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Research Article

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Parabolic sublinear operators with rough kernel generated by parabolic Calderón-Zygmund operators and parabolic local Campanato space estimates for their commutators on the parabolic generalized local Morrey spaces

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Abstract: In this paper, the author introduces parabolic generalized local Morrey spaces and gets the boundedness of a large class of parabolic rough operators on them. The author also establishes the parabolic local Campanato space estimates for their commutators on parabolic generalized local Morrey spaces. As its special cases, the corresponding results of parabolic sublinear operators with rough kernel and their commutators can be deduced, respectively. At last, parabolic Marcinkiewicz operator which satisfies the conditions of these theorems can be considered as an example.

Keywords: Parabolic singular integral operator, Parabolic sublinear operator, Parabolic maximal operator, Rough kernel, Parabolic generalized local Morrey space, Parabolic local Campanato spaces, Commutator

MSC: 42B20, 42B25, 42B35

1 Introduction

Let $\mathbb{R}^n$ be the $n-$dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norm $|x| = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Let $B = B(x_0, r_B)$ denote the ball with the center x_0 and radius r_B. For a given measurable set E, we also denote the Lebesgue measure of E by $|E|$. For any given $\Omega \subseteq \mathbb{R}^n$ and $0 < p < \infty$, denote by $L_p(\Omega)$ the spaces of all functions f satisfying

$$\|f\|_{L_p(\Omega)} = \left(\int\limits_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ denote the unit sphere on $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue measure $d\sigma(x')$, where x' denotes the unit vector in the direction of x.

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To study the existence and regularity results for an elliptic differential operator, i.e.

$$D=\sum_{i,j=1}^{n}a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}$$

with constant coefficients $\{a_{i,j}\}$, among some other estimates, one needs to study the singular integral operator $\overline{T}$ with a convolution kernel K (see [1] or [2]) satisfying

(a) $K(tx_1,\dots,tx_n)=t^{-n}K(x)$, for any $t>0$;

(b) $K\in C^\infty(\mathbb{R}^n\setminus\{0\})$;

(c) $\int_{S^{n-1}}K(x')d\sigma(x')=0$.

Similarly, for the heat operator

$$D=\frac{\partial}{\partial x_1}-\sum_{j=2}^{n}\frac{\partial^2}{\partial x_j^2},$$

the corresponding singular integral operator $\overline{T}$ has a kernel K satisfying

(a') $K\left(t^2x_1,\dots,tx_n\right)=t^{-n-1}K(x)$, for any $t>0$;

(b') $K\in C^\infty(\mathbb{R}^n\setminus\{0\})$;

(c') $\int_{S^{n-1}}K(x')\left(2x_1'^2+x_2'^2+\cdots+x_n'^2\right)d\sigma(x')=0$.

To study the regularity results for a more general parabolic differential operator with constant coefficients, in 1966, Fabes and Riviére [3] introduced the following parabolic singular integral operator

$$\overline{T}^{P}f(x)=p.v.\int_{\mathbb{R}^n}K(y)f(x-y)\,dy$$

with K satisfying

(i) $K(t^{\alpha_1}x_1,\dots,t^{\alpha_n}x_n)=t^{-\alpha}K(x_1,\dots,x_n)$, $t>0$, $x\neq 0$, $\alpha=\sum_{i=1}^{n}\alpha_i$;

(ii) $K\in C^\infty(\mathbb{R}^n\setminus\{0\})$;

(iii) $\int_{S^{n-1}}K(x')J(x')\,d\sigma(x')=0$, where $\alpha_i\geq 1$ $(i=1,\dots,n)$ and $J(x')=\alpha_1x_1'^2+\ldots+\alpha_nx_n'^2$.

Let $\rho\in(0,\infty)$ and $0\leq\varphi_{n-1}\leq 2\pi$, $0\leq\varphi_i\leq\pi$, $i=1,\dots,n-2$. For any $x\in\mathbb{R}^n$, set

$$\begin{aligned}x_1&=\rho^{\alpha_1}\cos\varphi_1\ldots\cos\varphi_{n-2}\cos\varphi_{n-1},\\ x_2&=\rho^{\alpha_2}\cos\varphi_1\ldots\cos\varphi_{n-2}\cos\varphi_{n-1},\\ &\ \vdots\\ x_{n-1}&=\rho^{\alpha_{n-1}}\cos\varphi_1\sin\varphi_2,\\ x_n&=\rho^{\alpha_n}\sin\varphi_1.\end{aligned}$$

Then $dx=\rho^{\alpha-1}J(x')\,d\rho d\sigma(x')$, where $\alpha=\sum_{i=1}^{n}\alpha_i$, $x'\in S^{n-1}$, $d\sigma$ is the element of area of S^{n-1} and $\rho^{\alpha-1}J$ is the Jacobian of the above transform. In [3] Fabes and Riviére have pointed out that $J(x')$ is a C^∞ function on S^{n-1} and $1\leq J(x')\leq M$, where M is a constant independent of x'. Without loss of generality, in this paper we may assume $\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_1\geq 1$. Notice that the above condition (i) can be written as (i') $K(A_tx)=|\det(A_t)|^{-1}K(x)$, where $A_t=diag\,[t^{\alpha_1},\dots,t^{\alpha_n}]=\begin{pmatrix}t^{\alpha_1}&&0\\&\ddots&\\0&&t^{\alpha_n}\end{pmatrix}$ is a diagonal matrix.

Note that for each fixed $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, the function

$$F(x,\rho)=\sum_{i=1}^{n}\frac{x_i^2}{\rho^{2\alpha_i}}$$

is a strictly decreasing function of $\rho>0$. Hence, there exists an unique t such that $F(x,t)=1$. It has been proved in [3] that if we set $\rho(0)=0$ and $\rho(x)=t$ such that $F(x,t)=1$, then ρ is a metric on $\mathbb{R}^n$, and $(\mathbb{R}^n,\rho)$ is called the mixed homogeneity space related to $\{\alpha_i\}_{i=1}^{n}$.

Remark 1.1. *Many works have been done for parabolic singular integral operators, including the weak type estimates and L_p (strong (p, p)) boundedness. For example, one can see references [4–6] for details.*

Let P be a real $n \times n$ matrix, whose all the eigenvalues have positive real part. Let $A_t = t^P$ $(t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that (see [7])

$(1-1)$ $\rho(A_t x) = t\rho(x)$, $t > 0$, for every $x \in \mathbb{R}^n$,

$(1-2)$ $\rho(0) = 0$, $\rho(x-y) = \rho(y-x) \geq 0$, and $\rho(x-y) \leq k(\rho(x-z) + \rho(y-z))$,

$(1-3)$ $dx = \rho^{\gamma-1} d\sigma(w)\, d\rho$, where $\rho = \rho(x)$, $w = A_{\rho^{-1}} x$ and $d\sigma(w)$ is a measure on the unit ellipsoid $\{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss (see [7]) and a homogeneous group in the sense of Folland-Stein (see [8]). Moreover, we always assume that there hold the following properties of the quasi-distance ρ:

$(1-4)$ For every x,

$$c_1 |x|^{\alpha_1} \leq \rho(x) \leq c_2 |x|^{\alpha_2} \text{ if } \rho(x) \geq 1;$$
$$c_3 |x|^{\alpha_3} \leq \rho(x) \leq c_4 |x|^{\alpha_4} \text{ if } \rho(x) \leq 1,$$

and

$$\rho(\theta x) \leq \rho(x) \text{ for } 0 < \theta < 1,$$

with some positive constants α_i and c_i $(i = 1, \ldots, 4)$. Similar properties also hold for the quasimetric ρ^* associated with the adjoint matrix P^*.

The following are some important examples of the above defined matrices P and distances ρ:

1. Let $(Px, x) \geq (x, x)$ $(x \in \mathbb{R}^n)$. In this case, $\rho(x)$ is defined by the unique solution of $|A_{t^{-1}} x| = 1$, and $k = 1$. This is the case studied by Calderón and Torchinsky in [9].

2. Let P be a diagonal matrix with positive diagonal entries, and let $t = \rho(x)$, $x \in \mathbb{R}^n$ be the unique solution of $|A_{t^{-1}} x| = 1$.

a) When all diagonal entries are greater than or equal to 1, Besov et al. in [10] and Fabes and Riviére in [3] have studied the weak (1.1) and L_p (strong (p, p)) estimates of the singular integral operators on this space.

b) If there are diagonal entries smaller than 1, then ρ satisfies the above $(1-1)-(1-4)$ with $k \geq 1$.

It is a simple matter to check that $\rho(x-y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space [3, 10]. Denote by $E(x, r)$ the ellipsoid with center at x and radius r, more precisely, $E(x, r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$. For $k > 0$, we denote $kE(x, r) = \{y \in \mathbb{R}^n : \rho(x-y) < kr\}$. Moreover, by the property of ρ and the polar coordinates transform above, we have

$$|E(x, r)| = \int\limits_{\rho(x-y)<r} dy = \upsilon_\rho r^{\alpha_1+\cdots+\alpha_n} = \upsilon_\rho r^\gamma,$$

where $|E(x,r)|$ stands for the Lebesgue measure of $E(x,r)$ and υ_ρ is the volume of the unit ellipsoid on $\mathbb{R}^n$. By $E^C(x, r) = \mathbb{R}^n \setminus E(x, r)$, we denote the complement of $E(x, r)$. Moreover, in the standard parabolic case $P_0 = diag[1, \ldots, 1, 2]$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Note that we deal not exactly with the parabolic metric, but with a general anisotropic metric ρ of generalized homogeneity, the parabolic metric being its particular case, but we keep the term parabolic in the title and text of the paper, the above existing tradition, see for instance [9].

Suppose that $\Omega(x)$ is a real-valued and measurable function defined on $\mathbb{R}^n$. Suppose that S^{n-1} is the unit sphere on $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$.

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \leq \infty$ be homogeneous of degree zero with respect to A_t ($\Omega(x)$ is A_t-homogeneous of degree zero). We define $s' = \frac{s}{s-1}$ for any $s > 1$. Suppose that T_Ω^P represents a parabolic linear or a parabolic sublinear operator, which satisfies that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin suppf$

$$|T_\Omega^P f(x)| \le c_0 \int\limits_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^\gamma} |f(y)| \, dy, \tag{1}$$

where c_0 is independent of f and x.

We point out that the condition (1) in the case $\Omega \equiv 1$ and $P = I$ was first introduced by Soria and Weiss in [11] . The condition (1) is satisfied by many interesting operators in harmonic analysis, such as the parabolic Calderón–Zygmund operators, parabolic Carleson's maximal operator, parabolic Hardy–Littlewood maximal operator, parabolic C. Fefferman's singular multipliers, parabolic R. Fefferman's singular integrals, parabolic Ricci–Stein's oscillatory singular integrals, parabolic the Bochner–Riesz means and so on (see [11, 12] for details).

