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Research Article

Huashui Zhan* and Shuping Chen

The BV solution of the parabolic equation with degeneracy on the boundary

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Abstract: Consider a parabolic equation which is degenerate on the boundary. By the degeneracy, to assure the well-posedness of the solutions, only a partial boundary condition is generally necessary. When $1 \le \alpha < p-1$, the existence of the local BV solution is proved. By choosing some kinds of test functions, the stability of the solutions based on a partial boundary condition is established.

Keywords: Local BV Solution, Boundary degeneracy, Partial boundary condition, Stability

MSC: 35L65, 35L85, 35R35

1 Introduction and the main results

Yin-Wang [1] first studied the equation

$$u_t = \operatorname{div}(\rho^{\alpha}|\nabla u|^{p-2}\nabla u), \quad (x,t) \in Q_T = \Omega \times (0,T). \tag{1}$$

where Ω is a bounded domain in R^N with appropriately smooth boundary, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, $p > 1$, $\alpha > 0$. An obvious character of the equation is that, the diffusion coefficient depends on the distance to the boundary. Since the diffusion coefficient vanishes on the boundary, it seems that there is no heat flux across the boundary. However, Yin-Wang [1] showed that the fact might not coincide with what we image. In fact, the exponent α, which characterizes the vanishing ratio of the diffusion coefficient near the boundary, does determine the behavior of the heat transfer near the boundary. One may refer to [1] for the details.

In our paper, we will consider the following equation

$$u_t = \operatorname{div}(\rho^{\alpha}|\nabla u|^{p-2}\nabla u) + \sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T. \tag{2}$$

The convection term $\sum_{i=1}^{N} \frac{\partial b_i(u)}{\partial x_i}$ not only brings the difference on operational skills, but more essentially, it makes the nature of boundary condition change. The equation (2) had been originally studied by the authors in [2, 3]. Instead of the whole boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

only a partial boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{4}$$

matching equation (2) is considered. Here, denoting $\{n_i(x)\}$ as the unit inner normal vector of $\partial\Omega$, when $b_i{}'(0)n_i(x) < 0$, $\forall x \in \partial\Omega$, then $\Sigma_p = \partial\Omega$. But generally, it is just a portion of $\partial\Omega$. However, we don't need

***Corresponding Author: Huashui Zhan:** School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China, E-mail: 2012111007@xmut.edu.cn, hszhan@jmu.edu.cn
Shuping Chen: School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China, E-mail: 2012111003@xmut.edu.cn

to pay too much attention to its explicit formula, we only need to remember it is just a subset of $\partial\Omega$. Certainly, the initial value is always necessary,

$$u(x,0) = u_0(x). \tag{5}$$

In [2, 3], we said a bounded domain Ω has the integral non-singularity, if the constants α, p, satisfy

$$\int_\Omega \rho^{-\frac{2\alpha}{p-2}} \mathrm{d}x \leqslant c.$$

We assumed that there are constants β, c such that

$$|b_i(s)| \leqslant c|s|^{1+\beta}, \quad \left|b'_i(s)\right| \leqslant c|s|^{\beta}. \tag{6}$$

If $p > 2$, we had obtained the existence of the solution of equation (2) with the initial boundary values (4)-(5), and if $\Sigma_p = \partial\Omega$, we also had obtained the stability of the weak solutions. In our paper, we will promote the existence of the solution without the condition (6) but limiting that $\alpha \geq 1$. The most innovation of our paper is that the stability of the weak solutions can be obtained only based on the partial boundary condition (4). Comparing with the case of that $\Sigma_p = \partial\Omega$ in [2, 3] or $\Sigma_p = \emptyset$ in [1] (when $\alpha \geq p-1$), how to obtain the stability of the weak solutions only based on the partial boundary condition (4) seems very difficult.

Let us give the basic definitions and the main results as following.

