0\}, L)$ is a Boolean algebra. In addition, we introduce the annihilators of a nonempty subset X of L with respect to an ideal I and study some properties of them. As an application, we show that if I and J are ideals in a BL-algebra L, then $J_I^{\perp}$ is the relative pseudo-complement of J with respect to I in the ideal lattice $(I(L), \subseteq)$. Moreover, we get some properties of the homomorphism image of annihilators, and also give the necessary and sufficient condition of the homomorphism image and the homomorphism pre-image of an annihilator to be an annihilator. Finally, we introduce the notion of α-ideal and give a notation $E(I)$. We show that $(E(I(L)), \wedge_E, \vee_E, E(0), E(L)$ is a pseudo-complemented lattice, a complete Brouwerian lattice and an algebraic lattice, when L is a BL-chain or a finite product of BL-chains.

Keywords: BL-algebra, MV-algebra, Ideal, Annihilator, Homomorphism

MSC: 08A72, 06B75

1 Introduction

It is well known that logic gives a technique for the artificial intelligence to make the computers simulate human being in dealing with certainty and uncertainty in information. And as uncertain information processing, non-classical logic has become a formal and useful tool for computer science to deal with uncertain information, fuzzy information and intelligent system. Various logical algebras have been proposed and researched as the semantical systems of non-classical logical systems. Among these logical algebras, residuated lattices were introduced by Ward and Dilworth in 1939 to constitute the semantics of Höhle Monoidal Logic which are the basis for the majority of formal fuzzy logic. Apart from their logical interest, residuated lattices have interesting algebraic properties and include two important classes of algebras: BL-algebras and MV-algebras. In order to study the basic logic framework of fuzzy set system, based on continuous triangle module and under the theoretical framework of residuated lattices theory, Hájek [1] proposed a new fuzzy logic system−BL-system and the corresponding logic algebraic system−BL-algebra. MV-algebras were introduced by Chang [2] in order to give an algebraic proof of the completeness theorem of Lukasiewice system of many valued logic.

The notion of ideals has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideals theory is a very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras the notion of ideals is at the center and deductive systems and ideals are dual notions, while in BL-algebra, with the lack of a suitable algebraic addition, the focus is shifted to deductive systems also called filters. So the notion of ideals is missing in BL-algebras. To fill this gap the paper [3] introduced the notion of ideals in BL-algebras, which

Yu Xi Zou: College of Mathematics, Northwest University, Xi'an, 710127, P.R. China
***Corresponding Author: Xiao Long Xin:** College of Mathematics, Northwest University, Xi'an, 710127, P.R. China,
E-mail: xlxin@nwu.edu.cn
Peng Fei He: School of Mathematics and Information Science, Shaanxi Normal University, Xi'an, 710119, P.R. China

generalized in a natural sense the existing notion in MV-algebras and subsequently all the results about ideals in MV-algebras. The paper also constructed some examples to show that, unlike in MV-algebras, ideals and filters are dual but behave quite differently in BL-algebra. So the notion of ideal from a purely algebraic point of view has a proper meaning in BL-algebras.

A lot of work has been done with respect to the co-annihilators and the annihilators. For example, in [4], Davery studied the relationship between minimal prime ideals conditions and annihilators conditions on distributive lattices. Turunen [5] defined co-annihilator of a non-empty set X of L and proved some of its properties on BL-algebras. They got $A^{\perp}$ as a prime filter if and only if A is linear and $A \neq \{1\}$. Also, in [6] B. A.Laurentiu Leustean introduced the notion of the co-annihilator relative to F on pseudo-BL-algebras, which is a generalization of the co-annihilator, and they also extended some results obtained in [4]. Moreover, in [7], B. L. Meng et al. defined the generalized co-annihilator of BL-algebras as a generalization of co-annihilator on BL-algebras. In [8] W.H. Cornish defined the notion of α-ideals in distributive lattices, where an ideal I is an α-ideal if $\bar{\bar{I}} = I$. Since the notion of ideals in BL-algebras has been defined in paper [3], we think it is a new direction to study the ideals by the concept of annihilators, which will enrich and develop the theory of ideals in BL-algebras.

This paper is organized as follows: In Section 2, we review some basic definitions and results about BL-algebras. In Section 3, we introduce the notion of the annihilators of BL-algebras and the notion of the annihilators of a nonempty subset X with respect to an ideal I. Also, we investigate the homomorphism image of annihilators. In section 4, we introduce the notion of α-ideals and give a notation $E(I)$. Then we focus on the algebraic structures of the set $(E(I(L))$.

2 Preliminaries

Definition 2.1 ([1]). *An algebra structure $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra , if it satisfies the following conditions: for all $x, y, z \in L$*
$BL1$ $(L, \wedge, \vee, 0, 1)$ is a bounded lattice relative to the order $\leq$;
$BL2$ $(L, \odot, 1)$ is a commutative monoid;
$BL3$ $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$;
$BL4$ $x \wedge y = x \odot (x \rightarrow y)$;
$BL5$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

In what follows, by L we denote the universe of a BL-algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$. For any $x \in L$ and a natural number n, we define $\overline{x} = x \rightarrow 0$, $\overline{\overline{x}} = \overline{(\overline{x})}$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

Proposition 2.2 ([1]). *Let L be a BL-algebra. For all $x, y, z \in L$, the followings hold:*
(1) $x \odot (x \rightarrow y) \leq y$,
(2) $x \odot y \leq x \wedge y \leq x \vee y$,
(3) $x \leq y \iff x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(5) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$ and $x \odot z \leq y \odot z$,
(6) $y \rightarrow x \leq (z \rightarrow y) \rightarrow (z \rightarrow x)$,
(7) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(8) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
(9) $(x \rightarrow y) \rightarrow z \leq x \rightarrow (y \rightarrow z)$,
(10) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \rightarrow 1 = 1$,
(11) $\overline{1} = 0$, $\overline{0} = 1, \overline{\overline{1}} = 1, \overline{\overline{0}} = 0$,
(12) $\overline{x \vee y} = \overline{x} \wedge \overline{y}$, $\overline{x \wedge y} = \overline{x} \vee \overline{y}$,
(13) $x \rightarrow y \leq x \odot z \rightarrow y \odot z$,
(14) $\overline{\overline{x \rightarrow y}} = x \rightarrow \overline{\overline{y}}$,

(15) $\overline{\overline{x \to y}} = \overline{\overline{x}} \to \overline{\overline{y}}$,
(16) $(L, \wedge, \vee)$ *is a distributive lattice.*

For every $x, y \in L$,we adopt the notation: $x \oslash y = \overline{x} \to y$.

Proposition 2.3 ([3]). *In every BL-algebra L, the following holds:*
(1) The operation $\oslash$ *is associative, that is, for every* $x, y, z \in L$, $(x \oslash y) \oslash z = x \oslash (y \oslash z)$;
(2) The operation $\oslash$ *is compatible with the order, that is, for every* $x, y, z, t \in L$, *such that* $x \leq y$ *and* $z \leq t$, *then* $x \oslash z \leq y \oslash t$.

Remark 2.4 ([3]). *If L is a BL-algebra that is not an MV-algebra, then there is an element* $x \in L$ *such that* $\overline{\overline{x}} \neq x$. *Hence* $x \oslash 0 \neq 0 \oslash x$ *and we conclude that the operation* $\oslash$ *is not commutative in general. The associative and noncommutative operation* $\oslash$ *will be called the pseudo-addition of the BL-algebra.*

Definition 2.5 ([9]). *Let L and M be two BL-algebras. A mapping* $f : L \to M$ *is said to be a homomorphism, if for any* $x, y \in L$, *we have (1)* $f(x \odot y) = f(x) \odot f(y)$; *(2)* $f(x \to y) = f(x) \to f(y)$; *(3)* $f(0) = 0$. *If f is an injection (a surjection), then f is said to be an injective (a surjection) homomorphism. If f is a bijection, then f is said to be an isomorphism.*

Let L, M be two BL-algebras, and $f : L \to M$ be a homomorphism. Then for any $x, y \in L$, (1) $f(x \wedge y) = f(x) \wedge f(y)$; (2) $f(x \vee y) = f(x) \vee f(y)$.

Theorem 2.6 ([9]). *Let* $(L, \wedge, \vee)$ *be a lattice and let* $f : L \to L$ *be a closure. Then Imf is a lattice in which the lattice operations are given by* $inf\{a, b\} = a \wedge b$, $sup\{a, b\} = f(a \vee b)$.

Definition 2.7 ([9]). *Let L be a BL-algebra and I be a nonempty subset of L. We say that I is an ideal of L if it satisfies:*
I1: for every $x, y \in L$, *if* $x \leq y$ *and* $y \in I$, *then* $x \in I$;
I2: for every $x, y \in I$, $x \oslash y \in I$.

Proposition 2.8 ([3]). *Let L be a BL-algebra and I be an ideal of L. Then for every* $x, y \in I$, *we have* $x \vee y \in I$ *and* $x \wedge y \in I$.

We recall that the smallest ideal containing A in L is called the ideal generated by the subset A in L and it is denoted by $\langle A \rangle$. It is also the intersection of all the ideals containing A.