Let $\Omega \in L_s(S^{n-1})$ with $1 < s \le \infty$ be homogeneous of degree zero with respect to A_t ($\Omega(x)$ is A_t-homogeneous of degree zero), that is,

$$\Omega(A_t x) = \Omega(x),$$

for any $t > 0$, $x \in \mathbb{R}^n$ and satisfies the cancellation(vanishing) condition

$$\int\limits_{S^{n-1}} \Omega(x') J\left(x'\right) d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

Let $f \in L^{loc}(\mathbb{R}^n)$. The parabolic homogeneous singular integral operator $\overline{T}_\Omega^P$ and the parabolic maximal operator M_Ω^P by with rough kernels are defined by

$$\overline{T}_\Omega^P f(x) = p.v. \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{\rho(x-y)^\gamma} f(y) dy, \tag{2}$$

$$M_\Omega^P f(x) = \sup_{t>0} |E(x,t)|^{-1} \int\limits_{E(x,t)} |\Omega(x-y)| \, |f(y)| dy,$$

satisfy condition (1).

It is obvious that when $\Omega \equiv 1$, $\overline{T}_\Omega^P \equiv \overline{T}^P$ and $M_\Omega^P \equiv M^P$ are the parabolic singular operator and the parabolic maximal operator, respectively. If $P = I$, then $M_\Omega^I \equiv M_\Omega$ is the Hardy-Littlewood maximal operator with rough kernel, and $\overline{T}_\Omega^I \equiv \overline{T}_\Omega$ is the homogeneous singular integral operator. It is well known that the parabolic maximal and singular operators play an important role in harmonic analysis (see [8, 9] and [13, 14]). In particular, the boundedness of $\overline{T}_\Omega^P$ on Lebesgue spaces has been obtained.

Theorem 1.2. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is A_t-homogeneous of degree zero having mean value zero on S^{n-1}. If $s' \le p$ or $p < s$, then the operator $\overline{T}_\Omega^P$ is bounded on $L_p(\mathbb{R}^n)$. Also, the operator $\overline{T}_\Omega^P$ is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Moreover, we have for $p > 1$*

$$\left\| \overline{T}_\Omega^P f \right\|_{L_p} \le C \, \|f\|_{L_p},$$

and for $p = 1$

$$\left\| \overline{T}_\Omega^P f \right\|_{WL_1} \le C \, \|f\|_{L_1}.$$

Corollary 1.3. *Under the assumptions of Theorem 1.2, the operator M_Ω^P is bounded on $L_p(\mathbb{R}^n)$. Also, the operator M_Ω^P is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Moreover, we have for $p > 1$*

$$\left\| M_\Omega^P f \right\|_{L_p} \le C \, \|f\|_{L_p},$$

and for $p = 1$

$$\left\| M_\Omega^P f \right\|_{WL_1} \le C \, \|f\|_{L_1}.$$

Proof. It suffices to refer to the known fact that

$$M_{\Omega}^{P} f(x) \le C_{\gamma} \overline{T}_{\Omega}^{P} f(x), \qquad C_{\gamma} = |E(0,1)|.$$

□

Note that in the isotropic case $P = I$ Theorem 1.2 has been proved in [15].

Let b be a locally integrable function on $\mathbb{R}^n$, then we define commutators generated by parabolic maximal and singular integral operators with rough kernels and b as follows, respectively.

$$M_{\Omega,b}^{P}(f)(x) = \sup_{t>0} |E(x,t)|^{-1} \int\limits_{E(x,t)} |b(x) - b(y)| \, |\Omega(x-y)| \, |f(y)| dy,$$

$$[b, \overline{T}_{\Omega}^{P}] f(x) \equiv b(x) \overline{T}_{\Omega}^{P} f(x) - \overline{T}_{\Omega}^{P}(bf)(x) = p.v. \int\limits_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{\rho(x-y)^{\gamma}} f(y) dy. \tag{3}$$

If we take $\alpha_1 = \cdots = \alpha_n = 1$ and $P = I$, then obviously $\rho(x) = |x| = \left(\sum\limits_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}}$, $\gamma = n$, $(\mathbb{R}^n, \rho) = (\mathbb{R}^n, |\cdot|)$, $E_I(x,r) = B(x,r)$, $A_t = tI$ and $J(x') \equiv 1$. In this case, $\overline{T}_{\Omega}^{P}$ defined as in (2) is the classical singular integral operator with rough kernel of convolution type whose boundedness in various function spaces has been well-studied by many authors (see [16-20], and so on). And also, in this case, $[b, \overline{T}_{\Omega}^{P}]$ defined as in (3) is the classical commutator of singular integral operator with rough kernel of convolution type whose boundedness in various function spaces has also been well-studied by many authors (see [16-20], and so on).

The classical Morrey spaces $L_{p,\lambda}$ have been introduced by Morrey in [21] to study the local behavior of solutions of second order elliptic partial differential equations(PDEs). In recent years there has been an explosion of interest in the study of the boundedness of operators on Morrey-type spaces. It has been shown that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, better inclusion between Morrey and Hölder spaces allows to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems.

Morrey has stated that many properties of solutions to PDEs can be attributed to the boundedness of some operators on Morrey spaces. For the boundedness of the Hardy–Littlewood maximal operator, the fractional integral operator and the Calderón–Zygmund singular integral operator on these spaces, we refer the readers to [22–24]. For the properties and applications of classical Morrey spaces, see [25-28] and references therein. The generalized Morrey spaces $M_{p,\varphi}$ are obtained by replacing r^{λ} with a function $\varphi(r)$ in the definition of the Morrey space. During the last decades various classical operators, such as maximal, singular and potential operators have been widely investigated in classical and generalized Morrey spaces.

We define the parabolic Morrey spaces $L_{p,\lambda,P}(\mathbb{R}^n)$ via the norm

$$\|f\|_{L_{p,\lambda,P}} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(E(x,r))} < \infty,$$

where $f \in L_p^{loc}(\mathbb{R}^n)$, $0 \le \lambda \le \gamma$ and $1 \le p \le \infty$.

Note that $L_{p,0,P} = L_p(\mathbb{R}^n)$ and $L_{p,\gamma,P} = L_{\infty}(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > \gamma$, then $L_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We also denote by $WL_{p,\lambda,P} \equiv WL_{p,\lambda,P}(\mathbb{R}^n)$ the weak parabolic Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WL_{p,\lambda,P}} \equiv \|f\|_{WL_{p,\lambda,P}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(E(x,r))} < \infty,$$

where $WL_p(E(x,r))$ denotes the weak L_p-space of measurable functions f for which

$$\begin{aligned} \|f\|_{WL_p(E(x,r))} &\equiv \|f \chi_{E(x,r)}\|_{WL_p(\mathbb{R}^n)} \\ &= \sup_{t>0} t \, |\{y \in E(x,r) : |f(y)| > t\}|^{1/p} \\ &= \sup_{0<t\le|E(x,r)|} t^{1/p} \left(f \chi_{E(x,r)} \right)^{*}(t) < \infty, \end{aligned}$$

where g^* denotes the non-increasing rearrangement of a function g.

Note that $WL_p(\mathbb{R}^n) = WL_{p,0,P}(\mathbb{R}^n)$,

$$L_{p,\lambda,P}(\mathbb{R}^n) \subset WL_{p,\lambda,P}(\mathbb{R}^n) \text{ and } \|f\|_{WL_{p,\lambda,P}} \le \|f\|_{L_{p,\lambda,P}}.$$

If $P = I$, then $L_{p,\lambda,I}(\mathbb{R}^n) \equiv L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space.

It is known that the parabolic maximal operator M^P is also bounded on $L_{p,\lambda,P}$ for all $1 < p < \infty$ and $0 < \lambda < \gamma$ (see, e.g. [29]), whose isotropic counterpart has been proved by Chiarenza and Frasca [23].

In this paper, we prove the boundedness of the parabolic sublinear operators with rough kernel T_Ω^P satisfying condition (1) generated by parabolic Calderón-Zygmund operators with rough kernel from one parabolic generalized local Morrey space $LM_{p,\varphi_1,P}^{\{x_0\}}$ to another one $LM_{p,\varphi_2,P}^{\{x_0\}}$, $1 < p < \infty$, and from the space $LM_{1,\varphi_1,P}^{\{x_0\}}$ to the weak space $WLM_{1,\varphi_2,P}^{\{x_0\}}$. In the case of $b \in LC_{p_2,\lambda,P}^{\{x_0\}}$ (parabolic local Campanato space) and $[b, T_\Omega^P]$ is a sublinear operator, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator operators $[b, T_\Omega^P]$ from $LM_{p_1,\varphi_1,P}^{\{x_0\}}$ to $LM_{p,\varphi_2,P}^{\{x_0\}}$, $1 < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $0 \le \lambda < \frac{1}{\gamma}$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Parabolic generalized local Morrey spaces

Let us define the parabolic generalized Morrey spaces as follows.

Definition 2.1 (parabolic generalized Morrey space). *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi,P} \equiv M_{p,\varphi,P}(\mathbb{R}^n)$ the parabolic generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |E(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(E(x,r))} < \infty.$$

Also by $WM_{p,\varphi,P} \equiv WM_{p,\varphi,P}(\mathbb{R}^n)$ we denote the weak parabolic generalized Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi,P}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |E(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(E(x,r))} < \infty.$$

According to this definition, we recover the parabolic Morrey space $L_{p,\lambda,P}$ and the weak parabolic Morrey space $WL_{p,\lambda,P}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-\gamma}{p}}$:

$$L_{p,\lambda,P} = M_{p,\varphi,P}\,\Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}, \qquad WL_{p,\lambda,P} = WM_{p,\varphi,P}\,\Big|_{\varphi(x,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Inspired by the above Definition 2.1, [16] and the Ph.D. thesis of Gurbuz [17], we introduce the parabolic generalized local Morrey spaces $LM_{p,\varphi,P}^{\{x_0\}}$ by the following definition.

Definition 2.2 (parabolic generalized local Morrey space). *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi,P}^{\{x_0\}} \equiv LM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi,P}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |E(x_0,r)|^{-\frac{1}{p}} \|f\|_{L_p(E(x_0,r))} < \infty.$$

Also by $WLM_{p,\varphi,P}^{\{x_0\}} \equiv WLM_{p,\varphi,P}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak parabolic generalized local Morrey space of all functions $f \in WL_p^{loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi,P}^{\{x_0\}}} = \sup_{r>0} \varphi(x_0,r)^{-1} |E(x_0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(E(x_0,r))} < \infty.$$

According to this definition, we recover the local parabolic Morrey space $LL^{\{x_0\}}_{p,\lambda,P}$ and weak local parabolic Morrey space $WLL^{\{x_0\}}_{p,\lambda,P}$ under the choice $\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}$:

$$LL^{\{x_0\}}_{p,\lambda,P}=LM^{\{x_0\}}_{p,\varphi,P}\,|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}},\qquad WLL^{\{x_0\}}_{p,\lambda,P}=WLM^{\{x_0\}}_{p,\varphi,P}\,|_{\varphi(x_0,r)=r^{\frac{\lambda-\gamma}{p}}}.$$

Furthermore, we have the following embeddings:

$$M_{p,\varphi,P}\subset LM^{\{x_0\}}_{p,\varphi,P},\qquad \|f\|_{LM^{\{x_0\}}_{p,\varphi,P}}\le\|f\|_{M_{p,\varphi,P}},$$

$$WM_{p,\varphi,P}\subset WLM^{\{x_0\}}_{p,\varphi,P},\qquad \|f\|_{WLM^{\{x_0\}}_{p,\varphi,P}}\le\|f\|_{WM_{p,\varphi,P}}.$$

In [30] the following statement has been proved for parabolic singular operators with rough kernel $\overline{T}^P_\Omega$, containing the result in [31–33].