Definition 1.1. *A function $u(x,t)$ is said to be a local BV solution of equation* (2) *with the initial value* (5)*, if*

$$u \in BV(Q_{T\lambda}) \bigcap L^\infty(Q_T),\ \rho^\alpha |\nabla u|^p \in L^1(Q_T),$$

and for any function $\varphi \in C_0^\infty(Q_T)$, the following integral equivalence holds

$$\iint_{Q_T} (-u\varphi_t + \rho^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi + \sum_{i=1}^N b_i(u) \cdot \varphi_{x_i}) dxdt = 0. \tag{7}$$

The initial value is satisfied in the sense of

$$\lim_{t\to 0} \int_\Omega |u(x,t) - u_0(x)| dx = 0. \tag{8}$$

Here, $Q_{T\lambda} = \{(x,t) \in Q_T : \rho(x) = dist(x, \partial\Omega) > \lambda\}$ for small enough $\lambda > 0$.

Definition 1.2. *A function $u(x,t)$ is said to be a local BV solution of equation* (2) *with the initial boundary values* (4)-(5)*, if u satisfies Definition 1.1, and it satisfies the partial boundary condition* (4) *in the sense of the trace.*

Definition 1.3. *If u is a local BV solution of equation* (2)*, satisfies that*

$$|u(x,t)| \leq c\rho(x),\ |\nabla u| \leq c\rho^{-\alpha}(x), \tag{9}$$

when x is near $\partial\Omega$, then we say u is a regular solution.

Remark 1.4. *If $b_i(s) \equiv 0$, we had proved that the solution of equation* (1) *is regular in [4].*

Theorem 1.5. *Let $1 < p$, $1 \leq \alpha < p-1$, $b_i(s) \in C^2(R^1)$. If*

$$u_0(x) \in C_0^\infty(\Omega), \tag{10}$$

then equation (2) *with initial boundary values* (4)-(5) *has a local BV solution u, and $u_t \in L^\infty(Q_T)$.*

Theorem 1.6. *Let $\alpha < p-1$, u and v be two local BV solutions of equation* (1) *with the same partial homogeneous boundary value*

$$u|_{\Sigma_p \times (0,T)} = 0 = v|_{\Sigma_p \times (0,T)}, \tag{11}$$

and with the different initial values $u(x,0) = v(x,0)$ *respectively. If* $b_i(s)$ *is a Lipschitz function, and moreover*

$$|\nabla u| \le c\rho^{-\alpha}(x), |\nabla v| \le c\rho^{-\alpha}(x) \tag{12}$$

then

$$\begin{aligned}\int_\Omega |u(x,t) - v(x,t)|\, dx \leqslant \int_\Omega |u_0 - v_0|\, dx + c\int_{\Sigma_p'} |u - v| d\Sigma \\ + \lim\sup_{n\to\infty} \int_{\Sigma_p'} g_n(u - v)|u - v| d\Sigma, \quad \forall t \in [0, T).\end{aligned} \tag{13}$$

Here, $n > 0$ is a nature number, the details of the definition and the properties of the function $g_n(s)$ is in Section 3, in particular, $| \ g_n(s)s \ | \le c$.

Theorem 1.7. *Let* $\alpha \ge p - 1$, *and* u, v *be two local BV solutions of* (1) *with the initial values* $u_0(x), v_0(x)$ *respectively. If* u *and* v *are regular, and*

$$|b_i(u) - b_i(v)| \le c|u - v|^{\alpha+2}, \tag{14}$$

then

$$\int_\Omega | \ u(x,t) - v(x,t) \ |^2 \, dx \le c \int_\Omega | \ u_0 - v_0 \ |^2 \, dx, a.e.t \in (0,T). \tag{15}$$

The most important character of Theorem 1.7 is in that we obtain the stability (15) without any boundary value condition. However, since the solutions considered in the theorem are regular, we can easily obtain the conclusion (15) in a similar way as in [1]. So we omit the details of the proof of the theorem in our paper.