Proposition 2.9. [3] *For every subset A of a BL-algebra L, we have*
(1) If A is empty, then $\langle A \rangle = \{0\}$;
(2) If A is not empty, then $\langle A \rangle = \{a \in L \mid a \leq x_1 \oslash x_2 \oslash \cdots \oslash x_n; x_1, x_2, \cdots x_n \in A\}$.

Definition 2.10 ([3]). *Let L be a BL-algebra and P be an ideal of L. We say P is a prime ideal if it satisfies for every* $x, y \in L$, $\overline{x \to y} \in P$ *or* $\overline{y \to x} \in P$.

Proposition 2.11 ([3]). *An ideal P of a BL-algebra L is prime if and only if for any* $x, y \in L$, $x \wedge y \in P$ *implies that* $x \in P$ *or* $y \in P$.

3 Annihilators in BL-algebras

Definition 3.1. *Let A be a non-void subset of a BL-algebra L, then we call the set* $A^{\perp} = \{x \in L \mid a \wedge x = 0$ *for all* $a \in A\}$ *is an annihilator of A.*

Example 3.2. *Let $L = \{0, a, b, c, d, 1\}$ be a set, where $0 \le a \le b \le 1$, $0 \le a \le d \le 1$ and $0 \le c \le d \le 1$. The Cayley tables are as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Now, we consider $A = \{0, a\}$, it is easy to check that $A^{\perp} = \{0, c\}$.

Proposition 3.3. *Let L be a BL-algebra and A be a subset of L, then $A^{\perp}$ is an ideal of L. Moreover, if $A \neq \{0\}$, then $A^{\perp}$ is proper.*

Proof. For every $a \in A$, we have $a \wedge 0 = 0$, hence $0 \in A^{\perp}$, which implies $A^{\perp}$ is nonempty.

I1: Assume that $y \in A^{\perp}$, $x \le y$, since for all $a \in A$, $x \wedge a \le y \wedge a = 0$, then we have $x \in A^{\perp}$;

I2: Assume that $x \in A^{\perp}$, $y \in A^{\perp}$. Let $a \in A$, then $a \wedge (\bar{x} \to y)$=$a \odot (a \to (\bar{x} \to y))$=$a \odot (\bar{x} \to (a \to y))$$\le$$a \odot (\bar{x} \odot a \to a \odot (a \to y))$=$a \odot ((\bar{x} \odot a) \to (a \wedge y))$=$a \odot ((\bar{x} \odot a) \to 0)$=$a \odot (\bar{x} \to \bar{a})$$\le$$a \odot (\bar{\bar{a}} \to \bar{\bar{x}})$$\le$$\bar{\bar{a}} \odot (\bar{\bar{a}} \to \bar{\bar{x}}$=$\bar{\bar{a}} \wedge \bar{\bar{x}}$=$\overline{\overline{a \wedge x}}$=0. Hence $x \oslash y = \bar{x} \to y \in A^{\perp}$, which implies $A^{\perp}$ is an ideal.

If $A \neq \{0\}$, then there is $a \in A$ such that $a \neq 0$, so $1 \wedge a = a \neq 0$, then we have $1 \notin A^{\perp}$. Therefore, $A^{\perp}$ is proper. □

Proposition 3.4. *Let L be a BL-algebra. Then the following conclusions hold: for all $x, a, b \in L$,*
(1) $\{1\}^{\perp} = \{0\}$, $\{0\}^{\perp} = L$;
(2) if $a \le b$, then $\{b\}^{\perp} \subseteq \{a\}^{\perp}$;
(3) $\{a\}^{\perp} \cap \{b\}^{\perp} = \{a \vee b\}^{\perp}$;
(4) $\{a\}^{\perp} \cup \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$;
(5) if $x \in \{a\}^{\perp}$, then $a \le \overline{x}$ and $x \le \overline{a}$.

Proof. (1) For all $x \in \{1\}^{\perp}$, $x = x \wedge 1 = 0$, so $x = 0$, which implies $\{1\}^{\perp} = \{0\}$. For all $x \in L$, since $x \wedge 0 = 0$, we have $L \subseteq \{0\}^{\perp}$, and evidently, $\{0\}^{\perp} \subseteq L$, so $\{0\}^{\perp} = L$.

(2) For all $x \in \{b\}^{\perp}$, we have $a \wedge x \le b \wedge x = 0$, so $a \wedge x = 0$, so we have $x \in \{a\}^{\perp}$.

(3) $x \in \{a\}^{\perp} \cap \{b\}^{\perp}$ iff $x \in \{a\}^{\perp}$ and $x \in \{b\}^{\perp}$ iff $x \wedge a = 0$ and $x \wedge b = 0$ iff $x \wedge (a \vee b) = 0$ iff $x \in \{a \vee b\}^{\perp}$.

(4) If $x \in \{a\}^{\perp} \cup \{b\}^{\perp}$, then $x \in \{a\}^{\perp}$ or $x \in \{b\}^{\perp}$, so $x \wedge a = 0$ or $x \wedge b = 0$, and so $x \wedge a \wedge b = 0$, therefore, $x \in \{a \wedge b\}^{\perp}$.

(5) If $x \in \{a\}^{\perp}$, then $x \wedge a = 0$, so we have $a \odot x \le x \wedge a = 0$, then $a \odot x = 0$, therefore, $a \le \overline{x}$ and $x \le \overline{a}$. □

Example 3.5. *Let L be the BL-algebra in Example 3.2. We have $a^{\perp} = \{0, c\}$, $c^{\perp} = \{0, a, b\}$, and $a \wedge c = 0$, so $a^{\perp} \cup c^{\perp} = \{0, a, b, c\}$. Hence $0^{\perp} = L \nsubseteq a^{\perp} \cup c^{\perp}$. Therefore, we do not have the equation for Proposition 3.4(4).*

Proposition 3.6. *Let L be a BL-algebra. Then the following conclusions hold: for all $x, y, a, b \in L$,*
(1) if $x \in \{a\}^{\perp}$, $y \in \{a \vee b\}^{\perp}$, then $x \wedge y \in \{a \wedge b\}^{\perp}$;
(2) if $x \in \{a\}^{\perp}$, $y \in \{a \to b\}^{\perp}$, then $x \wedge y \in \{a \wedge b\}^{\perp}$;
(3) if $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$, then $x \odot y, x \vee y, x \wedge y \in \{a \wedge b\}^{\perp}$.

Proof. (1) Since $y \in \{a \vee b\}^{\perp}$ and $b \le a \vee b$, by Proposition 3.4 (2), we have $\{a \vee b\}^{\perp} \subseteq \{b\}^{\perp}$, so $y \in \{b\}^{\perp}$, since $\{b\}^{\perp}$ is a down set, we get $x \wedge y \in \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$.

(2) Since $y \in \{a \to b\}^{\perp}$ and $b \leq a \to b$, by Proposition 3.4 (2), we get $\{a \to b\}^{\perp} \subseteq \{b\}^{\perp}$, so $y \in \{b\}^{\perp}$. Since $\{b\}^{\perp}$ is a down set, we get $x \wedge y \in \{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$.

(3) Since $x \in \{a\}^{\perp}$, $y \in \{b\}^{\perp}$ and $a \wedge b \leq a, b$, by Proposition 3.4 (2), we have $\{a\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$, $\{b\}^{\perp} \subseteq \{a \wedge b\}^{\perp}$, so $x, y \in \{a \wedge b\}^{\perp}$, and so $x \vee y \in \{a \wedge b\}^{\perp}$. Moreover, since $\{a \wedge b\}^{\perp}$ is a down set, we get $x \odot y, x \wedge y \in \{a \wedge b\}^{\perp}$. □

Proposition 3.7. *For any $\emptyset \neq X \subseteq L$, $\langle X \rangle \cap X^{\perp} = \{0\}$.*

Proof. Assume that $a \in \langle X \rangle \cap X^{\perp}$, then we have $a \leq x_1 \oslash x_2 \oslash \cdots \oslash x_n$ $x_i \in X$, and $a \wedge x_i = 0$ $i = 1, 2, \cdots n$.

Now we prove that $a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (a \wedge x_1) \oslash (a \wedge x_2) \oslash \cdots \oslash (a \wedge x_n)$. Firstly, $a \wedge (x \oslash y) \to ((a \wedge x) \oslash (a \wedge y)) \leq a \wedge x \to (\overline{(a \wedge x)} \to (a \wedge y)) = ((a \wedge x) \odot \overline{(a \wedge x)}) \to (a \wedge y) = 0 \to (a \wedge y) = 1$, so $a \wedge (x \oslash y) \leq (a \wedge x) \oslash (a \wedge y)$. Then assume that when $k = n$, $a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (a \wedge x_1) \oslash (a \wedge x_2) \oslash \cdots \oslash (a \wedge x_n)$ holds. When $k = n + 1$, $a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n \oslash x_{n+1}) \leq (a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \oslash (a \wedge x_{n+1}) \leq (a \wedge x_1) \oslash (a \wedge x_2) \oslash \cdots \oslash (a \wedge x_n) \oslash (a \wedge x_{n+1})$.