Theorem 2.3. *Suppose that $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, is A_t-homogeneous of degree zero and has mean value zero on S^{n-1}. Let $1\le s'<p<\infty$ $\left(s'=\frac{s}{s-1}\right)$ and $\varphi(x,r)$ satisfies conditions*

$$c^{-1}\varphi(x,r)\le\varphi(x,t)\le c\,\varphi(x,r) \tag{4}$$

whenever $r\le t\le 2r$, where c (≥ 1) does not depend on t, r, $x\in\mathbb{R}^n$ and

$$\int_r^\infty \varphi(x,t)^p\frac{dt}{t}\le C\,\varphi(x,r)^p, \tag{5}$$

where C does not depend on x and r. Then the operator $\overline{T}^P_\Omega$ is bounded on $M_{p,\varphi,P}$.

The results of [31–33] imply the following statement.

Theorem 2.4. *Let $1\le p<\infty$ and $\varphi(x,t)$ satisfies conditions (4) and (5). Then the operators M^P and $\overline{T}^P$ are bounded on $M_{p,\varphi,P}$ for $p>1$ and from $M_{1,\varphi,P}$ to $WM_{1,\varphi,P}$ and for $p=1$.*

The following statement, containing the results obtained in [31–33] has been proved in [34, 35] (see also [36-39] and [40, 41]).

Theorem 2.5. *Let $1\le p<\infty$ and the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \varphi_1(x,t)\frac{dt}{t}\le C\,\varphi_2(x,r), \tag{6}$$

where C does not depend on x and r. Then the operator $\overline{T}^P$ is bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$ for $p>1$ and from $M_{1,\varphi_1,P}$ to $WM_{1,\varphi_2,P}$ for $p=1$.

Finally, inspired by the Definition 2.2, [16] and the Ph.D. thesis of Gurbuz [17] in this paper we consider the boundedness of parabolic sublinear operators with rough kernel on the parabolic generalized local Morrey spaces and give the parabolic local Campanato space estimates for their commutators.

3 Parabolic sublinear operators with rough kernel generated by parabolic Calderón-Zygmund operators on the spaces $LM^{\{x_0\}}_{p,\varphi,P}$

In this section, we will prove the boundedness of the operator T^P_Ω on the parabolic generalized local Morrey spaces $LM^{\{x_0\}}_{p,\varphi,P}$ by using the following statement on the boundedness of the weighted Hardy operator

$$H_\omega g(t) := \int_t^\infty g(s)\omega(s)ds, \qquad 0 < t < \infty,$$

where ω is a fixed non-negative function and measurable on $(0,\infty)$.

Theorem 3.1 ([16, 17, 42]). *Let v_1, v_2 and ω be positive almost everywhere and measurable functions on $(0,\infty)$. The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_\omega g(t) \le C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{7}$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{\omega(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty. \tag{8}$$

Moreover, the value $C = B$ is the best constant for (7).

We first prove the following Theorem 3.2.

Theorem 3.2. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let T^P_Ω be a parabolic sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$.*

If $p > 1$ and $s' \le p$, then the inequality

$$\left\| T^P_\Omega f \right\|_{L_p(E(x_0,r))} \lesssim r^{\frac{\gamma}{p}} \int_{2kr}^\infty t^{-\frac{\gamma}{p}-1} \|f\|_{L_p(E(x_0,t))} dt$$

holds for any ellipsoid $E(x_0,r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

If $p > 1$ and $p < s$, then the inequality

$$\left\| T^P_\Omega f \right\|_{L_p(E(x_0,r))} \lesssim r^{\frac{\gamma}{p}-\frac{\gamma}{s}} \int_{2kr}^\infty t^{\frac{\gamma}{s}-\frac{\gamma}{p}-1} \|f\|_{L_p(E(x_0,t))} dt$$

holds for any ellipsoid $E(x_0,r)$ and for all $f \in L^{loc}_p(\mathbb{R}^n)$.

Moreover, for $s > 1$ the inequality

$$\left\| T^P_\Omega f \right\|_{WL_q(E(x_0,r))} \lesssim r^{\gamma} \int_{2kr}^\infty t^{-\gamma-1} \|f\|_{L_1(E(x_0,t))} dt \tag{9}$$

holds for any ellipsoid $E(x_0,r)$ and for all $f \in L^{loc}_1(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$ and $s' \le p$. Set $E = E(x_0,r)$ for the parabolic ball (ellipsoid) centered at x_0 and of radius r and $2kE = E(x_0, 2kr)$. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2kE}(y), \qquad f_2(y) = f(y)\chi_{(2kE)^C}(y), \qquad r > 0$$

and have

$$\left\| T_{\Omega}^{P} f \right\|_{L_p(E)} \leq \left\| T_{\Omega}^{P} f_1 \right\|_{L_p(E)} + \left\| T_{\Omega}^{P} f_2 \right\|_{L_p(E)}.$$

Since $f_1 \in L_p(\mathbb{R}^n)$, $T_{\Omega}^{P} f_1 \in L_p(\mathbb{R}^n)$ and from the boundedness of T_{Ω}^{P} on $L_p(\mathbb{R}^n)$ (see Theorem 1.2) it follows that:

$$\left\| T_{\Omega}^{P} f_1 \right\|_{L_p(E)} \leq \left\| T_{\Omega}^{P} f_1 \right\|_{L_p(\mathbb{R}^n)} \leq C \left\| f_1 \right\|_{L_p(\mathbb{R}^n)} = C \left\| f \right\|_{L_p(2kE)},$$

where constant $C > 0$ is independent of f.

It is clear that $x \in E$, $y \in (2kE)^C$ implies $\frac{1}{2k}\rho(x_0 - y) \leq \rho(x - y) \leq \frac{3k}{2}\rho(x_0 - y)$. We get

$$\left| T_{\Omega}^{P} f_2(x) \right| \leq 2^{\nu} c_1 \int\limits_{(2kE)^C} \frac{|f(y)|\,|\Omega(x-y)|}{\rho(x_0 - y)^{\nu}} dy.$$

By the Fubini's theorem, we have

$$\begin{aligned}
\int\limits_{(2kE)^C} \frac{|f(y)|\,|\Omega(x-y)|}{\rho(x_0 - y)^{\nu}} dy &\approx \int\limits_{(2kE)^C} |f(y)|\,|\Omega(x-y)| \int\limits_{\rho(x_0-y)}^{\infty} \frac{dt}{t^{\nu+1}} dy \\
&\approx \int\limits_{2kr}^{\infty} \int\limits_{2kr \leq \rho(x_0-y) \leq t} |f(y)|\,|\Omega(x-y)|\, dy \frac{dt}{t^{\nu+1}} \\
&\lesssim \int\limits_{2kr}^{\infty} \int\limits_{E(x_0,t)} |f(y)|\,|\Omega(x-y)|\, dy \frac{dt}{t^{\nu+1}}.
\end{aligned}$$

Applying the Hölder's inequality, we get

$$\begin{aligned}
&\int\limits_{(2kE)^C} \frac{|f(y)|\,|\Omega(x-y)|}{\rho(x_0 - y)^{\nu}} dy \\
&\lesssim \int\limits_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \|\Omega(x-\cdot)\|_{L_s(E(x_0,t))} |E(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{\nu+1}}.
\end{aligned} \tag{10}$$

For $x \in E(x_0,t)$, notice that Ω is A_t-homogenous of degree zero and $\Omega \in L_s(S^{n-1})$, $s > 1$. Then, we obtain

$$\begin{aligned}
\left(\int\limits_{E(x_0,t)} |\Omega(x-y)|^s dy \right)^{\frac{1}{s}} &= \left(\int\limits_{E(x-x_0,t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
&\leq \left(\int\limits_{E(0,t+|x-x_0|)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
&\leq \left(\int\limits_{E(0,2t)} |\Omega(z)|^s dz \right)^{\frac{1}{s}} \\
&= \left(\int\limits_{S^{n-1}} \int\limits_{0}^{2t} \left|\Omega(z')\right|^s d\sigma(z') r^{n-1} dr \right)^{\frac{1}{s}} \\
&= C \|\Omega\|_{L_s(S^{n-1})} |E(x_0,2t)|^{\frac{1}{s}}.
\end{aligned} \tag{11}$$

Thus, by (11), it follows that:

$$\left| T_{\Omega}^{P} f_2(x) \right| \lesssim \int\limits_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\nu}{p}+1}}.$$

Moreover, for all $p \in [1, \infty)$ the inequality

$$\left\| T_{\Omega}^{P} f_2 \right\|_{L_p(E)} \lesssim r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}} \tag{12}$$

is valid. Thus, we obtain

$$\left\| T_{\Omega}^{P} f \right\|_{L_p(E)} \lesssim \|f\|_{L_p(2kE)} + r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}.$$

On the other hand, we have

$$\begin{aligned} \|f\|_{L_p(2kE)} &\approx r^{\frac{\gamma}{p}} \|f\|_{L_p(2kE)} \int_{2kr}^{\infty} \frac{dt}{t^{\frac{\gamma}{p}+1}} \\ &\leq r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}. \end{aligned} \tag{13}$$

By combining the above inequalities, we obtain

$$\left\| T_{\Omega}^{P} f \right\|_{L_p(E)} \lesssim r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}}.$$

Let $1 < p < s$. Similarly to (11), when $y \in B(x_0, t)$, it is true that

$$\left(\int_{E(x_0,r)} |\Omega(x-y)|^s \, dy \right)^{\frac{1}{s}} \leq C \|\Omega\|_{L_s(S^{n-1})} \left| E\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}}. \tag{14}$$

By the Fubini's theorem, the Minkowski inequality and (14), we get

$$\begin{aligned} \left\| T_{\Omega}^{P} f_2 \right\|_{L_p(E)} &\leq \left(\int_E \left| \int_{2kr}^{\infty} \int_{E(x_0,t)} |f(y)| \, |\Omega(x-y)| \, dy \frac{dt}{t^{\gamma+1}} \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \int_{2kr}^{\infty} \int_{E(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_p(E)} \, dy \frac{dt}{t^{\gamma+1}} \\ &\leq |E(x_0,r)|^{\frac{1}{p}-\frac{1}{s}} \int_{2kr}^{\infty} \int_{E(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_s(E)} \, dy \frac{dt}{t^{\gamma+1}} \\ &\lesssim r^{\frac{\gamma}{p}-\frac{\gamma}{s}} \int_{2kr}^{\infty} \|f\|_{L_1(E(x_0,t))} \left| E\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{\gamma+1}} \\ &\lesssim r^{\frac{\gamma}{p}-\frac{\gamma}{s}} \int_{2kr}^{\infty} t^{\frac{\gamma}{s}-\frac{\gamma}{p}-1} \|f\|_{L_p(E(x_0,t))} \, dt. \end{aligned}$$

Let $p = 1 < s \leq \infty$. From the weak $(1,1)$ boundedness of T_{Ω}^{P} and (13) it follows that:

$$\begin{aligned} \left\| T_{\Omega}^{P} f_1 \right\|_{WL_1(E)} &\leq \left\| T_{\Omega}^{P} f_1 \right\|_{WL_1(\mathbb{R}^n)} \lesssim \|f_1\|_{L_1(\mathbb{R}^n)} \\ &= \|f\|_{L_1(2kE)} \lesssim r^{\gamma} \int_{2kr}^{\infty} \|f\|_{L_1(E(x_0,t))} \frac{dt}{t^{\gamma+1}}. \end{aligned} \tag{15}$$

Then from (12) and (15) we get the inequality (9), which completes the proof. □

In the following theorem (our main result), we get the boundedness of the operator T_{Ω}^{P} satisfying condition (1) on the parabolic generalized local Morrey spaces $LM_{p,\varphi,P}^{\{x_0\}}$.