Recently, the author has been interested in the initial-boundary value problem of the following strongly degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i}(a^{ij}(u,x,t)\frac{\partial u}{\partial x_j}) + \frac{\partial b_i(u,x,t)}{\partial x_i}, \ (x,t) \in Q_T. \tag{16}$$

The stability of the solutions based on a partial boundary condition (4) has been established in [5–7] et. al. Actually, many mathematicians have been interested in the problem, and have obtained many important results of the the stability of the solutions based on a partial boundary condition, one may see the Refs. [8–11]. Unlike the equation (16), to the best knowledge of the authors, considering the parabolic equation related to the $p-$Laplacian, our paper is the first one to study the stability of the solutions based on a partial boundary condition (4). Of course, whether the condition (12) in Theorem 1.6 and the assumption that u, v are regular in Theorem 1.7 are necessary or not? This is a very interesting problem to be studied in the future. Some other related references, one can refer to Refs. [12–16]. The paper is arranged as following. In Section 1, we have introduced the problem and given the main results of the paper. In Section 2, we prove the existence of the local BV solution. In Section 3, only based on a partial boundary condition, we prove the Theorem 1.6.

2 The local BV solution

To study equation (2), we consider the following regularized problem

$$u_{\varepsilon t} - \text{div}(\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon) - \sum_{i=1}^N \frac{\partial b_i(u_\varepsilon)}{\partial x_i} = 0, (x,t) \in Q_T, \tag{17}$$

$$u_\varepsilon(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{18}$$

$$u_\varepsilon(x,0) = u_0(x), \qquad x \in \Omega. \tag{19}$$

where $\rho_\varepsilon = \rho * \delta_\varepsilon + \varepsilon,\ \varepsilon > 0$, δ_ε is the mollifier as usual. It is well-known that the above problem has a unique classical solution [17, 18]. Hence, for any $\varphi \in C_0^\infty(Q_T)$, u_ε satisfies

$$\iint_{Q_T} (-u_\varepsilon\varphi_t + \rho_\varepsilon^\alpha|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\varphi + \sum_{i=1}^N b_i(u_\varepsilon)\cdot\varphi_{x_i})\mathrm{d}x\mathrm{d}t = 0. \tag{20}$$

Lemma 2.1. *If $u_0 \in C_0^\infty(\Omega)$, $\alpha \geq 1$, then the solution u_ε of the initial boundary value problem* (17)-(19) *converges locally in $BV(Q_T)$, and its limit function u is the local BV solution of equation* (2) *with the initial value* (5).

Proof. By the maximum principle, there is a constant c only dependent on $\|u_0\|_{L^\infty}(\Omega)$ but independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leqslant c. \tag{21}$$

Multiplying (17) by u_ε and integrating it over Q_T, we have

$$\frac{1}{2}\int_\Omega u_\varepsilon^2\mathrm{d}x + \iint_{Q_T} \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\mathrm{d}x\mathrm{d}t + \iint_{Q_T} u_\varepsilon\sum_{i=1}^N\frac{\partial b_i(u_\varepsilon)}{\partial x_i}\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_\Omega u_0^2\mathrm{d}x.$$

By the fact

$$\iint_{Q_T} u_\varepsilon\sum_{i=1}^N\frac{\partial b_i(u_\varepsilon)}{\partial x_i}\mathrm{d}x\mathrm{d}t = -\sum_{i=1}^N\iint_{Q_T}\frac{\partial u_\varepsilon}{\partial x_i}b_i(u_\varepsilon)\mathrm{d}x\mathrm{d}t = -\sum_{i=1}^N\int_\Omega\frac{\partial}{\partial x_i}\int_0^{u_\varepsilon} b_i(s)\mathrm{d}s\mathrm{d}x = 0,$$

then

$$\frac{1}{2}\int_\Omega u_\varepsilon^2\mathrm{d}x + \iint_{Q_T}\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\mathrm{d}x\mathrm{d}t \leqslant c. \tag{22}$$

For small enough $\lambda > 0$, let $\Omega_\lambda = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \lambda\}$. Since $p > 1$, by (22),

$$\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|dxdt \leq c\left(\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|^p dxdt\right)^{\frac{1}{p}} \leq c(\lambda). \tag{23}$$