Considering the above we have $a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (a \wedge x_1) \oslash (a \wedge x_2) \oslash \cdots \oslash (a \wedge x_n)$. Then $a \wedge a \leq a \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (a \wedge x_1) \oslash (a \wedge x_2) \oslash \cdots \oslash (a \wedge x_n) = 0 \oslash 0 \oslash \cdots \oslash 0 = 0$. Therefore, $\langle X \rangle \cap X^{\perp} = \{0\}$. □

Proposition 3.8. *Let A be an ideal of L and A be linear(which means that A is totally ordered). Then $A^{\perp}$ is a prime ideal.*

Proof. Assume that A is an ideal which is linear, and $a \wedge b \in A^{\perp}$ but $a, b \notin A^{\perp}$. Then there are $x', x'' \in A$, such that $a \wedge x' \neq 0$, and $b \wedge x'' \neq 0$. Set $x = x' \vee x''$. Then $x \in A$ as A is an ideal. Clearly, $a \wedge x = a \wedge (x' \vee x'') = (a \wedge x') \vee (a \wedge x'') \neq 0$. Similarly, we have $b \wedge x \neq 0$. Since $a \wedge x \leq x, b \wedge x \leq x$, we conclude $a \wedge x, b \wedge x \in A$. As A is linear, we may assume that $a \wedge x \leq b \wedge x$. Now, $0 = (a \wedge b) \wedge x = a \wedge (b \wedge x) \geq a \wedge (a \wedge x) = a \wedge x$, which contradicts the fact $a \wedge x \neq 0$, which implies that $a \in A^{\perp}$ or $b \in A^{\perp}$. Therefore, $A^{\perp}$ is prime. □

Proposition 3.9. *Let L be a BL- algebra, if $X \subseteq Y \subseteq L$, then $Y^{\perp} \subseteq X^{\perp}$;*

Proof. If $z \in Y^{\perp}$, we have $z \wedge y = 0$. Then for any $x \in X \subseteq Y$, $z \wedge x = 0$, and so $z \in \cap_{x \in X}\{x\}^{\perp} = X^{\perp}$. This means that $Y^{\perp} \subseteq X^{\perp}$. □

Proposition 3.10. *Let L be a BL- algebra, for any $\emptyset \neq X \subseteq L$, the following hold:*
(1) $X^{\perp} = \cap_{x \in X}\{x\}^{\perp}$;
(2) $X \subseteq X^{\perp\perp}$;
(3) $X^{\perp} = X^{\perp\perp\perp}$;
(4) $X^{\perp} = \langle X \rangle^{\perp}$.

Proof. (1) $a \in X^{\perp} \Longleftrightarrow$ for all $x \in X$, $a \wedge x = 0 \Longleftrightarrow$ for all $x \in X$, $a \in x^{\perp} \Longleftrightarrow a \in \cap_{x \in X}\{x\}^{\perp}$.

(2) By the definition of annihilator, we have $X^{\perp\perp} = \{a \in L \mid a \wedge x = 0 \ for \ all \ x \in X^{\perp}\}$. So for all $x \in X^{\perp}$, if $b \in X$, then $b \wedge x = 0$, hence $b \in X^{\perp\perp}$.

(3) By (2) taking $X = X^{\perp}$ we have $X^{\perp} \subseteq X^{\perp\perp\perp}$. Conversely, by (1) and Proposition 3.8, we have $X^{\perp\perp\perp} \subseteq X^{\perp}$, therefore, $X^{\perp} = X^{\perp\perp\perp}$.

(4) Since $X \subseteq \langle X \rangle$, by Proposition 3.9, we have $\langle X \rangle^{\perp} \subseteq X^{\perp}$. Now we prove $X^{\perp} \subseteq \langle X \rangle^{\perp}$. Let $y \in X^{\perp}$, so for any $x \in X$, we have $x \wedge y = 0$. For any $z \in \langle X \rangle$, there are $x_1, x_2, \cdots x_n \in X$, such that $z \leq x_1 \oslash x_2 \oslash \cdots \oslash x_n$. So $y \wedge z \leq y \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (y \wedge x_1) \oslash (y \wedge x_2) \oslash \cdots \oslash (y \wedge x_n) = 0 \oslash 0 \oslash \cdots \oslash 0 = 0$. Hence $y \in \langle X \rangle^{\perp}$, that is, $X^{\perp} \subseteq \langle X \rangle^{\perp}$. Therefore, $X^{\perp} = \langle X \rangle^{\perp}$. □

Proposition 3.11. *Let L be a BL-algebra, and $X, Y \subseteq L$. Then*
(1) $L^{\perp} = \{0\}$;
(2) $X^{\perp\perp} \cap X^{\perp} = \{0\}$;

(3) $(X \cup Y)^{\perp} = X^{\perp} \cap Y^{\perp}$*;*
(4) $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$.

Proof. (1) If there is $a \in L$ such that $a \neq 0$ and $a \in L^{\perp}$, then $1 \wedge a = a \neq 0$, this is a contradiction, so $a = 0$.

(2) If $x \in X^{\perp\perp} \cap X^{\perp}$, by the definition of annihilator, we have $x = x \wedge x = 0$, so $X^{\perp\perp} \cap X^{\perp} \subseteq 0$. Conversely, since for every $Y \subseteq L$ there is $0 \in Y^{\perp}$, we get $0 \in X^{\perp\perp}$ and $0 \in X^{\perp}$, so $0 \in X^{\perp\perp} \cap X^{\perp}$, finally $X^{\perp\perp} \cap X^{\perp} = \{0\}$.

(3) Since $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$, we have $(X \cup Y)^{\perp} \subseteq X^{\perp}$ and $(X \cup Y)^{\perp} \subseteq Y^{\perp}$, so $(X \cup Y)^{\perp} \subseteq X^{\perp} \cap Y^{\perp}$, conversely, for any $a \in X^{\perp} \cap Y^{\perp}$, we have $a \in X^{\perp}$ and $a \in Y^{\perp}$, ie. for any $x \in X, y \in Y$, we have $a \wedge x = 0$ and $a \wedge y = 0$. So for any $t \in X \cup Y$, we always have $a \wedge t = 0$, hence $a \in (X \cup Y)^{\perp}$.

(4) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we have $X^{\perp} \subseteq (X \cap Y)^{\perp}$ and $Y^{\perp} \subseteq (X \cap Y)^{\perp}$, so $X^{\perp} \cup Y^{\perp} \subseteq (X \cap Y)^{\perp}$. □

Example 3.12. *Let L be the BL-algebra in Example 3.2. Take* $X = \{a, c\}$, $Y = \{c, b\}$*, then* $X \cap Y = \{c\}$*, so we have* $X^{\perp} = \{0\}$, $Y^{\perp} = \{0\}$*, and* $(X \cap Y)^{\perp} = \{0, a, b\}$*. Hence* $(X \cap Y)^{\perp} \nsubseteq X^{\perp} \cup Y^{\perp}$*. Therefore, we do not have the equation for Proposition 3.11(4)*

Proposition 3.13. *Let L be a BL-algebra, and* $\emptyset \neq X, Y \subseteq L$*. For all* $a, b \in L$*, if* $a \in X^{\perp}, b \in Y^{\perp}$*, then (1)* $a \wedge b \in (X \cup Y)^{\perp}$*, (2)* $a \vee b \in (X \cap Y)^{\perp}$.

Proof. (1) If $a \in X^{\perp}, b \in Y^{\perp}$, then $a \wedge b \in X^{\perp}$ and $a \wedge b \in Y^{\perp}$, so $a \wedge b \in X^{\perp} \cap Y^{\perp} = (X \cup Y)^{\perp}$.

(2) If $a \in X^{\perp}, b \in Y^{\perp}$, then $a \in X^{\perp} \subseteq (X \cap Y)^{\perp}$ and $b \in Y^{\perp} \subseteq (X \cap Y)^{\perp}$, so $a \vee b \in (X \cap Y)^{\perp}$. □

Proposition 3.14. *Let L be a BL-algebra,* $X, Y \subseteq L$*. Then* $X^{\perp} \cap Y^{\perp} = \{0\}$ *if and only if* $X^{\perp} \subseteq Y^{\perp\perp}$ *and* $Y^{\perp} \subseteq X^{\perp\perp}$.

Proof. $\Longrightarrow$ For all $a \in X^{\perp}, b \in Y^{\perp}$, we have $a \wedge b \in X^{\perp} \cap Y^{\perp}$, since $X^{\perp} \cap Y^{\perp} = \{0\}$, we get $a \wedge b = 0$, by definition of annihilator, we get $a \in Y^{\perp\perp}$ and $b \in X^{\perp\perp}$, so $X^{\perp} \subseteq Y^{\perp\perp}$ and $Y^{\perp} \subseteq X^{\perp\perp}$.

$\Longleftarrow$ If $X^{\perp} \subseteq Y^{\perp\perp}$ and $Y^{\perp} \subseteq X^{\perp\perp}$, then $X^{\perp} \cap Y^{\perp} \subseteq Y^{\perp\perp} \cap Y^{\perp} = \{0\}$, so $X^{\perp} \cap Y^{\perp} \subseteq \{0\}$. Clearly, $\{0\} \subseteq X^{\perp} \cap Y^{\perp}$, so $X^{\perp} \cap Y^{\perp} = \{0\}$. □

Proposition 3.15. *Let L be a BL-algebra, I be a non-empty subset of L. Then there is a* $X \subseteq L$*, satisfied* $I = X^{\perp}$ *if and only if I is a down set, and* $I^{\perp\perp} = I$, $I \cap X^{\perp\perp} = \{0\}$, $X^{\perp} \cap I^{\perp} = \{0\}$.