Theorem 3.3. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let T_{Ω}^{P} be a parabolic sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}+1}} dt \le C\, \varphi_2(x_0,r), \tag{16}$$

and for $1 < p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^{\infty} \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}-\frac{\gamma}{s}+1}} dt \le C\, \varphi_2(x_0,r) r^{\frac{\gamma}{s}}, \tag{17}$$

where C does not depend on r.

Then the operator T_{Ω}^{P} is bounded from $LM_{p,\varphi_1,P}^{\{x_0\}}$ to $LM_{p,\varphi_2,P}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1,P}^{\{x_0\}}$ to $WLM_{1,\varphi_2,P}^{\{x_0\}}$ for $p = 1$. Moreover, we have for $p > 1$

$$\left\| T_{\Omega}^{P} f \right\|_{LM_{p,\varphi_2,P}^{\{x_0\}}} \lesssim \|f\|_{LM_{p,\varphi_1,P}^{\{x_0\}}}, \tag{18}$$

and for $p = 1$

$$\left\| T_{\Omega}^{P} f \right\|_{WLM_{1,\varphi_2,P}^{\{x_0\}}} \lesssim \|f\|_{LM_{1,\varphi_1,P}^{\{x_0\}}}. \tag{19}$$

Proof. Let $1 < p < \infty$ and $s' \le p$. By Theorem 3.2 and Theorem 3.1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1 = \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}}$, $w(r) = r^{-\frac{\gamma}{p}-1}$ and $g(r) = \|f\|_{L_p(E(x_0,r))}$, we have

$$\begin{aligned}\left\| T_{\Omega}^{P} f \right\|_{LM_{p,\varphi_2,P}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(E(x_0,r))} = \|f\|_{LM_{p,\varphi_1,P}^{\{x_0\}}},\end{aligned}$$

where the condition (8) is equivalent to (16), then we obtain (18).

Let $1 < p < s$. By Theorem 3.2 and Theorem 3.1 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1 = \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}+\frac{\gamma}{s}}$, $w(r) = r^{-\frac{\gamma}{p}+\frac{\gamma}{s}-1}$ and $g(r) = \|f\|_{L_p(E(x_0,r))}$, we have

$$\begin{aligned}\left\| T_{\Omega}^{P} f \right\|_{LM_{p,\varphi_2,P}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} r^{-\frac{\gamma}{s}} \int_r^{\infty} \|f\|_{L_p(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p}-\frac{\gamma}{s}+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p}} \|f\|_{L_p(E(x_0,r))} = \|f\|_{LM_{p,\varphi_1,P}^{\{x_0\}}},\end{aligned}$$

where the condition (8) is equivalent to (17). Thus, we obtain (18).

Also, for $p = 1$ we have

$$\begin{aligned}\left\| T_{\Omega}^{P} f \right\|_{WLM_{1,\varphi_2,P}^{\{x_0\}}} &\lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} \int_r^{\infty} \|f\|_{L_1(E(x_0,t))} \frac{dt}{t^{\gamma+1}} \\ &\lesssim \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\gamma} \|f\|_{L_1(E(x_0,r))} = \|f\|_{LM_{1,\varphi_1,P}^{\{x_0\}}}.\end{aligned}$$

Hence, the proof is completed. □

In the case of $s = \infty$ from Theorem 3.3, we get

Corollary 3.4. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and the pair (φ_1, φ_2) satisfies condition (16). Then the operators M^P and $\overline{T}^P$ are bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{1,\varphi_2,P}$ for $p = 1$.*

Corollary 3.5. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. For $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies condition (16) and for $1 < p < s$ the pair (φ_1, φ_2) satisfies condition (17). Then the operators M^P_Ω and $\overline{T}^P_\Omega$ are bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{1,\varphi_2,P}$ for $p = 1$.*

Corollary 3.6. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, is homogeneous of degree zero. Let T^P_Ω be a parabolic sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\,\varphi_2(x_0, r),$$

and for $1 < p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0, \tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}} dt \le C\,\varphi_2(x_0, r) r^{\frac{n}{s}},$$

where C does not depend on r.

Then the operator T^P_Ω is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{1,\varphi_2,P}$ for $p = 1$. Moreover, we have for $p > 1$

$$\left\| T^P_\Omega f \right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \|f\|_{LM^{\{x_0\}}_{p,\varphi_1,P}},$$

and for $p = 1$

$$\left\| T^P_\Omega f \right\|_{WLM^{\{x_0\}}_{1,\varphi_2,P}} \lesssim \|f\|_{LM^{\{x_0\}}_{1,\varphi_1,P}}.$$

Remark 3.7. *Note that, in the case of $P = I$ Corollary 3.6 has been proved in [16, 17]. Also, in the case of $P = I$ and $s = \infty$ Corollary 3.6 has been proved in [16, 17].*

Corollary 3.8. *Let $1 \le p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let T^P_Ω be a parabolic sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p > 1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x, \tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}+1}} dt \le C\,\varphi_2(x, r), \tag{20}$$

and for $1 < p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x, \tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}-\frac{\gamma}{s}+1}} dt \le C\,\varphi_2(x, r) r^{\frac{\gamma}{s}}, \tag{21}$$

where C does not depend on x and r.

Then the operator T^P_Ω is bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$ for $p > 1$ and from $M_{1,\varphi_1,P}$ to $WM_{1,\varphi_2,P}$ for $p = 1$. Moreover, we have for $p > 1$

$$\left\| T^P_\Omega f \right\|_{M_{p,\varphi_2,P}} \lesssim \|f\|_{M_{p,\varphi_1,P}},$$

and for $p = 1$

$$\left\| T^P_\Omega f \right\|_{WM_{1,\varphi_2,P}} \lesssim \|f\|_{M_{1,\varphi_1,P}}.$$

In the case of $s=\infty$ from Corollary 3.8, we get

Corollary 3.9. *Let $1\le p<\infty$ and the pair (φ_1,φ_2) satisfies condition (20). Then the operators M^P and $\overline{T}^P$ are bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$ for $p>1$ and from $M_{1,\varphi_1,P}$ to $WM_{1,\varphi_2,P}$ for $p=1$.*

Corollary 3.10. *Let $1\le p<\infty$ and $\Omega\in L_s\left(S^{n-1}\right)$, $1<s\le\infty$, be A_t-homogeneous of degree zero. Let also, for $s'\le p$, $p\neq 1$, the pair (φ_1,φ_2) satisfies condition (20) and for $1<p<q$ the pair (φ_1,φ_2) satisfies condition (21). Then the operators M_Ω^P and $\overline{T}_\Omega^P$ are bounded from M_{p,φ_1} to M_{p,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p=1$.*

Remark 3.11. *Condition (20) in Corollary 3.8 is weaker than condition (6) in Theorem 2.5. Indeed, if condition (6) holds, then*

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}+1}}dt \le \int_r^\infty \varphi_1(x,t)\frac{dt}{t}, \qquad r\in(0,\infty),$$

so condition (20) holds.

On the other hand, the functions

$$\varphi_1(r)=\frac{1}{\chi_{(1,\infty)}(r)\, r^{\frac{\gamma}{p}-\beta}}, \qquad \varphi_2(r)=r^{-\frac{\gamma}{p}}\left(1+r^\beta\right), \qquad 0<\beta<\frac{\gamma}{p}$$

satisfy condition (20) but do not satisfy condition (6) (see [41, 43]).

Corollary 3.12. *Let $1\le p<\infty$ and $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, be homogeneous of degree zero. Let T_Ω^P be a parabolic sublinear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$. Let also, for $s'\le p$, $p\neq 1$, the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}}dt \le C\,\varphi_2(x,r),$$

and for $1<p<s$ the pair (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}}dt \le C\,\varphi_2(x,r)r^{\frac{n}{s}},$$

where C does not depend on x and r.

Then the operator T_Ω^P is bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$ for $p>1$ and from $M_{1,\varphi_1,P}$ to $WM_{1,\varphi_2,P}$ for $p=1$. Moreover, we have for $p>1$

$$\left\|T_\Omega^P f\right\|_{M_{p,\varphi_2,P}} \lesssim \|f\|_{M_{p,\varphi_1,P}},$$

and for $p=1$

$$\left\|T_\Omega^P f\right\|_{WM_{1,\varphi_2,P}} \lesssim \|f\|_{M_{1,\varphi_1,P}}.$$

Remark 3.13. *Note that, in the case of $P=I$ Corollary 3.12 has been proved in [16–18]. Also, in the case of $P=I$ and $s=\infty$ Corollary 3.12 has been proved in [16–18] and [41, 43].*

4 Commutators of parabolic linear operators with rough kernel generated by parabolic Calderón-Zygmund operators and parabolic local Campanato functions on the spaces $LM^{\{x_0\}}_{p,\varphi,P}$

In this section, we will prove the boundedness of the operators $[b, T^P_\Omega]$ with $b \in LC^{\{x_0\}}_{p_2,\lambda,P}$ on the parabolic generalized local Morrey spaces $LM^{\{x_0\}}_{p,\varphi,P}$ by using the following weighted Hardy operator

$$H_\omega g(r) := \int_r^\infty \left(1 + \ln \frac{t}{r}\right) g(t)\omega(t)dt, \qquad r \in (0, \infty),$$

where ω is a weight function.

Let T be a linear operator. For a locally integrable function b on $\mathbb{R}^n$, we define the commutator $[b, T]$ by

$$[b, T] f(x) = b(x)\, T f(x) - T(bf)(x)$$

for any suitable function f. Let $\overline{T}$ be a Calderón–Zygmund operator. A well known result of Coifman et al. [44] states that when $K(x) = \frac{\Omega(x')}{|x|^n}$ and Ω is smooth, the commutator $[b, \overline{T}] f = b\, \overline{T} f - \overline{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$.