Differentiating (17) with t, and denoting $w = u_{\varepsilon t}$, then

$$\begin{aligned}\frac{\partial w}{\partial t} &= (p-2)\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}u_{x_k}u_{x_i}w_{x_kx_i} + \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}w_{x_ix_i}\\
&+(p-2)\nabla\rho_\varepsilon^\alpha\cdot\nabla u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}u_{x_k}w_{x_k} + \nabla\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\cdot\nabla w\\
&+(p-2)(p-4)\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-6}{2}}u_{x_j}u_{x_ix_j}u_{x_i}w_{x_k}u_{x_k}\\
&+(p-2)\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}(u_{x_i}u_{x_ix_k}w_{x_k} + u_{x_k}u_{x_kx_i}w_{x_i} + u_{x_k}u_{x_ix_i}w_{x_k})\\
&+b_i''(u)u_{x_i}w + b_i'(u)w_{x_i},\end{aligned}$$

rewriting it as

$$\frac{\partial w}{\partial t} = a_{ij}\frac{\partial^2 w}{\partial x_i\partial x_j} + \sum_{i=1}^N f_i(x,t,w)w_{x_i} + b_i''(u)u_{x_i}w,$$

where

$$a_{ij} = \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(\delta_{ij} + (p-2)(|\nabla u_\varepsilon|^2+\varepsilon)^{-1}u_{x_i}u_{x_j}).$$

$$f_i(x,t,w) = (p-2)\nabla\rho_\varepsilon^\alpha \cdot \nabla u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}} u_{x_i} + (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}(\rho_\varepsilon^\alpha)_{x_i}$$
$$+(p-2)(p-4)\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-6}{2}} u_{x_j}u_{x_kx_j}u_{x_k}u_{x_i}$$
$$+(p-2)\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}(u_{x_k}u_{x_kx_i}+u_{x_i}u_{x_ix_k}+u_{x_i}u_{x_kx_k}) + b_i'(u),$$

Clearly, w satisfies that

$$w(x,t)=0, (x,t)\in\partial\Omega\times[0,T),$$
$$w(x,0) = \mathrm{div}(\rho_\varepsilon^\alpha(|\nabla u_0|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_0) + \frac{\partial b_i(u_0)}{\partial x_i}, x\in\Omega.$$

Denoting that

$$a = (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}},$$

then

$$\min\{p-1,1\}a|\xi|^2 \le a_{ij}\xi_i\xi_j \le \max\{p-1,1\}a|\xi|^2.$$

By the maximum principle, due to $\alpha \ge 1$, we have

$$\sup_{\Omega\times(0,T)}|u_{\varepsilon t}| \le \sup_{\Omega}|\mathrm{div}(\rho_\varepsilon^\alpha(|\nabla u_0|^2+\varepsilon)^{\frac{p-2}{2}} + \frac{\partial b_i(u_0)}{\partial x_i}| \le c. \tag{24}$$

By (23)-(24), we know that $u_\varepsilon \in BV(Q_{T\lambda})$, and

$$\int_0^T\int_{\Omega_\lambda}|\nabla u_\varepsilon|dxdt \le c, \int_0^T\int_{\Omega_\lambda}|u_{\varepsilon t}|dxdt \le c. \tag{25}$$

Then by Kolmogoroff's theorem, there exists a subsequence (still denoted as u_ε) of u_ε, which is strongly convergent to $u\in BV(Q_{T\lambda})$. In particular, by the arbitrary of λ, $u_\varepsilon \to u$ a.e. in Q_T.

Hence, by (22), (25), there exists a function u and $n-$dimensional vector function $\overrightarrow{\zeta} = (\zeta_1,\cdots,\zeta_n)$ satisfying that

$$u\in BV(Q_{T\lambda})\bigcap L^\infty(Q_T), \quad \left|\overrightarrow{\zeta}\right| \in L^{\frac{p}{p-1}}(Q_T),$$

and

$$u_\varepsilon \rightharpoonup *u, \quad \text{in} \quad L^\infty(Q_T),$$
$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in} \quad L^p_{loc}(Q_T),$$
$$\rho_\varepsilon^\alpha|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup \overrightarrow{\zeta} \quad \text{in} \quad L^{\frac{p}{p-1}}(Q_T).$$