Proof. $\Longrightarrow$ If $I = X^{\perp}$, it is clear that I is a down set, $I^{\perp\perp} = X^{\perp\perp\perp} = X^{\perp} = I$, $I \cap X^{\perp\perp} = X^{\perp} \cap X^{\perp\perp} = \{0\}$. Then we have $X^{\perp} \cap I^{\perp} = X^{\perp} \cap X^{\perp\perp} = \{0\}$ by Proposition 3.11 (2).

$\Longleftarrow$ Since $I \cap X^{\perp\perp} = \{0\}$, for any $i \in I$, for any $x \in X^{\perp\perp}$, then $i \wedge x \in I \cap X^{\perp\perp} = \{0\}$, so $i \in X^{\perp\perp\perp}$. We have already known $X^{\perp\perp\perp} = X^{\perp}$, so $I \subseteq X^{\perp}$. Since $X^{\perp} \cap I^{\perp} = \{0\}$, then similarly, we get $X^{\perp} \subseteq I^{\perp\perp}$, since $I^{\perp\perp} = I$, we get $X^{\perp} \subseteq I$. Therefore, $I = X^{\perp}$. □

Proposition 3.16. *Let L be a BL-algebra, if an ideal I which is linear contains an element* $x \neq 0$ *and* $x \vee \overline{x} = 1$*, then x is the largest element of I.*

Proof. By $x \vee \overline{x} = 1$, we have $\overline{x} \wedge \overline{\overline{x}} = 0$. So $x \wedge \bar{x} \leq \overline{x} \wedge \overline{\overline{x}} = 0$. Let $a \in I$, then $a = a \wedge 1 = a \wedge (x \vee \overline{x}) = (a \wedge x) \vee (a \wedge \overline{x})$, where the last equation follows by the distributive of L. Since I is linear, by Proposition 3.8, we have $I^{\perp}$ is a prime ideal. Since $x \wedge \overline{x} = 0 \in I^{\perp}$, either $x \in I^{\perp}$ or $\overline{x} \in I^{\perp}$. As $x \wedge x = x \neq 0$, we necessarily have $\overline{x} \in I^{\perp}$. Hence for any $a \in I$, we have $a \wedge \overline{x} = 0$, which implies that $a = a \wedge x$, thus $a \leq x$. Therefore, x is the largest element of I. □

Proposition 3.17. *Let L be a BL-algebra. Then the ideal lattice* $I(L)$ *is pseudo-complemented and for any ideal I of L, its pseudo-complement is* $I^{\perp}$.

Proof. By Proposition 3.7, we have $I \cap I^{\perp} = \{0\}$. Let G be an ideal of L such that $I \cap G = \{0\}$, we shall prove that $G \subseteq I^{\perp}$. Let $a \in G$, for any $x \in I$, then we have $x \wedge a \leq x \in I$, $x \wedge a \leq a \in G$, so $x \wedge a \in I \cap G = \{0\}$. Hence $x \wedge a = 0$ for any $x \in I$, then we have $a \in I^{\perp}$. So $I^{\perp}$ is the largest ideal such that $I \cap G = \{0\}$. It follows that $I^{\perp}$ is the pseudo-complement of I. □

Let $An(L) = \{X^{\perp} \mid X \subseteq L\}$ be the set of annihilators of L. Since $X^{\perp} = \langle X\rangle^{\perp}$, we get that $An(L) = \{I^{\perp} \mid I \in I(L)\}$. Hence $An(L)$ is the set of pseudo-complements of the pseudo-complemented lattice $I(L)$.

Proposition 3.18. *Let L be a BL-algebra, $I, J \in I(L)$. Then*
(1) $\{0\}, L \in An(L)$;
(2) $I \in An(L) \Longleftrightarrow I^{\perp\perp} = I$;
(3) $\perp\perp: X \to X^{\perp\perp}$ is a closure map;
(4) $I \cap (I \cap J)^{\perp} = I \cap J^{\perp}$;
(5) $(I \cap J)^{\perp\perp} = I^{\perp\perp} \cap J^{\perp\perp}$;
(6) $I, J \in An(L)$, then $I \wedge_{An(L)} J = I \cap J$;
(7) $(I \vee J)^{\perp} = I^{\perp} \cap J^{\perp}$;
(8) if $I, J \in An(L)$, then $I \vee_{An(L)} J = (I^{\perp} \cap J^{\perp})^{\perp}$.

Proof. (1) By Propositions 3.4 and 3.11.

(2) Assume that $I \in An(L)$, then there exists $X \subseteq L$ such that $X^{\perp} = I$, so we get $I^{\perp\perp} = X^{\perp\perp\perp} = X^{\perp} = I$. The converse is clear.

(3) By Propositions 3.9, we know the function $f : X \to X^{\perp\perp}$ is isotone and by Propositions 3.10, we get that $f = f^2 \geq id_L$. So, $X \to X^{\perp\perp}$ is a closure map.

(4) Since $(I \cap J) \cap (I \cap J)^{\perp} = \{0\}$, by Proposition 3.17, we get $I \cap (I \cap J)^{\perp} \subseteq J^{\perp}$ and so $I \cap (I \cap J)^{\perp} \subseteq I \cap J^{\perp}$. Conversely, by $I \cap J \subseteq J$, we get $J^{\perp} \subseteq (I \cap J)^{\perp}$, so $I \cap J^{\perp} \subseteq I \cap (I \cap J)^{\perp}$. Therefore, $I \cap (I \cap J)^{\perp} = I \cap J^{\perp}$.

(5) Since $I \cap J \subseteq I, J$, we get $(I \cap J)^{\perp\perp} \subseteq I^{\perp\perp} \cap J^{\perp\perp}$. Conversely, $(I \cap J) \cap (I \cap J)^{\perp} = \{0\} \Rightarrow I \cap (I \cap J)^{\perp} \subseteq J^{\perp} = J^{\perp\perp\perp} \Rightarrow I \cap J^{\perp\perp} \cap (I \cap J)^{\perp} = \{0\} \Rightarrow J^{\perp\perp} \cap (I \cap J)^{\perp} \subseteq I^{\perp} = I^{\perp\perp\perp} \Rightarrow I^{\perp\perp} \cap J^{\perp\perp} \cap (I \cap J)^{\perp} = \{0\} \Rightarrow I^{\perp\perp} \cap J^{\perp\perp} \subseteq (I \cap J)^{\perp\perp}$. So we get $(I \cap J)^{\perp\perp} = I^{\perp\perp} \cap J^{\perp\perp}$.

(6) By (3) and Proposition 2.6, we have $I \wedge_{An(L)} J = I \cap J$.

(7) Since $I, J \subseteq I \vee J$, we get $(I \vee J)^{\perp} \subseteq I^{\perp} \cap J^{\perp} = I^{\perp\perp\perp} \cap J^{\perp\perp\perp} = (I^{\perp} \cap J^{\perp})^{\perp\perp}$. Conversely, $I \subseteq I^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$, similarly, we have $J \subseteq J^{\perp\perp} \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$, so $I \vee J \subseteq (I^{\perp} \cap J^{\perp})^{\perp}$, hence $(I^{\perp} \cap J^{\perp})^{\perp\perp} \subseteq (I \vee J)^{\perp}$. Therefore, $(I \vee J)^{\perp} = (I^{\perp} \cap J^{\perp})^{\perp\perp} = I^{\perp\perp\perp} \cap J^{\perp\perp\perp} = I^{\perp} \cap J^{\perp}$.

(8) By (3) and Proposition 2.6, we have $I \vee_{An(L)} J = (I \vee J)^{\perp\perp}$, then by (7) we have $I \vee_{An(L)} J = (I^{\perp} \cap J^{\perp})^{\perp}$. □

Theorem 3.19. *Let L be a BL-algebra. Then $(An(L); \cap, \vee_{An(L)}, \perp, \{0\}, L)$ is a Boolean algebra.*

Proof. Firstly, we show that $An(L)$ is distributive, it suffices to prove that: for all $I, J, H \in An(L)$, $H \cap (I \vee_{An(L)} J) \subseteq (H \cap I) \vee_{An(L)} (H \cap J)$. Now, let $K = (H \cap I) \vee_{An(L)} (H \cap J)$, then $H \cap I \subseteq K = K^{\perp\perp}$ gives $H \cap I \cap K^{\perp} = \{0\}$ and so $H \cap K^{\perp} \subseteq I^{\perp}$. Similarly, $H \cap K^{\perp} \subseteq J^{\perp}$ and therefore $H \cap K^{\perp} \subseteq I^{\perp} \cap J^{\perp} = (I^{\perp} \cap J^{\perp})^{\perp\perp}$. It follows that $H \cap K^{\perp} \cap (I^{\perp} \cap J^{\perp})^{\perp} = \{0\}$ and hence $H \cap (I \vee_{An(L)} J) = H \cap (I^{\perp} \cap J^{\perp})^{\perp} \subseteq K^{\perp\perp} = K = (H \cap I) \vee_{An(L)} (H \cap J)$.