Since $BMO(\mathbb{R}^n) \subset \bigcap\limits_{p>1} LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$, if we only assume $b \in LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$, or more generally $b \in LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$, then $[b, \overline{T}]$ may not be a bounded operator on $L_p(\mathbb{R}^n)$, $1 < p < \infty$. However, it has some boundedness properties on other spaces. As a matter of fact, Grafakos et al. [45, 46] have considered the commutator with $b \in LC^{\{x_0\}}_{p,I}(\mathbb{R}^n)$ on Herz spaces for the first time. Morever, in [16, 17] and [19, 46], they have considered the commutators with $b \in LC^{\{x_0\}}_{p,\lambda,I}(\mathbb{R}^n)$. The commutator of Calderón–Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [25–27]). The boundedness of the commutator has been generalized to other contexts and important applications to some non-linear PDEs have been given by Coifman et al. [47].

We introduce the parabolic local Campanato space $LC^{\{x_0\}}_{p,\lambda,P}$ following the known ideas of defining local Campanato space (see [16, 17, 42] etc).

Definition 4.1. *Let $1 \le p < \infty$ and $0 \le \lambda < \frac{1}{\gamma}$. A function $f \in L^{loc}_p(\mathbb{R}^n)$ is said to belong to the $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$ (parabolic local Campanato space), if*

$$\|f\|_{LC^{\{x_0\}}_{p,\lambda,P}} = \sup_{r>0} \left(\frac{1}{|E(x_0, r)|^{1+\lambda p}} \int_{E(x_0,r)} \left| f(y) - f_{E(x_0,r)} \right|^p dy \right)^{\frac{1}{p}} < \infty, \tag{22}$$

where

$$f_{E(x_0,r)} = \frac{1}{|E(x_0, r)|} \int_{E(x_0,r)} f(y)\, dy.$$

Define

$$LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n) = \left\{ f \in L^{loc}_p(\mathbb{R}^n) : \|f\|_{LC^{\{x_0\}}_{p,\lambda,P}} < \infty \right\}.$$

Remark 4.2. *If two functions which differ by a constant are regarded as a function in the space $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$, then $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$ becomes a Banach space. The space $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$ when $\lambda = 0$ is just the $LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$. Apparently, (22) is equivalent to the following condition:*

$$\sup_{r>0} \inf_{c\in\mathbb{C}} \left(\frac{1}{|E(x_0, r)|^{1+\lambda p}} \int_{E(x_0,r)} |f(y) - c|^p\, dy \right)^{\frac{1}{p}} < \infty.$$

In [48], Lu and Yang have introduced the central BMO space $CBMO_p(\mathbb{R}^n) = LC^{\{0\}}_{p,0,I}(\mathbb{R}^n)$. Also the space $CBMO^{\{x_0\}}(\mathbb{R}^n) = LC^{\{x_0\}}_{1,0,I}(\mathbb{R}^n)$ has been considered in other denotes in [49]. The space $LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of parabolic bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \leq p < \infty$, the functions in $BMO(\mathbb{R}^n)$ (parabolic BMO) can be described by means of the condition:

$$\sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|E(x,r)|} \int\limits_{E(x,r)} |f(y) - f_{E(x,r)}|^p dy \right)^{\frac{1}{p}} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$ depends on p. If $p_1 < p_2$, then $LC^{\{x_0\}}_{p_2,P}(\mathbb{R}^n) \subsetneqq LC^{\{x_0\}}_{p_1,P}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$. One can imagine that the behavior of $LC^{\{x_0\}}_{p,P}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Theorem 4.3 ([16, 17, 42])**.** *Let v_1, v_2 and ω be weigths on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighbourhood of the origin. The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_\omega g(r) \leq C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{23}$$

holds for some $C > 0$ for all non-negative and non-decreasing functions g on $(0,\infty)$ if and only if

$$B := \sup_{r>0} v_2(r) \int\limits_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\omega(t)dt}{\operatorname*{ess\,sup}_{t<s<\infty} v_1(s)} < \infty. \tag{24}$$

Moreover, the value $C = B$ is the best constant for (23).

Remark 4.4. *In (23) and (24) it is assumed that $\frac{1}{\infty} = 0$ and $0 \cdot \infty = 0$.*

Lemma 4.5. *Let b be function in $LC^{\{x_0\}}_{p,\lambda,P}(\mathbb{R}^n)$, $1 \leq p < \infty$, $0 \leq \lambda < \frac{1}{\gamma}$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|E(x_0,r_1)|^{1+\lambda p}} \int\limits_{E(x_0,r_1)} \left|b(y) - b_{E(x_0,r_2)}\right|^p dy \right)^{\frac{1}{p}} \leq C \left(1 + \ln \frac{r_1}{r_2}\right) \|b\|_{LC^{\{x_0\}}_{p,\lambda,P}}, \tag{25}$$

where $C > 0$ is independent of b, r_1 and r_2.

From this inequality (25), we have

$$\left|b_{E(x_0,r_1)} - b_{E(x_0,r_2)}\right| \leq C \left(1 + \ln \frac{r_1}{r_2}\right) |E(x_0,r_1)|^\lambda \|b\|_{LC^{\{x_0\}}_{p,\lambda,P}}, \tag{26}$$

and it is easy to see that

$$\|b - b_E\|_{L_p(E)} \leq C \left(1 + \ln \frac{r_1}{r_2}\right) r^{\frac{\gamma}{p}+\gamma\lambda} \|b\|_{LC^{\{x_0\}}_{p,\lambda,P}}. \tag{27}$$

In [30] the following statements have been proved for the parabolic commutators of parabolic singular integral operators with rough kernel $\overline{T}^P_\Omega$, containing the result in [31–33].

Theorem 4.6. *Suppose that $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, is A_t-homogeneous of degree zero and $b \in BMO(\mathbb{R}^n)$. Let $1 \leq s' < p < \infty$ $\left(s' = \frac{s}{s-1}\right)$ and $\varphi(x,r)$ satisfies the conditions (4) and (5). If the commutator operator $[b, \overline{T}^P_\Omega]$ is bounded on $L_p(\mathbb{R}^n)$, then the operator $[b, \overline{T}^P_\Omega]$ is bounded on $M_{p,\varphi,P}$.*

Theorem 4.7. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $\varphi(x,t)$ satisfies conditions (4) and (5). Then the operators M^P_b and $[b, \overline{T}^P]$ are bounded on $M_{p,\varphi,P}$.*

As in the proof of Theorem 3.3, it suffices to prove the following Theorem 4.8.

Theorem 4.8. *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let T_Ω^P be a parabolic linear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. Let also, $b \in LC_{p_2,\lambda,P}^{\{x_0\}}(\mathbb{R}^n)$, $0 \le \lambda < \frac{1}{\gamma}$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.*

Then, for $s' \le p$, the inequality

$$\left\|[b, T_\Omega^P]f\right\|_{L_p(E(x_0,r))} \lesssim \|b\|_{LC_{p_2,\lambda,P}^{\{x_0\}}} r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{\gamma\lambda - \frac{\gamma}{p_1} - 1} \|f\|_{L_{p_1}(E(x_0,t))} dt$$

holds for any ellipsoid $E(x_0, r)$ and for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

Also, for $p_1 < s$, the inequality

$$\left\|[b, T_\Omega^P]f\right\|_{L_p(E(x_0,r))} \lesssim \|b\|_{LC_{p_2,\lambda,P}^{\{x_0\}}} r^{\frac{\gamma}{p} - \frac{\gamma}{s}} \int_{2kr}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{\gamma\lambda - \frac{\gamma}{p_1} + \frac{\gamma}{s} - 1} \|f\|_{L_{p_1}(E(x_0,t))} dt$$

holds for any ellipsoid $E(x_0, r)$ and for all $f \in L_{p_1}^{loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$, $b \in LC_{p_2,\lambda,P}^{\{x_0\}}(\mathbb{R}^n)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Set $E = E(x_0, r)$ for the parabolic ball (ellipsoid) centered at x_0 and of radius r and $2kE = E(x_0, 2kr)$. We represent f as

$$f = f_1 + f_2, \qquad f_1(y) = f(y)\chi_{2kE}(y), \qquad f_2(y) = f(y)\chi_{(2kE)^C}(y), \qquad r > 0$$

and have

$$\begin{aligned}[b, T_\Omega^P]f(x) &= (b(x) - b_E) T_\Omega^P f_1(x) - T_\Omega^P((b(\cdot) - b_E) f_1)(x) \\ &\quad + (b(x) - b_E) T_\Omega^P f_2(x) - T_\Omega^P((b(\cdot) - b_E) f_2)(x) \\ &\equiv J_1 + J_2 + J_3 + J_4.\end{aligned}$$

Hence we get

$$\left\|[b, T_\Omega^P]f\right\|_{L_p(E)} \le \|J_1\|_{L_p(E)} + \|J_2\|_{L_p(E)} + \|J_3\|_{L_p(E)} + \|J_4\|_{L_p(E)}.$$

By the Hölder's inequality, the boundedness of T_Ω^P on $L_{p_1}(\mathbb{R}^n)$ (see Theorem 1.2) it follows that:

$$\begin{aligned}\|J_1\|_{L_p(E)} &\le \left\|(b(\cdot) - b_E) T_\Omega^P f_1(\cdot)\right\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|(b(\cdot) - b_E)\|_{L_{p_2}(\mathbb{R}^n)} \left\|T_\Omega^P f_1(\cdot)\right\|_{L_{p_1}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{LC_{p_2,\lambda,P}^{\{x_0\}}} r^{\frac{\gamma}{p_2} + \gamma\lambda} \|f_1\|_{L_{p_1}(\mathbb{R}^n)} \\ &= \|b\|_{LC_{p_2,\lambda,P}^{\{x_0\}}} r^{\frac{\gamma}{p_2} + \frac{\gamma}{p_1} + \gamma\lambda} \|f\|_{L_{p_1}(2kE)} \int_{2kr}^{\infty} t^{-1-\frac{\gamma}{p_1}} dt \\ &\lesssim \|b\|_{LC_{p_2,\lambda,P}^{\{x_0\}}} r^{\frac{\gamma}{p}} \int_{2kr}^{\infty} \left(1 + \ln\frac{t}{r}\right) t^{\gamma\lambda - \frac{\gamma}{p_1} - 1} \|f\|_{L_{p_1}(E(x_0,t))} dt.\end{aligned}$$

Using the the boundedness of T_Ω^P on $L_p(\mathbb{R}^n)$ (see Theorem 1.2), by the Hölder's inequality for J_2 we have

$$\begin{aligned}\|J_2\|_{L_p(E)} &\le \left\|T_\Omega^P(b(\cdot) - b_E) f_1\right\|_{L_p(\mathbb{R}^n)} \\ &\lesssim \|(b(\cdot) - b_E) f_1\|_{L_p(\mathbb{R}^n)}\end{aligned}$$

$$\lesssim \|b(\cdot)-b_E\|_{L_{p_2}(\mathbb{R}^n)}\|f_1\|_{L_{p_1}(\mathbb{R}^n)}$$
$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p_2}+\frac{\gamma}{p_1}+\gamma\lambda}\|f\|_{L_{p_1}(2kE)}\int_{2kr}^{\infty} t^{-1-\frac{\gamma}{p_1}}dt$$
$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p}}\int_{2kr}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{\gamma\lambda-\frac{\gamma}{p_1}-1}\|f\|_{L_{p_1}(E(x_0,t))}dt.$$

For J_3, it is known that $x\in E$, $y\in(2kE)^C$, which implies $\frac{1}{2k}\rho(x_0-y)\le\rho(x-y)\le\frac{3k}{2}\rho(x_0-y)$.