In order to prove u satisfies equation (2), we notice that for any function $\varphi\in C_0^\infty(Q_T)$,

$$\iint_{Q_T}(-u_\varepsilon\varphi_t + \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\varphi + \sum_{i=1}^N b_i(u_\varepsilon)\cdot\varphi_{x_i})\mathrm{d}x\mathrm{d}t = 0. \tag{26}$$

and $u_\varepsilon \to u$ is almost everywhere convergent, so $b_i(u_\varepsilon)\to b_i(u)$ is true. Then

$$\iint_{Q_T}(\frac{\partial u}{\partial t}\varphi + \vec{\zeta}\cdot\nabla\varphi + \sum_{i=1}^N b_i(u)\cdot\varphi_{x_i})\mathrm{d}x\mathrm{d}t = 0. \tag{27}$$

Now, if we can prove that

$$\iint_{Q_T}\rho^\alpha\,|\nabla u|^{p-2}\,\nabla u\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t = \iint_{Q_T}\overrightarrow{\zeta}\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t. \tag{28}$$

for any function $\varphi\in C_0^\infty(Q_T)$, then u satisfies equation (2).

Let $0 \leqslant \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ in $supp\varphi$. Let $v \in BV(Q_{T\lambda}) \cap L^\infty(Q_T), \rho^\alpha|\nabla v|^p \in L^1(Q_T)$. It is well-known that

$$\iint\limits_{Q_T} \psi\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u_\varepsilon - \nabla v)\mathrm{d}x\mathrm{d}t \geqslant 0. \tag{29}$$

By choosing $\varphi = \psi u_\varepsilon$ in (26), we can obtain

$$\begin{aligned}\iint\limits_{Q_T} \psi\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2\mathrm{d}x\mathrm{d}t =& \frac{1}{2}\iint\limits_{Q_T}\psi_t u_\varepsilon^2\mathrm{d}x\mathrm{d}t - \iint\limits_{Q_T}\rho_\varepsilon^\alpha u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\ &-\sum_{i=1}^N\iint\limits_{Q_T} b_i(u_\varepsilon)(u_{\varepsilon x_i}\psi + u_\varepsilon\psi_{x_i})\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{30}$$

Noticing that when $p \geq 2$,

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2 \geq |\nabla u_\varepsilon|^p,$$
$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon| \leq (|\nabla u_\varepsilon|^{p-1}+1),$$

and when $1 < p < 2$,

$$\begin{aligned}(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon|^2 &\geq (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p}{2}} - \varepsilon^{\frac{p}{2}},\\ (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}|\nabla u_\varepsilon| &\leq (|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-1}{2}},\end{aligned} \tag{31}$$

then in both cases, by (29), we have

$$\begin{aligned}&\frac{1}{2}\iint\limits_{Q_T}\psi_t u_\varepsilon^2\mathrm{d}x\mathrm{d}t - \iint\limits_{Q_T}\rho_\varepsilon^\alpha u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\psi\mathrm{d}x\mathrm{d}t - \sum_{i=1}^N\iint\limits_{Q_T} b_i(u_\varepsilon)(u_{\varepsilon x_i}\psi + u_\varepsilon\psi_{x_i})\mathrm{d}x\mathrm{d}t\\ &+\varepsilon^{\frac{p}{2}}c(\Omega) - \iint\limits_{Q_T}\rho_\varepsilon^\alpha\psi|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v dxdt - \iint\limits_{Q_T}\rho_\varepsilon^\alpha\psi|\nabla v|^{p-2}\nabla(u_\varepsilon - v)dxdt \geq 0.\end{aligned} \tag{32}$$

Thus

$$\begin{aligned}&\frac{1}{2}\iint\limits_{Q_T}\psi_t u_\varepsilon^2\mathrm{d}x\mathrm{d}t - \iint\limits_{Q_T}\rho_\varepsilon^\alpha u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon\cdot\nabla\psi\mathrm{d}x\mathrm{d}t - \sum_{i=1}^N\iint\limits_{Q_T} b_i(u_\varepsilon)(u_{\varepsilon x_i}\psi + u_\varepsilon\psi_{x_i})\mathrm{d}x\mathrm{d}t\\ &+\varepsilon^{\frac{p}{2}}c(\Omega) - \iint\limits_{Q_T}\rho_\varepsilon^\alpha\psi|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla v dxdt - \iint\limits_{Q_T}\psi\rho^\alpha|\nabla v|^{p-2}\nabla v\cdot(\nabla u_\varepsilon - \nabla v)\mathrm{d}x\mathrm{d}t\\ &+\iint\limits_{Q_T}\psi(\rho^\alpha - \rho_\varepsilon^\alpha)|\nabla v|^{p-2}\nabla v\cdot(\nabla u_\varepsilon - \nabla v)\mathrm{d}x\mathrm{d}t \geqslant 0.\end{aligned} \tag{33}$$