Secondly, we show that $An(L)$ is complemented, observe that $L = \{0\}^{\perp} \in An(L)$ and $\{0\} = L^{\perp} \in An(L)$. Since for every $I \in An(L)$ we have $I \cap I^{\perp} = \{0\}$ and $I \vee_{An(L)} I^{\perp} = (I^{\perp} \cap I^{\perp\perp})^{\perp} = \{0\}^{\perp} = L$. So the complement of $I \in An(L)$ is $I^{\perp}$. Therefore, $An(L)$ is a Boolean algebra. □

Definition 3.20. *Let L be a BL-algebra, $\emptyset \neq X \subseteq L$, I be an ideal of L and f be an endomorphism. We define the annihilator of X with respect to I to be the set $X_I^{\perp f} = \{a \in L \mid f(a) \wedge x \in I, \forall x \in X\}$.*

If $f = id_L$, we denote $X_I^{\perp} := X_I^{\perp id_L} = \{a \in L \mid a \wedge x \in I, \forall x \in X\}$.

Example 3.21. *Let $L = \{0, a, b, c, 1\}$ be a set, where $0 \le a \le c \le 1$ and $0 \le b \le c \le 1$. The Cayley tables are as follows:*

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Now we define a map f as follows: $f(0) = 0$, $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(1) = 1$, then we can check that f is an endomorphism. Let $X = \{a, c\}$, and $I = \{0, a\}$, then we get $X_I^{\perp_f} = \{0, b\}$.

Example 3.22. *Let $L = \{0, a, b, 1\}$ be a set, where $0 \le a, b \le 1$. The Cayley tables are as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Let $X = \{0, b\}$, and $I = \{0, a\}$, then we get $X_I^{\perp} = \{0, a\}$.

Proposition 3.23. *Let L be a BL-algebra, $\emptyset \ne X \subseteq L$, I be an ideal of L and f be an endomorphism. Then $X_I^{\perp_f}$ is an ideal of L.*

Proof. Clearly, $0 \in X_I^{\perp_f}$, so $X_I^{\perp_f}$ is nonempty.

I1: Let $a \in X_I^{\perp_f}$ and $b \in L$ such that $b \le a$. Then $f(b) \le f(a)$. It follows that $f(b) \wedge x \le f(a) \wedge x$. Since $f(a) \wedge x \in I$, we have $f(b) \wedge x \in I$, which implies $b \in X_I^{\perp_f}$;

I2: Let $a, b \in X_I^{\perp_f}$, then $f(a) \wedge x \in I$ and $f(b) \wedge x \in I$ for all $x \in X$. Since $f(a \oslash b) \wedge x = (f(a) \oslash f(b)) \wedge x \le (f(a) \wedge x) \oslash (f(b) \wedge x)$, we have $f(a \oslash b) \wedge x \in I$, that is, $a \oslash b \in X_I^{\perp_f}$. □

Proposition 3.24. *Let L be a BL-algebra, f be an endomorphism, $I, J \in I(L)$ and $\emptyset \ne X, X' \subseteq L$. Then we have:*
(1) $I \subseteq J$ implies $X_I^{\perp_f} \subseteq X_J^{\perp_f}$;
(2) $X \subseteq X'$ implies $(X')_I^{\perp_f} \subseteq X_I^{\perp_f}$;
(3) $(\cup_{\lambda \in \Lambda} X_\lambda)_I^{\perp_f} = \cap_{\lambda \in \Lambda} (X_\lambda)_I^{\perp_f}$;
(4) $X_I^{\perp_f} = \cap_{x \in X} x_I^{\perp_f}$;
(5) $X_{\cap_{\lambda \in \Lambda} I_\lambda}^{\perp_f} = \cap_{\lambda \in \Lambda} X_{I_\lambda}^{\perp_f}$;
(6) $\langle X \rangle_I^{\perp_f} = X_I^{\perp_f}$;
(7) Ker(f)$\subseteq X_{\{0\}}^{\perp_f}$, $L_{\{0\}}^{\perp_f}$ =Ker(f).

Proof. (1) Let $I \subseteq J$ and $a \in X_I^{\perp_f}$. Then $f(a) \wedge x \in I$ for all $x \in X$. So we have $f(a) \wedge x \in J$ for all $x \in X$, that is, $a \in X_J^{\perp_f}$. Therefore, $X_I^{\perp_f} \subseteq X_J^{\perp_f}$.

(2) Let $X \subseteq X'$ and $a \in (X')_I^{\perp_f}$. Then $f(a) \wedge x \in I$ for all $x \in X'$. Hence $f(a) \wedge x \in I$ for all $x \in X$, which implies $a \in X_I^{\perp_f}$. Therefore, $(X')_I^{\perp_f} \subseteq X_I^{\perp_f}$.

(3) By (2) we have $(\cup_{\lambda \in \Lambda} X_\lambda)_I^{\perp_f} \subseteq (X_\lambda)_I^{\perp_f}$ for all $\lambda \in \Lambda$, so we get $(\cup_{\lambda \in \Lambda} X_\lambda)_I^{\perp_f} \subseteq \cap_{\lambda \in \Lambda} (X_\lambda)_I^{\perp_f}$. Conversely, let $a \in \cap_{\lambda \in \Lambda} (X_\lambda)_I^{\perp_f}$, we have $a \in (X_\lambda)_I^{\perp_f}$ for all $\lambda \in \Lambda$. Hence $f(a) \wedge x_\lambda \in I$ for all $x_\lambda \in X_\lambda$ and $\lambda \in \Lambda$, which implies $a \in (\cup_{\lambda \in \Lambda} X_\lambda)_I^{\perp_f}$. Therefore, $(\cup_{\lambda \in \Lambda} X_\lambda)_I^{\perp_f} = \cap_{\lambda \in \Lambda} (X_\lambda)_I^{\perp_f}$.

(4) It is clear by (3).

(5) We have $a \in X^{\perp_f}_{\cap_{\lambda\in\Lambda} I_\lambda}$ if and only if $f(a) \wedge x \in \cap_{\lambda\in\Lambda} I_\lambda$ for all $x \in X$, and if and only if $f(a) \wedge x \in I_\lambda$ for all $x \in X$ and $\lambda \in \Lambda$, which is equivalent to $a \in X^{\perp_f}_{I_\lambda}$ for all $\lambda \in \Lambda$, that is, $a \in \cap_{\lambda\in\Lambda} X^{\perp_f}_{I_\lambda}$.

(6) Since $X \subseteq \langle X \rangle$, by (2) we get $\langle X \rangle^{\perp_f}_I \subseteq X^{\perp_f}_I$. Conversely, let $a \in X^{\perp_f}_I$ and $z \in \langle X \rangle$. Then $f(a) \wedge x \in I$ for all $x \in X$. Since $z \in \langle X \rangle$, then there exist $x_1, x_2, \cdots, x_n \in X$ such that $z \leq x_1 \oslash x_2 \oslash \cdots \oslash x_n$. It follows that $f(a) \wedge z \leq f(a) \wedge (x_1 \oslash x_2 \oslash \cdots \oslash x_n) \leq (f(a) \wedge x_1) \oslash (f(a) \wedge x_2) \oslash \cdots \oslash (f(a) \wedge x_n)$, since $f(a) \wedge x_i \in I$ for all $1 \leq i \leq n$, we get $f(a) \wedge z \in I$, which implies that $a \in \langle X \rangle^{\perp_f}_I$. Therefore, $\langle X \rangle^{\perp_f}_I = X^{\perp_f}_I$.

(7) Let $a \in$ Ker(f), we have $f(a) = 0$, then $f(a) \wedge x = 0$ for all $x \in X$, that is, $a \in X^{\perp_f}_{\{0\}}$, hence Ker(f)$\subseteq X^{\perp_f}_{\{0\}}$. Let $a \in L^{\perp_f}_{\{0\}}$, then $f(a) \wedge x = 0$ for all $x \in L$. In particular, taking $x = f(a)$, we have $f(a) = 0$, which implies $a \in$ Ker(f). Hence $L^{\perp_f}_{\{0\}} \subseteq$Ker(f). Conversely, by Ker(f)$\subseteq X^{\perp_f}_{\{0\}}$, for any $\emptyset \neq X \subseteq L$, taking $X = L$, so Ker(f)$\subseteq L^{\perp_f}_{\{0\}}$. Therefore, $L^{\perp_f}_{\{0\}}$ =Ker(f). □

Proposition 3.25. *Let I be an ideal of a BL-algebra L, f be an endomorphism and $a, b \in L$. Then (1) $a \leq b$ implies $b^{\perp_f}_I \subseteq a^{\perp_f}_I$; (2) $(a \vee b)^{\perp_f}_I = a^{\perp_f}_I \cap b^{\perp_f}_I$.*

Proof. (1) Let $x \in b^{\perp_f}_I$, then $f(x) \wedge b \in I$ and $f(x) \wedge a \leq f(x) \wedge b$, since $a \leq b$. It follows that $f(x) \wedge a \in I$, that is, $x \in a^{\perp_f}_I$;

(2) Since $a, b \leq a \vee b$, by (1) we have that $(a \vee b)^{\perp_f}_I \subseteq a^{\perp_f}_I, b^{\perp_f}_I$, so $(a \vee b)^{\perp_f}_I \subseteq a^{\perp_f}_I \cap b^{\perp_f}_I$. Conversely, let $x \in a^{\perp_f}_I \cap b^{\perp_f}_I$, that is, $a \wedge f(x), b \wedge f(x) \in I$, it follows $f(x) \wedge (a \vee b) = (f(x) \wedge a) \vee (f(x) \wedge b) \in I$, as I is an ideal of L, and L is a distributive lattice. Therefore, $x \in (a \vee b)^{\perp_f}_I$; □

Proposition 3.26. *Let L be a BL-algebra, $\emptyset \neq X \subseteq L$, I be an ideal of L. Then*
(1) $I \subseteq X^{\perp}_I$;
(2) $X^{\perp}_I = L$ if and only if $X \subseteq I$.