When $s'\le p_1$, by the Fubini's theorem, the Hölder's inequality and (11) we have

$$\left|T^P_\Omega f_2(x)\right|\le c_0\int_{(2kE)^C}|\Omega(x-y)|\frac{|f(y)|}{\rho(x_0-y)^{\gamma}}dy$$
$$\approx\int_{2kr}^{\infty}\int_{2kr<\rho(x_0-y)<t}|\Omega(x-y)||f(y)|dyt^{-1-\gamma}dt$$
$$\lesssim\int_{2kr}^{\infty}\int_{E(x_0,t)}|\Omega(x-y)||f(y)|dyt^{-1-\gamma}dt$$
$$\lesssim\int_{2kr}^{\infty}\|f\|_{L_{p_1}(E(x_0,t))}\|\Omega(x-\cdot)\|_{L_s(E(x_0,t))}|E(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}}t^{-1-\gamma}dt$$
$$\lesssim\int_{2kr}^{\infty}\|f\|_{L_{p_1}(E(x_0,t))}t^{-1-\frac{\gamma}{p_1}}dt.$$

Hence, we get

$$\|J_3\|_{L_p(E)}\le\left\|(b(\cdot)-b_B)T^P_\Omega f_2(\cdot)\right\|_{L_p(\mathbb{R}^n)}$$
$$\lesssim\|(b(\cdot)-b_E)\|_{L_p(\mathbb{R}^n)}\int_{2kr}^{\infty}t^{-1-\frac{\gamma}{p_1}}\|f\|_{L_{p_1}(E(x_0,t))}dt$$
$$\lesssim\|(b(\cdot)-b_E)\|_{L_{p_2}(\mathbb{R}^n)}r^{\frac{\gamma}{p_1}}\int_{2kr}^{\infty}t^{-1-\frac{\gamma}{p_1}}\|f\|_{L_{p_1}(E(x_0,t))}dt$$
$$\lesssim\|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}}r^{\frac{\gamma}{p}+\gamma\lambda}\int_{2kr}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{-1-\frac{\gamma}{p_1}}\|f\|_{L_{p_1}(E(x_0,t))}dt$$
$$\lesssim\|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}}r^{\frac{\gamma}{p}}\int_{2kr}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{\gamma\lambda-\frac{\gamma}{p_1}-1}\|f\|_{L_{p_1}(E(x_0,t))}dt.$$

When $p_1<s$, by the Fubini's theorem, the Minkowski inequality, the Hölder's inequality and from (27), (14) we get

$$\|J_3\|_{L_p(E)}\le\left(\int_E\left|\int_{2kr}^{\infty}\int_{E(x_0,t)}|f(y)||b(x)-b_E||\Omega(x-y)|dy\frac{dt}{t^{\gamma+1}}\right|^p dx\right)^{\frac{1}{p}}$$
$$\le\int_{2kr}^{\infty}\int_{E(x_0,t)}|f(y)|\|(b(\cdot)-b_E)\Omega(\cdot-y)\|_{L_p(E)}dy\frac{dt}{t^{\gamma+1}}$$

$$\le \int\limits_{2kr}^{\infty} \int\limits_{E(x_0,t)} |f(y)| \, \|b(\cdot) - b_E\|_{L_{p_2}(E)} \, \|\Omega(\cdot - y)\|_{L_{p_1}(E)} \, dy \frac{dt}{t^{\gamma+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p_2}+\gamma\lambda} |E|^{\frac{1}{p_1}-\frac{1}{s}} \int\limits_{2kr}^{\infty} \int\limits_{E(x_0,t)} |f(y)| \, \|\Omega(\cdot - y)\|_{L_s(E)} \, dy \frac{dt}{t^{\gamma+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p}-\frac{\gamma}{s}+\gamma\lambda} \int\limits_{2kr}^{\infty} \|f\|_{L_1(E(x_0,t))} \left| E\left(x_0, \frac{3}{2}t\right) \right|^{\frac{1}{s}} \frac{dt}{t^{\gamma+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p}-\frac{\gamma}{s}+\gamma\lambda} \int\limits_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p_1}(E(x_0,t))} \frac{dt}{t^{\frac{\gamma}{p_1}-\frac{\gamma}{s}+1}}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{\gamma}{p}-\frac{\gamma}{s}} \int\limits_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{\gamma\lambda - \frac{\gamma}{p_1} + \frac{\gamma}{s} - 1} \|f\|_{L_{p_1}(E(x_0,t))} \, dt.$$

On the other hand, for J_4, when $s' \le p$, for $x \in E$, by the Fubini's theorem, applying the Hölder's inequality and from (26), (27), (11) we have

$$\left| T^P_\Omega \left((b(\cdot) - b_B) f_2 \right)(x) \right| \lesssim \int\limits_{(2kE)^C} |b(y) - b_E| \, |\Omega(x-y)| \frac{|f(y)|}{\rho(x-y)^\gamma} dy$$

$$\lesssim \int\limits_{(2kE)^C} |b(y) - b_E| \, |\Omega(x-y)| \frac{|f(y)|}{\rho(x_0-y)^\gamma} dy$$

$$\approx \int\limits_{2kr}^{\infty} \int\limits_{2kr<\rho(x_0-y)<t} |b(y) - b_E| \, |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{\gamma+1}}$$

$$\lesssim \int\limits_{2kr}^{\infty} \int\limits_{E(x_0,t)} \left| b(y) - b_{E(x_0,t)} \right| |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{\gamma+1}}$$

$$+ \int\limits_{2kr}^{\infty} \left| b_{E(x_0,r)} - b_{E(x_0,t)} \right| \int\limits_{E(x_0,t)} |\Omega(x-y)| \, |f(y)| \, dy \frac{dt}{t^{\gamma+1}}$$

$$\lesssim \int\limits_{2kr}^{\infty} \left\| \left(b(\cdot) - b_{E(x_0,t)} \right) f \right\|_{L_p(E(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(E(x_0,t))} |E(x_0,t)|^{1-\frac{1}{p}-\frac{1}{s}} \frac{dt}{t^{\gamma+1}}$$

$$+ \int\limits_{2kr}^{\infty} \left| b_{E(x_0,r)} - b_{E(x_0,t)} \right| \|f\|_{L_{p_1}(E(x_0,t))} \|\Omega(\cdot - y)\|_{L_s(E(x_0,t))} |E(x_0,t)|^{1-\frac{1}{p_1}-\frac{1}{s}} t^{-\gamma-1} dt$$

$$\lesssim \int\limits_{2kr}^{\infty} \left\| \left(b(\cdot) - b_{E(x_0,t)} \right) \right\|_{L_{p_2}(E(x_0,t))} \|f\|_{L_{p_1}(E(x_0,t))} t^{-1-\frac{\gamma}{p_1}} dt$$

$$+ \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \int\limits_{2kr}^{\infty} \left(1 + \ln \tfrac{t}{r}\right) \|f\|_{L_{p_1}(E(x_0,t))} t^{-1-\frac{\gamma}{p_1}+\gamma\lambda} dt$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \int\limits_{2kr}^{\infty} \left(1 + \ln \tfrac{t}{r}\right) \|f\|_{L_{p_1}(E(x_0,t))} t^{-1-\frac{\gamma}{p_1}+\gamma\lambda} dt.$$

Then, we have

$$\|J_4\|_{L_p(E)} = \left\| T^P_\Omega \left(b(\cdot) - b_E \right) f_2 \right\|_{L_p(E)}$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} r^{\frac{n}{p}} \int\limits_{2kr}^{\infty} \left(1 + \ln \frac{t}{r}\right) t^{\gamma\lambda - \frac{\gamma}{p_1} - 1} \|f\|_{L_{p_1}(E(x_0,t))} \, dt.$$

When $p_1 < s$, by the Minkowski inequality, applying the Hölder's inequality and from (26), (27), (14) we have

$$\|J_4\|_{L_p(E)} \le \left(\int\limits_E \left| \int\limits_{2kr}^{\infty} \int\limits_{E(x_0,t)} \left| b(y) - b_{E(x_0,t)} \right| |f(y)| \, |\Omega(x-y)| \, dy \frac{dt}{t^{\gamma+1}} \right|^p dx \right)^{\frac{1}{p}}$$

$$+\left(\int\limits_{E}\left|\int\limits_{2kr}^{\infty}\left|b_{E(x_0,r)}-b_{E(x_0,t)}\right|\int\limits_{E(x_0,t)}|f(y)||\Omega(x-y)|\,dy\frac{dt}{t^{\gamma+1}}\right|^{p}dx\right)^{\frac{1}{p}}$$

$$\lesssim\int\limits_{2kr}^{\infty}\int\limits_{E(x_0,t)}\left|b(y)-b_{E(x_0,t)}\right||f(y)|\,\|\Omega(\cdot-y)\|_{L_p(E(x_0,t))}\,dy\frac{dt}{t^{\gamma+1}}$$

$$+\int\limits_{2kr}^{\infty}\left|b_{E(x_0,r)}-b_{E(x_0,t)}\right|\int\limits_{E(x_0,t)}|f(y)|\,\|\Omega(\cdot-y)\|_{L_p(E(x_0,t))}\,dy\frac{dt}{t^{\gamma+1}}$$

$$\lesssim|E|^{\frac{1}{p}-\frac{1}{s}}\int\limits_{2kr}^{\infty}\int\limits_{E(x_0,t)}\left|b(y)-b_{E(x_0,t)}\right||f(y)|\,\|\Omega(\cdot-y)\|_{L_s(E(x_0,t))}\,dy\frac{dt}{t^{\gamma+1}}$$

$$+|E|^{\frac{1}{p}-\frac{1}{s}}\int\limits_{2kr}^{\infty}\left|b_{E(x_0,r)}-b_{E(x_0,t)}\right|\int\limits_{E(x_0,t)}|f(y)|\,\|\Omega(\cdot-y)\|_{L_s(E(x_0,t))}\,dy\frac{dt}{t^{\gamma+1}}$$

$$\lesssim r^{\frac{\gamma}{p}-\frac{\gamma}{s}}\int\limits_{2kr}^{\infty}\left\|\left(b(\cdot)-b_{E(x_0,t)}\right)\right\|_{L_{p_2}(E(x_0,t))}\|f\|_{L_{p_1}(E(x_0,t))}|E(x_0,t)|^{1-\frac{1}{p}}\left|E\left(x_0,\frac{3}{2}t\right)\right|^{\frac{1}{s}}\frac{dt}{t^{\gamma+1}}$$

$$+r^{\frac{\gamma}{p}-\frac{\gamma}{s}}\int\limits_{2kr}^{\infty}\left|b_{E(x_0,r)}-b_{E(x_0,t)}\right|\|f\|_{L_{p_1}(E(x_0,t))}\left|E\left(x_0,\frac{3}{2}t\right)\right|^{\frac{1}{s}}\frac{dt}{t^{\frac{\gamma}{p_1}+1}}$$

$$\lesssim r^{\frac{\gamma}{p}-\frac{\gamma}{s}}\|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}}\int\limits_{2kr}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{\gamma\lambda-\frac{\gamma}{p_1}+\frac{\gamma}{s}-1}\|f\|_{L_{p_1}(E(x_0,t))}\,dt.$$

Now combined by all the above estimates, we end the proof of this Theorem 4.8. □

Now we can give the following theorem (our main result).