Noticing

$$\begin{aligned}&\left|\iint\limits_{Q_T}\psi(\rho^\alpha - \rho_\varepsilon^\alpha)|\nabla v|^{p-2}\nabla v\cdot(\nabla u_\varepsilon - \nabla v)\mathrm{d}x\mathrm{d}t\right|\\ \leqslant& \sup_{(x,t)\in Q_T}\frac{|\psi(\rho^\alpha - \rho_\varepsilon^\alpha)|}{\rho^\alpha}\iint\limits_{Q_T}\rho^\alpha|\nabla v|^{p-1}|\nabla u_\varepsilon - \nabla v|\,\mathrm{d}x\mathrm{d}t\\ \leqslant& \sup_{(x,t)\in Q_T}\frac{|\psi(\rho^\alpha - \rho_\varepsilon^\alpha)|}{\rho^\alpha}(\iint\limits_{Q_T}\rho^\alpha|\nabla v|^p\mathrm{d}x\mathrm{d}t + \iint\limits_{Q_T}\rho^\alpha|\nabla v|^{p-1}|\nabla u_\varepsilon|\,\mathrm{d}x\mathrm{d}t)\end{aligned} \tag{34}$$

and

$$(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-2}{2}}\nabla u_\varepsilon = |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon + \frac{p-2}{2}\varepsilon\int\limits_0^1(|\nabla u_\varepsilon|^2+\varepsilon s)^{\frac{p-4}{2}}ds\nabla u_\varepsilon$$

then,

$$\lim_{\varepsilon\to 0}\iint_{Q_T}\frac{p-2}{2}\varepsilon\int_0^1(|\nabla u_\varepsilon|^2+\varepsilon s)^{\frac{p-4}{2}}ds\nabla u_\varepsilon\nabla\psi u_\varepsilon dxdt=0. \tag{35}$$

By Hölder inequality, there holds

$$\iint_{Q_T}\rho^\alpha|\nabla v|^{p-1}|\nabla u_\varepsilon|\,\mathrm{d}x\mathrm{d}t\leqslant(\iint_{Q_T}(\rho^m|\nabla v|^{p-1})^s\mathrm{d}x\mathrm{d}t)^{1/s}\cdot(\iint_{Q_T}(\rho^n|\nabla u_\varepsilon|)^p\mathrm{d}x\mathrm{d}t)^{1/p},$$

where $m=\frac{\alpha(p-1)}{p}$, $n=\frac{\alpha}{p}$, $s=\frac{p}{p-1}$. Due to $\rho^\alpha|\nabla u|^p,\rho^\alpha|\nabla v|^p\in L^1(Q_T)$, then

$$\iint_{Q_T}\rho^\alpha|\nabla v|^p\mathrm{d}x\mathrm{d}t+\iint_{Q_T}\rho^\alpha|\nabla v|^{p-1}|\nabla u_\varepsilon|\,\mathrm{d}x\mathrm{d}t\leqslant c.$$

Let $\varepsilon\to 0$ in (33). It converges to 0.

Thus, we have

$$\frac{1}{2}\iint_{Q_T}\psi_t u^2\mathrm{d}x\mathrm{d}t-\iint_{Q_T}u\overrightarrow{\zeta}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t-\sum_{i=1}^N\iint_{Q_T}b_i(u)(u_{x_i}\psi+u\psi_{x_i})\mathrm{d}x\mathrm{d}t$$
$$-\iint_{Q_T}\psi\overrightarrow{\zeta}\cdot\nabla v\mathrm{d}x\mathrm{d}t-\iint_{Q_T}\psi\rho^\alpha|\nabla v|^{p-2}\nabla v\cdot(\nabla u_\varepsilon-\nabla v)\mathrm{d}x\mathrm{d}t\geqslant 0.$$