Proof. (1) Let $i \in I$, then $i \wedge x \leq i \in I$ for all $x \in X$, hence $i \in X^{\perp}_I$, that is, $I \subseteq X^{\perp}_I$;

(2) If $X^{\perp}_I = L$, then $1 \in X^{\perp}_I$, so for all $x \in X$ we have $x = x \wedge 1 \in I$. Conversely, if $X \subseteq I$, then for any $a \in L$ and for all $x \in X$ we have $a \wedge x \leq x \in I$, so $a \in X^{\perp}_I$. That is, $L \subseteq X^{\perp}_I$, which implies that $X^{\perp}_I = L$. □

Proposition 3.27. *Let L be a BL-algebra, $I, J, H \in I(L)$. Then we have:*
(1) $J^{\perp}_I \cap J \subseteq I$;
(2) $J \cap H \subseteq I$ if and only if $H \subseteq J^{\perp}_I$.

Proof. (1) Let $x \in J^{\perp}_I \cap J$, then $x \in J^{\perp}_I$ and $x \in J$, so we get $x = x \wedge x \in I$. That is, $J^{\perp}_I \cap J \subseteq I$;

(2) If $J \cap H \subseteq I$ and $x \in H$, then for any $y \in J$, we have $x \wedge y \in J \cap H$, it follows that $x \wedge y \in I$. Hence $x \in J^{\perp}_I$, that is $H \subseteq J^{\perp}_I$. Conversely, let $H \subseteq J^{\perp}_I$, by (1), we have $J \cap H \subseteq J \cap J^{\perp}_I \subseteq I$. □

Theorem 3.28. *Let L be a BL-algebra, $I, J \in I(L)$. Then $J^{\perp}_I$ is the relative pseudo-complement of J with respect to I in the lattice $(I(L), \subseteq)$.*

Proof. We have already known $J^{\perp}_I$ is an ideal, and by Proposition 3.27 (1), we have $J^{\perp}_I \cap J \subseteq I$. Now we show that $J^{\perp}_I$ is the greatest ideal of L such that $H \cap J \subseteq I$, where $H \in I(L)$. Assume that H is an ideal of L such that $H \cap J \subseteq I$, If $a \in H$, then $a \wedge x \leq a, x$ for all $x \in J$. Since J and H are ideals of L, we have $a \wedge x \in J \cap H$, Hence $a \wedge x \in I$ for all $x \in J$, that is, $a \in J^{\perp}_I$. Therefore, $J^{\perp}_I$ is the relative pseudo-complement of J with respect to I in the lattice $(I(L), \subseteq)$. □

Proposition 3.29. *Let L, M be two BL-algebras, $f : L \to M$ be a homomorphism, $\emptyset \neq A \subseteq L$. Then $f(A^{\perp}) \subseteq (f(A))^{\perp}$.*

Proof. For all $x \in f(A^{\perp})$, there is a $y \in A^{\perp}$, such that $x = f(y)$. For all $z \in f(A)$, there is a $t \in A$, such that $z = f(t)$. So we have $x \wedge z = f(y) \wedge f(t) = f(y \wedge t) = f(0) = 0$, therefore, $x \in (f(A))^{\perp}$. □

The next example shows the following: let L, M be two BL-algebras, $f : L \to M$ be a homomorphism, $\emptyset \neq A \subseteq L$, then $f(A^{\perp})$ may not be an annihilator of a subset of M.

Example 3.30. *Let L be the BL-algebra of Example 3.22.*

Let $M = \{0, \frac{1}{2}, 1\}$, such that $0 \leq \frac{1}{2} \leq 1$. The Cayley tables are as follows:

$\odot$	0	$\frac{1}{2}$	1
0	0	0	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\to$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1
1	0	$\frac{1}{2}$	1

Then $(M, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Let $f(1) = f(a) = 1$, $f(0) = f(b) = 0$, then $f : L \to M$ is a homomorphism. Let $A = \{b\}$, then $A^{\perp} = \{0, a\}$, $f(A^{\perp}) = \{0, 1\}$, clearly $\{0, 1\}$is not a down set, so there is no $B \subseteq M$, such that $f(A^{\perp}) = B^{\perp}$.

Proposition 3.31. *Let L, M be two BL-algebras, $f : L \to M$ be a surjective homomorphism, $\emptyset \neq B \subseteq M$. Then $(f^{-1}(B))^{\perp} \subseteq f^{-1}(B^{\perp})$.*

Proof. For all $x \in (f^{-1}(B))^{\perp}$, and for all $b \in B$, there is a $a \in L$, such that $b = f(a)$, so $x \wedge a = 0$, then $f(x) \wedge b = f(x) \wedge f(a) = f(x \wedge a) = f(0) = 0$. Therefore, $f(x) \in B^{\perp}$, which implies that $x \in f^{-1}(B^{\perp})$. □

The next example shows the following: let L, M be two BL-algebras, $f : L \to M$ be a surjective homomorphism, $\emptyset \neq B \subseteq M$, then $f^{-1}(B^{\perp})$ may not be an annihilator of a subset of L.

Example 3.32. *Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq c \leq 1$, $0 \leq b \leq d \leq 1$ and $0 \leq b \leq c \leq 1$. The Cayley tables are as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra.

Let $M = \{0, 1\}$. The Cayley tables are as follows:

$\odot$	0	1
0	0	0
1	0	1

$\to$	0	1
0	1	1
1	0	1

Then $(M, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Let $f(0) = f(b) = f(d) = 0$, $f(a) = f(c) = f(1) = 1$, then $f : L \to M$ is a homomorphism. Let $B = \{1\}$. Then $B^{\perp} = \{0\}$, $f^{-1}(B^{\perp}) = f^{-1}(0) = \{0, b, d\}$, we can check that there is no $A \subseteq L$, such that $f^{-1}(B^{\perp}) = A^{\perp}$.

Theorem 3.33. *Let L, M be two BL-algebras, $f : L \to M$ be a homomorphism, $\emptyset \neq A \subseteq L$. Then $f(A^{\perp}) = (f(A))^{\perp}$ if and only if $(f(A^{\perp}))^{\perp\perp} = f(A^{\perp})$ and $(f(A))^{\perp} \cap (f(A^{\perp}))^{\perp} = \{0\}$.*

Proof. $\Longrightarrow$ If $f(A^{\perp}) = (f(A))^{\perp}$, then $(f(A^{\perp}))^{\perp\perp} = (f(A))^{\perp\perp\perp} = (f(A))^{\perp} = f(A^{\perp})$, $(f(A))^{\perp} \cap (f(A^{\perp}))^{\perp} = (f(A))^{\perp} \cap (f(A))^{\perp\perp} = \{0\}$.

$\Longleftarrow$ By Proposition 3.27, we have $f(A^{\perp}) \subseteq (f(A))^{\perp}$. Now we prove that $f(A^{\perp}) \supseteq (f(A))^{\perp}$. Since $(f(A))^{\perp} \cap (f(A^{\perp}))^{\perp} = \{0\}$, we have $(f(A))^{\perp} \subseteq (f(A^{\perp}))^{\perp\perp} = f(A^{\perp})$. Therefore, $f(A^{\perp}) = (f(A))^{\perp}$. □

Theorem 3.34. *Let L, M be two BL-algebras, $f : L \to M$ be a surjective homomorphism, $\emptyset \neq B \subseteq M$. Then $(f^{-1}(B))^{\perp} = f^{-1}(B^{\perp})$ if and only if $f^{-1}(B^{\perp}) \cap (f^{-1}(B))^{\perp\perp} = \{0\}$.*

Proof. $\Longrightarrow$ If $(f^{-1}(B))^{\perp} = f^{-1}(B^{\perp})$, then $f^{-1}(B^{\perp}) \cap (f^{-1}(B))^{\perp\perp} = (f^{-1}(B))^{\perp} \cap (f^{-1}(B))^{\perp\perp} = \{0\}$.