Theorem 4.9. *Let $x_0\in\mathbb{R}^n$, $1<p<\infty$ and $\Omega\in L_s(S^{n-1})$, $1<s\le\infty$, be A_t-homogeneous of degree zero. Let T^P_Ω be a parabolic linear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. Let $b\in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0\le\lambda<\frac{1}{\gamma}$ and $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. Let also, for $s'\le p$ the pair (φ_1,φ_2) satisfies the condition*

$$\int\limits_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}+1-\gamma\lambda}}dt\le C\,\varphi_2(x_0,r),\tag{28}$$

and for $p_1<s$ the pair (φ_1,φ_2) satisfies the condition

$$\int\limits_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{essinf}\limits_{t<\tau<\infty}\varphi_1(x_0,\tau)\tau^{\frac{\gamma}{p_1}}}{t^{\frac{\gamma}{p_1}-\frac{\gamma}{s}+1-\gamma\lambda}}dt\le C\,\varphi_2(x_0,r)r^{\frac{\gamma}{s}},\tag{29}$$

where C does not depend on r.

Then the operator $[b,T^P_\Omega]$ is bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$. Moreover,

$$\left\|[b,T^P_\Omega]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}}\lesssim\|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}}\|f\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.\tag{30}$$

Proof. Let $p>1$ and $s'\le p$. By Theorem 4.8 and Theorem 4.3 with $v_2(r)=\varphi_2(x_0,r)^{-1}$, $v_1=\varphi_1(x_0,r)^{-1}r^{-\frac{\gamma}{p_1}}$, $w(r)=r^{\gamma\lambda-\frac{\gamma}{p_1}-1}$ and $g(r)=\|f\|_{L_{p_1}(E(x_0,r))}$, we have

$$\left\|[b,T^P_\Omega]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}}\lesssim\sup_{r>0}\varphi_2(x_0,r)^{-1}\|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}}\int\limits_{r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{\gamma\lambda-\frac{\gamma}{p_1}-1}\|f\|_{L_{p_1}(E(x_0,t))}\,dt$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p_1}} \|f\|_{L_{p_1}(E(x_0,r))} = \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \|f\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}},$$

where the condition (24) is equivalent to (28), then we obtain (30).

Let $p > 1$ and $p_1 < s$. By Theorem 4.8 and Theorem 4.3 with $v_2(r) = \varphi_2(x_0,r)^{-1}$, $v_1 = \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p_1}+\frac{\gamma}{s}}$, $w(r) = r^{\gamma\lambda-\frac{\gamma}{p_1}+\frac{\gamma}{s}-1}$ and $g(r) = \|f\|_{L_{p_1}(E(x_0,r))}$, we have

$$\left\|[b,T^P_\Omega]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \sup_{r>0} \varphi_2(x_0,r)^{-1} r^{-\frac{\gamma}{s}} \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \int_r^\infty \left(1+\ln\frac{t}{r}\right) t^{\gamma\lambda-\frac{\gamma}{p_1}+\frac{\gamma}{s}-1} \|f\|_{L_{p_1}(E(x_0,t))}\,dt$$

$$\lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \sup_{r>0} \varphi_1(x_0,r)^{-1} r^{-\frac{\gamma}{p_1}} \|f\|_{L_{p_1}(E(x_0,r))} = \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \|f\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}},$$

where the condition (24) is equivalent to (29). Thus, we obtain (30).

Hence, the proof is completed. □

In the case of $s = \infty$ from Theorem 4.9, we get

Corollary 4.10. *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$, $b \in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \frac{1}{\gamma}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and the pair (φ_1,φ_2) satisfies condition (28). Then the operators M^P_b and $[b,\overline{T}^P]$ are bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$.*

Corollary 4.11. *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let $b \in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \frac{1}{\gamma}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let also, for $s' \le p$, the pair (φ_1,φ_2) satisfies condition (28) and for $p < s$, the pair (φ_1,φ_2) satisfies condition (29). Then the operators $M^P_{\Omega,b}$ and $[b,\overline{T}^P_\Omega]$ are bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$.*

Corollary 4.12. *Let $x_0 \in \mathbb{R}^n$, $1 < p < \infty$ and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be homogeneous of degree zero. Let T^P_Ω be a parabolic linear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $b \in LC^{\{x_0\}}_{p_2,\lambda,P}(\mathbb{R}^n)$, $0 \le \lambda < \frac{1}{n}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let also, for $s' \le p$ the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}+1-n\lambda}}\,dt \le C\,\varphi_2(x_0,r), \tag{31}$$

and for $p_1 < s$ the pair (φ_1,φ_2) satisfies the condition

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x_0,\tau)\tau^{\frac{n}{p_1}}}{t^{\frac{n}{p_1}-\frac{n}{s}+1-n\lambda}}\,dt \le C\,\varphi_2(x_0,r) r^{\frac{n}{s}},$$

where C does not depend on r.

Then the operator $[b,T^P_\Omega]$ is bounded from $LM^{\{x_0\}}_{p_1,\varphi_1,P}$ to $LM^{\{x_0\}}_{p,\varphi_2,P}$. Moreover,

$$\left\|[b,T^P_\Omega]f\right\|_{LM^{\{x_0\}}_{p,\varphi_2,P}} \lesssim \|b\|_{LC^{\{x_0\}}_{p_2,\lambda,P}} \|f\|_{LM^{\{x_0\}}_{p_1,\varphi_1,P}}.$$

Remark 4.13. *Note that, in the case of $P = I$ Corollary 4.12 has been proved in [16, 17]. Also, in the case of $P = I$ and $s = \infty$ Corollary 4.12 has been proved in [16, 17].*

Corollary 4.14. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$, be A_t-homogeneous of degree zero. Let T^P_Ω be a parabolic linear operator satisfying condition (1), bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$ (parabolic bounded mean oscillation space). Let also, for $s' \le p$ the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}+1}}\,dt \le C\,\varphi_2(x,r), \tag{32}$$

and for $p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{\gamma}{p}}}{t^{\frac{\gamma}{p}-\frac{\gamma}{s}+1}} dt \leq C\, \varphi_2(x,r) r^{\frac{\gamma}{s}}, \tag{33}$$

where C does not depend on x and r.

Then the operator $[b, T_\Omega^P]$ is bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$. Moreover,

$$\left\| [b, T_\Omega^P] f \right\|_{M_{p,\varphi_2,P}} \lesssim \|b\|_{BMO} \|f\|_{M_{p,\varphi_1,P}}.$$

In the case of $s = \infty$ from Corollary 4.14, we get

Corollary 4.15. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfies condition (32). Then the operators M_b^P and $[b, \overline{T}^P]$ are bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$.*

Corollary 4.16. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be A_t-homogeneous of degree zero. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$, the pair (φ_1, φ_2) satisfies condition (32) and for $p < s$, the pair (φ_1, φ_2) satisfies condition (33). Then the operators $M_{\Omega,b}^P$ and $[b, \overline{T}_\Omega^P]$ are bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$.*

Corollary 4.17. *Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$, be homogeneous of degree zero. Let $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \leq p$ the pair (φ_1, φ_2) satisfies the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C\, \varphi_2(x,r),$$

and for $p < s$ the pair (φ_1, φ_2) satisfies the condition

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{essinf}_{t<\tau<\infty} \varphi_1(x,\tau)\tau^{\frac{n}{p}}}{t^{\frac{n}{p}-\frac{n}{s}+1}} dt \leq C\, \varphi_2(x,r) r^{\frac{n}{s}},$$

where C does not depend on x and r.

Then the operator $[b, T_\Omega^P]$ is bounded from $M_{p,\varphi_1,P}$ to $M_{p,\varphi_2,P}$. Moreover,

$$\left\| [b, T_\Omega^P] f \right\|_{M_{p,\varphi_2,P}} \lesssim \|b\|_{BMO} \|f\|_{M_{p,\varphi_1,P}}.$$

Remark 4.18. *Note that, in the case of $P = I$ Corollary 4.17 has been proved in [16–18]. Also, in the case of $P = I$ and $s = \infty$ Corollary 4.17 has been proved in [16–18] and [41, 43].*

Now, we give the applications of Theorem 3.3 and Theorem 4.9 for the parabolic Marcinkiewicz operator.

Suppose that $\Omega(x)$ is a real-valued and measurable function defined on $\mathbb{R}^n$ satisfying the following conditions:

(a) $\Omega(x)$ is homogeneous of degree zero with respect to A_t, that is,

$$\Omega(A_t x) = \Omega(x), \text{ for any } t > 0,\ x \in \mathbb{R}^n \setminus \{0\};$$

(b) $\Omega(x)$ has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x') J(x')\, d\sigma(x') = 0,$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.
(c) $\Omega \in L_1(S^{n-1})$.

Then the parabolic Marcinkiewicz integral of higher dimension μ_Ω^γ is defined by

$$\mu_\Omega^\gamma(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{\rho(x-y)\le t} \frac{\Omega(x-y)}{\rho(x-y)^{\gamma-1}} f(y)dy.$$

On the other hand, for a suitable function b, the commutator of the parabolic Marcinkiewicz integral μ_Ω^γ is defined by

$$[b,\mu_\Omega^\gamma](f)(x) = \left(\int_0^\infty |F_{\Omega,t,b}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t,b}(f)(x) = \int_{\rho(x-y)\le t} \frac{\Omega(x-y)}{\rho(x-y)^{\gamma-1}} [b(x)-b(y)] f(y)dy.$$

We consider the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 \frac{dt}{t^3})^{1/2} < \infty\}$. Then, it is clear that $\mu_\Omega^\gamma(f)(x) = \|F_{\Omega,t}(x)\|$.