$\Longleftarrow$ Firstly, if $x \in f^{-1}(B^{\perp})$, $y \in L$, such that $y \leq x$, then $f(y) \leq f(x)$, since $f(x) \in B^{\perp}$, we get $f(y) \in B^{\perp}$, so $y \in f^{-1}(B^{\perp})$, and so we have $f^{-1}(B^{\perp})$ is a down set. Since $f^{-1}(B^{\perp}) \cap (f^{-1}(B))^{\perp\perp} = \{0\}$, we get $f^{-1}(B^{\perp}) \subseteq (f^{-1}(B))^{\perp\perp\perp} = (f^{-1}(B))^{\perp}$, by Proposition 3.29, we already have $(f^{-1}(B))^{\perp} \subseteq f^{-1}(B^{\perp})$. Therefore, $(f^{-1}(B))^{\perp} = f^{-1}(B^{\perp})$. □

Theorem 3.35. *Let L, M be two BL-algebras, $f : L \to M$ be an isomorphism, $\emptyset \neq A \subseteq L$, $\emptyset \neq B \subseteq M$. Then $f(A^{\perp}) = (f(A))^{\perp}$ and $(f^{-1}(B))^{\perp} = f^{-1}(B^{\perp})$.*

Proof. (1) By Proposition 3.29, we have already known $f(A^{\perp}) \subseteq (f(A))^{\perp}$, we now prove $f(A^{\perp}) \supseteq (f(A))^{\perp}$. For any $y \in (f(A))^{\perp}$, since f is surjective, there is a $x \in L$, such that $f(x) = y$. For any $a \in A$, we have $f(a) \in f(A)$, so $f(x \wedge a) = f(x) \wedge f(a) = y \wedge f(a) = 0$. Since f is injective, we get $x \wedge a = 0$, so $x \in A^{\perp}$, that is $y \in f(A^{\perp})$. Therefore, $f(A^{\perp}) \supseteq (f(A))^{\perp}$.

(2) By Proposition 3.31, we have already known $(f^{-1}(B))^{\perp} \subseteq f^{-1}(B^{\perp})$, we now prove $(f^{-1}(B))^{\perp} \supseteq f^{-1}(B^{\perp})$. For any $x \in f^{-1}(B^{\perp})$, then $f(x) \in B^{\perp}$. For all $a \in f^{-1}(B)$, so $f(a) \in B$, $f(x \wedge a) = f(x) \wedge f(a) = 0$, since f is injective, we get $x \wedge a = 0$, so $x \in (f^{-1}(B))^{\perp}$. Therefore, $f^{-1}(B^{\perp}) \subseteq (f^{-1}(B))^{\perp}$. □

4 $\alpha-$ideals in BL-algebras

Definition 4.1. *An ideal I of a BL-algebra L is said to be an α-ideal if $i^{\perp\perp} \subseteq I$, for all $i \in I$.*

Example 4.2. *Let L be the BL-algebra of Example 3.22. Consider $I = \{0, a\}$, since $0^{\perp\perp} = 0 \subseteq I$ and $a^{\perp\perp} = \{0, a\} \subseteq I$, so I is an α-ideal.*

Let I be an ideal of a BL-algebra L, we define $E(I) = \{x \in L \mid \exists i \in I, i^{\perp} \subseteq x^{\perp}\}$.

Example 4.3. *Let $L = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq c \leq 1$, $0 \leq b \leq d \leq 1$ and $0 \leq b \leq c \leq 1$, The Cayley tables are as follows:*

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	a	b	b	b	b
c	0	a	b	c	b	c
d	0	a	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	a	1	1	1	1	1
b	0	a	1	1	1	1
c	0	a	d	1	d	1
d	0	a	c	c	1	1
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Consider $I = \{0, a\}$, then $E(I) = L$.

Theorem 4.4. *Let L be a BL-algebra, then $E(I)$ is the smallest $\alpha - ideal$ containing I, for any ideal I of L.*

Proof. Clearly, $I \subseteq E(I)$.

I1: Assume that $x \leq y$ and $y \in E(I)$. Since $y \in E(I)$, we get there is a $i \in I$ such that $i^{\perp} \subseteq y^{\perp}$. And since $x \leq y$, we get $y^{\perp} \subseteq x^{\perp}$, so $i^{\perp} \subseteq y^{\perp} \subseteq x^{\perp}$, finally we get $x \in E(I)$.

I2: Assume that $x, y \in E(I)$. Since $x, y \in E(I)$, we get there are $i, j \in I$ such that $i^{\perp} \subseteq x^{\perp}$ and $j^{\perp} \subseteq y^{\perp}$. For any $t \in (i \oslash j)^{\perp}$, we have $t \wedge (i \oslash j) = 0$, so $t \wedge i = 0$ and $t \wedge j = 0$, Since $i^{\perp} \subseteq x^{\perp}$ and $j^{\perp} \subseteq y^{\perp}$ we deduce that $t \wedge x = 0$ and $t \wedge y = 0$. Since $t \wedge (x \oslash y) \leq (t \wedge x) \oslash (t \wedge y) = 0 \oslash 0 = 0$, so $t \in (x \oslash y)^{\perp}$, so $(i \oslash j)^{\perp} \subseteq (x \oslash y)^{\perp}$. Since I is an ideal, we have $i \oslash j \in I$, therefore $x \oslash y \in E(I)$.

By I1 and I2 we get $E(I)$ is an ideal.

Now we prove $E(I)$ is an α-ideal. For any $x \in E(I)$, we get there is a $i \in I$ such that $i^{\perp} \subseteq x^{\perp}$. For any $t \in x^{\perp\perp}$, by Proposition 3.9 we have $x^{\perp} = x^{\perp\perp\perp} \subseteq t^{\perp}$, so $i^{\perp} \subseteq x^{\perp} \subseteq t^{\perp}$, it follows that $t \in E(I)$, which implies that $x^{\perp\perp} \subseteq E(I)$. Therefore, $E(I)$ is an α-ideal.

Next we prove $E(I)$ is the smallest α-ideal containing I. Let K be an α-ideal such that $I \subseteq K$. For any $x \in E(I)$, we get there is $i \in I$ such that $i^{\perp} \subseteq x^{\perp}$. Then by Propositions 3.9 and 3.10, we have $x \in \langle x \rangle \subseteq \langle x \rangle^{\perp\perp} = x^{\perp\perp} \subseteq i^{\perp\perp} \subseteq K$. Therefore, $E(I) \subseteq K$. □

Proposition 4.5. *Let L be a BL-algebra, then the following hold:*
(1) $E(I)$ is the intersection of all α-ideal containing I, for any ideal I of L;
(2) For any ideal I of L, I is an α-ideal if and only if $E(I) = I$;
(3) $\cap\{I \mid I \text{ is an } \alpha - ideal \text{ of } L\} = \{0\}$;
(4) Let I_1 and I_2 are ideals of L. Then $E(I_1) = E(I_2)$ if and only if $I_1 \subseteq E(I_2)$ and $I_2 \subseteq E(I_1)$.

Proof. (1) and (2) By Theorem 4.4, they are clear.

(3) Clearly, since $\{0\}$ is an α-ideal.

(4) $\Rightarrow$ Let $E(I_1) = E(I_2)$, so $I_1 \subseteq E(I_1) = E(I_2)$, and $I_2 \subseteq E(I_2) = E(I_1)$.

$\Leftarrow$ Since $E(I)$ is the smallest α-ideal containing I, by $I_1 \subseteq E(I_2)$ we get $E(I_1) \subseteq E(I_2)$ and by $I_2 \subseteq E(I_1)$ we get $E(I_2) \subseteq E(I_1)$. Therefore, $E(I_1) = E(I_2)$. □

Proposition 4.6. *Let I and J be ideals of BL-algebras L and M, respectively. Then $E(I \times J) = E(I) \times E(J)$.*

Proof. Let $x \in L$ and $y \in M$, we define $(x, y)^{\perp} = x^{\perp} \times y^{\perp}$. Then $E(I \times J) = \{(x, y) \mid \exists (a, b) \in I \times J : (a, b)^{\perp} \subseteq (x, y)^{\perp}\} = \{(x, y) \mid \exists a \in I, \exists b \in J : a^{\perp} \subseteq x^{\perp}, b^{\perp} \subseteq y^{\perp}\} = \{(x, y) \mid x \in E(I), y \in E(J)\} = E(I) \times E(J)$. □

Proposition 4.7. *If I_λ are ideals of BL-algebras L_λ, for all $\lambda \in \Lambda$, then $E(\prod_{\lambda\in\Lambda} I_\lambda)$ $= \prod_{\lambda\in\Lambda} E(I_\lambda)$.*

Proof. Clearly, by Proposition 4.6.

□

Proposition 4.8. *Let L be a BL-algebra, $f : L \to M$ be an isomorphism and I be an ideal of L. Then $E(f(I)) = f(E(I))$.*

Proof. Let $z \in E(f(I))$. Then there is $a \in f(I)$ such that $a^{\perp} \subseteq z^{\perp}$. Hence there exists $a_0 \in I$, $z_0 \in L$ such that $a = f(a_0)$ and $z = f(z_0)$. By Proposition 3.35, we have $f(a_0^{\perp}) = (f(a_0))^{\perp} = a^{\perp} \subseteq z^{\perp} = (f(z_0))^{\perp} = f(z_0^{\perp})$. So $a_0^{\perp} \subseteq z_0^{\perp}$. Which means $z_0 \in E(I)$ and so $z = f(z_0) \in f(E(I))$.