By the Minkowski inequality and the conditions on Ω, we get

$$\mu_\Omega^\gamma(f)(x) \le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^{\gamma-1}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3}\right)^{1/2} dy \le C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{\rho(x-y)^{\gamma}} |f(y)|dy.$$

Thus, μ_Ω^γ satisfies the condition (1). When $\Omega \in L_s\left(S^{n-1}\right)$ $(s>1)$, It is known that μ_Ω is bounded on $L_p(\mathbb{R}^n)$ for $p>1$, and bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ for $p=1$ (see [50]), then from Theorems 3.3, 4.9 we get

Corollary 4.19. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$, $\Omega \in L_s\left(S^{n-1}\right)$, $1 < s \le \infty$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1,φ_2) satisfies condition (16) and for $1<p<s$ the pair (φ_1,φ_2) satisfies condition (17) and Ω satisfies conditions (a)–(c). Then the operator μ_Ω^γ is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$ for $p>1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$.*

Corollary 4.20. *Let $1 \le p < \infty$, $\Omega \in L_s\left(S^{n-1}\right)$, $1 < s \le \infty$. Let also, for $s' \le p$, $p \ne 1$, the pair (φ_1,φ_2) satisfies condition (20) and for $1<p<s$ the pair (φ_1,φ_2) satisfies condition (21) and Ω satisfies conditions (a)–(c). Then the operator μ_Ω^γ is bounded from M_{p,φ_1} to M_{p,φ_2} for $p>1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p=1$.*

Corollary 4.21. *Let $x_0 \in \mathbb{R}^n$, $\Omega \in L_s(S^{n-1})$, $1<s\le\infty$. Let $1<p<\infty$, $b \in LC_{p_2,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $0 \le \lambda < \frac{1}{n}$. Let also, for $s' \le p$ the pair (φ_1,φ_2) satisfies condition (28) and for $p_1 < s$ the pair (φ_1,φ_2) satisfies condition (29) and Ω satisfies conditions (a)–(c). Then, the operator $[b,\mu_\Omega^\gamma]$ is bounded from $LM_{p_1,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$.*

Corollary 4.22. *Let $\Omega \in L_s(S^{n-1})$, $1<s\le\infty$, $1<p<\infty$ and $b \in BMO(\mathbb{R}^n)$. Let also, for $s' \le p$ the pair (φ_1,φ_2) satisfies condition (32) and for $p<s$ the pair (φ_1,φ_2) satisfies condition (33) and Ω satisfies conditions (a)–(c). Then, the operator $[b,\mu_\Omega^\gamma]$ is bounded from M_{p,φ_1} to M_{p,φ_2}.*

Remark 4.23. *Obviously, if we take $\alpha_1 = \cdots = \alpha_n = 1$ and $P = I$, then $\rho(x) = |x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$, $\gamma = n$, $(\mathbb{R}^n,\rho) = (\mathbb{R}^n,|\cdot|)$, $E_I(x,r) = B(x,r)$. In this case, μ_Ω^γ is just the classical Marcinkiewicz integral operator μ_Ω, which was first defined by Stein in 1958. In [51], Stein has proved that if Ω satisfies the Lipshitz condition of degree of α $(0<\alpha\le 1)$ on S^{n-1} and the conditions (a), (b) (obviously, in the case $A_t = tI$ and $J(x') \equiv 1$), then μ_Ω is both of the type (p,p) $(1<p\le 2)$ and the weak type (1.1). (See also [52] for the boundedness of the classical Marcinkiewicz integral μ_Ω.)*

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References

[1] Calderón A.P. and Zygmund A., On the existence of certain singular integrals, Acta Math., 1952, 88, 85-139.

[2] Calderón A.P. and Zygmund A., Singular integral operators and differential equations, Amer. J. Math., 1957, 79, 901-921.

[3] Fabes E. and Riviére N., Singular integrals with mixed homogeneity, Stud. Math., 1966, 27, 19-38.

[4] Hofmann S., A characterization of commutators of parabolic singular integrals, Fourier analysis and partial differential equations (Miraflores de la Siera, 1992), Stud. Adv. Math. CRC, Boca Raton, FL 1995, 195-210.

[5] Hofmann S., Parabolic singular integrals of Calderón-type, rough operators, and caloric layer potentials, Duke Math. J., 1997, 90(2), 209-259.

[6] Tao T., The weak-type $(1,1)$ of $L \log L$ homogeneous convolution operators, Indiana Univ. Math. J., 1999, 48, 1547-1584.

[7] Coifman R.R., Weiss G., Analyse harmonique non-commutative sur certains espaces homogènes, (French) Étude de certaines intégrales singuliàres, Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971.

[8] Folland G.B., Stein E.M., Hardy Spaces on homogeneous groups, Math. Notes, 28, Princeton Univ. Press, Princeton, 1982.

[9] Calderón A.P. and Torchinsky A., Parabolic maximal functions associated with a distribution, Adv. Math., 1975, 16, 1-64.

[10] Besov O.V., Il'in V.P., Lizorkin P.I., The L_p-estimates of a certain class of non-isotropically singular integrals, (Russian) Dokl. Akad. Nauk SSSR., 1966, 169, 1250-1253.

[11] Soria F., Weiss G., A remark on singular integrals and power weights, Indiana Univ. Math. J., 1994, 43, 187-204.

[12] Lu G., Lu S.Z., Yang D.C., Singular integrals and commutators on homogeneous groups, Anal. Math., 2002, 28, 103-134.

[13] Palagachev D.K., Softova L., Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's, Potential Anal., 2004, 20, 237-263.

[14] Stein E.M., Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals, Princeton Univ. Press, Princeton NJ, 1993.

[15] Muckenhoupt B. and Wheeden R.L., Weighted norm inequalities for singular and fractional integral. Trans. Amer. Math. Soc., 1971, 161, 249-258.

[16] Balakishiyev A.S., Guliyev V.S., Gurbuz F. and Serbetci A., Sublinear operators with rough kernel generated by Calderon-Zygmund operators and their commutators on generalized local Morrey spaces, J. Inequal. Appl. 2015, 2015:61. doi:10.1186/s13660-015-0582-y.

[17] Gurbuz F., Boundedness of some potential type sublinear operators and their commutators with rough kernels on generalized local Morrey spaces [*Ph.D. thesis*], Ankara University, Ankara, Turkey, 2015.

[18] Gurbuz F., Some estimates on generalized Morrey spaces for certain sublinear operators with rough kernel generated by Calderón-Zygmund operators and their commutators, submitted.

[19] Fu Z.W., Lin Y. and Lu S.Z., λ-Central BMO estimates for commutators of singular integral operators with rough kernel, Acta Math. Sin. (Engl. Ser.), 2008, 24, 373-386.

[20] Lu S.Z., Ding Y., Yan D.Y., Singular Integrals and Related Topics, World Scientific, Singapore, 2007.

[21] Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 1938, 43, 126-166.

[22] Adams D.R., A note on Riesz potentials, Duke Math. J., 1975, 42, 765-778.

[23] Chiarenza F., Frasca M., Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat., 1987, 7, 273-279.

[24] Peetre J., On the theory of $M_{p,\lambda}$, J. Funct. Anal., 1969, 4, 71-87.

[25] Chiarenza F., Frasca M., Longo P., Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients, Ricerche Mat., 1991, 40, 149-168.

[26] Chiarenza F., Frasca M., Longo P., $W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc., 1993, 336, 841-853.

[27] Fazio G.Di, Ragusa M.A., Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, J. Funct. Anal., 1993, 112, 241-256.

[28] Fazio G.Di, Palagachev D.K. and Ragusa M.A., Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, J. Funct. Anal., 1999, 166, 179-196.

[29] Meskhi A., Maximal functions, potentials and singular integrals in grand Morrey spaces, Complex Var. Elliptic Equ., 2011, 56, 1003-1019.

[30] Ding Y., Yang D.C., Zhou Z. , Boundedness of sublinear operators and commutators on $L^{p,\omega}(\mathbb{R}^n)$, Yokohama Math. J., 1998, 46, 15-27.

[31] Mizuhara T., Boundedness of some classical operators on generalized Morrey spaces, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo (1991), 183-189.

[32] Nakai E., Hardy–Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces, Math. Nachr., 1994, 166, 95-103.

[33] Nakai E., The Campanato, Morrey and Hölder spaces on spaces of homogeneous type, Studia Math., 2006, 176, 1, 1-19.
[34] Guliyev V.S., Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian).
[35] Guliyev V.S., Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.
[36] Burenkov V.I., Guliyev H.V., Guliyev V.S., Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces, J. Comput. Appl. Math., 2007, 208(1), 280-301.
[37] Burenkov V.I., Guliyev V.S., Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces, Potential Anal., 2009, 30(3), 211-249.
[38] Burenkov V.I., Gogatishvili A., Guliyev V.S., Mustafayev R.Ch., Boundedness of the fractional maximal operator in local Morrey-type spaces, Complex Var. Elliptic Equ., 2010, 55(8-10), 739-758.
[39] Burenkov V.I., Gogatishvili A., Guliyev V.S., Mustafayev R.Ch., Boundedness of the Riesz potential in local Morrey-type spaces, Potential Anal., 2011, 35(1), 67-87.
[40] Guliyev V.S., Mustafayev R.Ch., Boundedness of the anisotropic maximal and anisotropic singular integral operators in generalized Morrey spaces, Acta Math. Sin. (Engl. Ser.), 2011, 27(12), 2361-2370.
[41] Guliyev V.S., Aliyev S.S., Karaman T., Shukurov P., Boundedness of sublinear operators and commutators on generalized Morrey Space, Int. Eq. Op. Theory., 2011, 71(3), 327-355.
[42] Guliyev V.S., Generalized local Morrey spaces and fractional integral operators with rough kernel, J. Math. Sci. (N.Y.), 2013, 193(2), 211-227.
[43] Karaman T., Boundedness of some classes of sublinear operators on generalized weighted Morrey spaces and some applications [*Ph.D. thesis*], Ankara University, Ankara, Turkey, 2012.
[44] Coifman R.R., Rochberg R., Weiss G., Factorization theorems for Hardy spaces in several variables, Ann. Math., 1976, 103(3), 611-635.
[45] Grafakos L., Li X.W., Yang D.C., Bilinear operators on Herz-type Hardy spaces. Trans. Amer. Math. Soc., 1998, 350, 1249-1275.
[46] Tao X.X., Shi Y.L., Multilinear commutators of Calderón-Zygmund operator on λ-central Morrey spaces, Adv. Math., 2011, 40, 47-59.
[47] Coifman R.R., Lions P., Meyer Y., Semmes S., Compensated compactness and Hardy spaces. J. Math. Pures Appl., 1993, 72, 247-286.
[48] Lu S.Z. and Yang D.C., The central BMO spaces and Littlewood-Paley operators, Approx. Theory Appl. (N.S.), 1995, 11, 72-94.
[49] Rzaev R.M., On approximation of local summary functions by singular integrals in terms of mean oscillation and some applications, Preprint No. 1 of Inst. Physics of NAS of Azerb., 1992, pp. 1-43 (Russian).
[50] Xue Q.Y., Ding Y. and Yabuta K., Parabolic Littlewood-Paley g-function with rough kernel, Acta Math. Sin. (Engl. Ser.), 2008, 24(12) 2049-2060.
[51] Stein E.M., On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz, Trans. Amer. Math. Soc., 1958, 88, 430-466.
[52] Torchinsky A. and Wang S., A note on the Marcinkiewicz integral, Colloq. Math., 1990, 60/61, 235-243.