Conversely, let $z \in f(E(I))$. Then $z = f(z_0)$, for some $z_0 \in E(I)$. Then we have there exists $a_0 \in I$ such that $a_0^{\perp} \subseteq z_0^{\perp}$, so $(f(a_0))^{\perp} = f(a_0^{\perp}) \subseteq f(z_0^{\perp}) = (f(z_0))^{\perp} = z^{\perp}$. It follows that $z \in E(f(I))$. □

Theorem 4.9. *Let L be a BL-algebra, I and J be ideals of BL-chain or a finite product of BL-chains. Then $E(I \cap J) = E(I) \cap E(J)$.*

Proof. Firstly, we note that if I is an ideal of the BL-chain L then $E(I) = L$ or $E(I) = \{0\}$. If $I \neq \{0\}$, then there exists $0 \neq a \in I$. Since L is a BL-chain, we get $a^{\perp} = \{0\}$ and so $E(I) = L$. If $I = \{0\}$, then $E(I) = \{x \mid 0^{\perp} \subseteq x^{\perp}\} = \{x \mid x^{\perp} = L\} = \{0\}$.

Now, we show that if I and J are ideals of BL-chain L, then $E(I \cap J) = E(I) \cap E(J)$. If $I \neq \{0\}$ and $J \neq \{0\}$, then $I \cap J \neq \{0\}$. Then there exist $0 \neq a \in I$ and $0 \neq b \in J$ and since L is a BL-chain we have $a \leq b$ or $b \leq a$. Assume that $a \leq b$, then $0 \neq a \in I \cap J$. So $E(I \cap J) = L$, $E(I) = L$ and $E(J) = L$. If $I = \{0\}$ or $J = \{0\}$, then $I \cap J = \{0\}$. Then $E(I) = \{0\}$ or $E(J) = \{0\}$ and $E(I \cap J) = \{0\}$. Therefore, $E(I \cap J) = E(I) \cap E(J)$.

Let I and J be ideals of $\prod_{i=1}^{n} L_i$, where L_i are BL-chain for all $1 \leq i \leq n$, so we have $I = \prod_{i=1}^{n} I_i$ and $J = \prod_{i=1}^{n} J_i$, where I_i and J_i are ideals of L_i, for all $1 \leq i \leq n$. So $E(I \cap J) = E(\prod_{i=1}^{n} I_i \cap \prod_{i=1}^{n} J_i) = E(\prod_{i=1}^{n}(I_i \cap J_i)) = \prod_{i=1}^{n}(E(I_i) \cap E(J_i)) = \prod_{i=1}^{n} E(I_i) \cap \prod_{i=1}^{n} E(J_i) = E(\prod_{i=1}^{n} I_i) \cap E(\prod_{i=1}^{n} J_i) = E(I) \cap E(J)$. □

Proposition 4.10. *Let L be a BL-algebra, for any $I, J \in I(L)$, the following hold:*
(1) $I^{\perp}$ is an $\alpha - ideal$;
(2) $E(E(I)) = E(I)$;
(3) $E(I^{\perp}) = I^{\perp}$;
(4) If $I \subseteq J$, then $E(I) \subseteq E(J)$.

Proof. (1) For any $a \in I^{\perp}$, we have $I^{\perp\perp} \subseteq a^{\perp}$, then we get $a^{\perp\perp} \subseteq I^{\perp\perp\perp} = I^{\perp}$, so $I^{\perp}$ is an α-ideal.

(2) and (3) By Proposition 4.4 and (1), we can easily prove them.

(4) Clearly, by the definition of $E(I)$. □

We denote $E(I(L)) = \{E(I) \mid I \in I(L)\}$. And we know that the set of all ideals of L is a complete lattice and for every family $\{F_i\}_{i \in I}$ of ideals of L we have: $\wedge_{i \in I} F_i = \cap_{i \in I} F_i$ and $\vee_{i \in I} F_i = \langle \cup_{i \in I} F_i \rangle$.

Theorem 4.11. *$(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is a complete Brouwerian lattice, where $E(I) \wedge_E E(J) = E(I \cap J)$ and $E(I) \vee_E E(J) = E(I \vee J)$ and L is a BL-chain or a finite product of BL-chains.*

Proof. By Theorem 4.9, we have $E(I \cap J) = E(I) \cap E(J)$. Hence $E(I) \wedge_E E(J) = E(I) \cap E(J)$. Since $I, J \subseteq I \vee J$, by Proposition 4.10, we have $E(I), E(J) \subseteq E(I \vee J)$. This means that $E(I \vee J)$ is an upper bound of $E(I), E(J)$. Now let $E(I), E(J) \subseteq E(K)$, for some $K \in I(L)$. Then $I, J \subseteq E(K)$, hence $I \vee J \subseteq E(K)$ and so $E(I \vee J) \subseteq E(E(K)) = E(K)$, therefore $E(I \vee J)$ is the least upper bound of $E(I)$ and $E(J)$.

Now we prove that for any family of ideals G_i, $i \in I$, we have that $\vee_E(E(G_i)) = E(\vee(G_i))$. Since $E(G_i) \subseteq E(\vee(G_i))$, we get $E(\vee(G_i))$ is an upper bound of $E(G_i)$, for all $i \in I$. Also if $E(G_i) \subseteq E(K)$, for all $i \in I$, then $G_i \subseteq E(K)$. Then $\vee(G_i) \subseteq E(K)$, hence $E(\vee(G_i)) \subseteq E(E(K)) = E(K)$. Therefore $\vee_E(E(G_i))$ is the least upper bound of $E(G_i)$, for all $i \in I$. So $(E(I(L)), \wedge, \vee_E, E(0), E(L))$is a complete lattice. Then we have $\vee_E(E(I) \wedge E(G_i)) = \vee_E(E(I) \cap E(G_i)) = \vee_E(E(I \cap G_i)) = E(\vee(I \cap G_i)) = E(I \cap (\vee(G_i))) = E(I) \wedge E(\vee(G_i)) = E(I) \wedge (\vee_E(E(G_i)))$.

Therefore, $(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is a complete Brouwerian lattice. □

Theorem 4.12. *The lattice $(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is pseudo-complemented, where L is a BL-chain or a finite product of BL-chains.*

Proof. Let $I, J \in I(L)$, we get $E(I) \wedge_E E(I^{\perp}) = E(I \cap I^{\perp}) = E(0)$. Now let $E(I) \wedge_E E(K) = E(0) = \{0\}$, by Propositions 3.17, 4.10, $E(K) \subseteq (E(I))^{\perp} \subseteq I^{\perp} = E(I^{\perp})$, so for every ideal $E(I)$, its pseudo-complement is $E(I^{\perp})$. Therefore, $(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is pseudo-complemented. □

Theorem 4.13. *The lattice $(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is an algebraic lattice, where L is a BL-chain or a finite product of BL-chains.*

Proof. Firstly, we show $E(\langle z \rangle)$ is a compact element in the lattice $E(I(L))$. Assume that $E(\langle z \rangle) \subseteq \vee_E E(G_i)$, where $i \in I$ and $G_i \in I(L)$. Then $z \in E(\langle z \rangle) \subseteq \vee_E E(G_i)$, so there is $a \in \vee_{i \in I} G_i$, such that $a^{\perp} \subseteq z^{\perp}$, this means that there exists $x_i \in G_i (1 \le i \le n)$ such that $a \le x_1 \oslash x_2 \oslash \cdots \oslash x_n$. Consider $X = \{G_1, G_2, \cdots, G_n\} \subseteq \cup_{i \in I} G_i$, so $(x_1 \oslash x_2 \oslash \cdots \oslash x_n)^{\perp} \subseteq a^{\perp} \subseteq z^{\perp}$, so $z \in E(\vee_{G_i \in X} G_i)$, so we get $\langle z\rangle \subseteq E(\vee_{G_i \in X} G_i)$, and $E(\langle z\rangle) \subseteq E(E(\vee_{G_i \in X} G_i)) = E(\vee_{G_i \in X} G_i) = E(G_1) \vee_E E(G_2) \vee_E \cdots \vee_E E(G_n)$. Therefore $E(\langle z\rangle)$ is a compact element in the lattice $(E(I(L))$. Now consider $E(I) \in E(I(L))$. Since $I = \vee_{a \in I} \langle a\rangle$. we get $E(I) = E(\vee_{a \in I} \langle a\rangle) = \vee_E\{E(\langle a\rangle) \mid a \in I\}$. Therefore, $(E(I(L)), \wedge_E, \vee_E, E(0), E(L))$ is an algebraic lattice. □

5 Conclusions

In this paper, motivating by the previous research on co-annihilators and ideals in BL-algebras, we introduce the concept of annihilators to BL-algebras. We conclude that the ideal lattice $(I(L), \subseteq)$ is pseudo-complemented, and for any ideal I, its pseudo-complement is $I^{\perp}$. Also, using the notion of the annihilator of a nonempty set X with respect to an ideal I, we show that $J_I^{\perp}$ is the relative pseudo-complement of J with respect to I in the ideal lattice $(I(L), \subseteq)$. Moreover, we give the necessary and sufficient condition under which the homomorphism image and the homomorphism preimage of annihilator become an annihilator. Finally, we introduce the notion of $E(I)$, and we get that $(E(I(L)), \wedge_E, \vee_E, E(0), E(L)$ is a pseudo-complemented lattice, a complete Brouwerian lattice and an algebraic lattice, when L is a BL-chain or a finite product of BL-chains.

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