Let $\varphi=\psi u$ in (2), we get

$$\iint_{Q_T}\psi\overrightarrow{\zeta}\cdot\nabla u\mathrm{d}x\mathrm{d}t-\frac{1}{2}\iint_{Q_T}u^2\psi_t\mathrm{d}x\mathrm{d}t+\iint_{Q_T}u\overrightarrow{\zeta}\cdot\nabla\psi\mathrm{d}x\mathrm{d}t+\sum_{i=1}^N\iint_{Q_T}b_i(u)(u_{x_i}\psi+u\psi_{x_i})\mathrm{d}x\mathrm{d}t=0.$$

Thus

$$\iint_{Q_T}\psi(\overrightarrow{\zeta}-\rho^\alpha|\nabla v|^{p-2}\nabla v)\cdot(\nabla u-\nabla v)\mathrm{d}x\mathrm{d}t\geqslant 0. \tag{36}$$

Let $v=u-\lambda\varphi,\lambda>0,\varphi\in C_0^\infty(Q_T)$, then

$$\iint_{Q_T}\psi(\overrightarrow{\zeta}-\rho^\alpha|\nabla(u-\lambda\varphi)|^{p-2}\nabla(u-\lambda\varphi))\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\geqslant 0,$$

If $\lambda\to 0$, then

$$\iint_{Q_T}\psi(\overrightarrow{\zeta}-\rho^\alpha|\nabla u|^{p-2}\nabla u)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\geqslant 0.$$

Moreover, if $\lambda<0$, similarly we can get

$$\iint_{Q_T}\psi(\overrightarrow{\zeta}-\rho^\alpha|\nabla u|^{p-2}\nabla u)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t\leqslant 0.$$

Thus

$$\iint_{Q_T}\psi(\overrightarrow{\zeta}-\rho^\alpha|\nabla u|^{p-2}\nabla u)\cdot\nabla\varphi\mathrm{d}x\mathrm{d}t=0,$$

Noticing that $\psi=1$ on suppφ, (28) holds. At the same time, we can prove (5) as in [19], we omit the details here. The lemma is proved. □

Corollary 2.2. *If $u_0 \in C_0^{\infty}(\Omega)$, $\alpha \geq 1$, there exists a local BV solution u of equation* (2) *with the initial value* (5), *such that $u_t \in L^{\infty}(Q_T)$.*

Lemma 2.3. *If $\alpha < p-1$, u is a weak solution of equation* (1) *(also equation* (2)*) with the initial value* (5). *Then u has trace on the boundary $\partial\Omega$.*

Lemma 2.3 had been proved in [1, 2]. Clearly, Theorem 1.5 is the directly corollary of Lemma 2.1-2.3.

Remark 2.4. *The condition $\alpha \geq 1$ is only used to prove that $|u_{\varepsilon t}| \leq c$, which implies that $\int_{Q_{T\lambda}} |u_{\varepsilon t}| dxdt \leq c$. Maybe one can prove the later conclusion directly. $u_0 \in C_0^{\infty}(\Omega)$ is the simplest condition, but it is not the most general condition. However, we mainly concern with how the degeneracy of diffusion coefficient ρ^{α} affects the boundary value condition.*

3 The stability when $\alpha < p-1$

Proof of Theorem 1.6. For a small positive constant $\lambda > 0$, let

$$\Omega_\lambda = \{x \in \Omega : \rho(x) = \mathrm{dist}(x, \partial\Omega) > \lambda\},$$

and let

$$\phi(x) = \begin{cases} 1, & \text{if } x \in \Omega_{2\lambda}, \\ \frac{1}{\lambda}(\rho(x) - \lambda), & x \in \Omega_\lambda \setminus \Omega_{2\lambda} \\ 0, & \text{if } x \in \Omega \setminus \Omega_\lambda. \end{cases} \tag{37}$$

For any given positive integer n, let $g_n(s)$ be an odd function, and

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 \mathrm{e}^{1-n^2 s^2}, & 0 \leq s \leqslant \frac{1}{n}. \end{cases} \tag{38}$$

Clearly,

$$\lim_{n \to 0} g_n(s) = \mathrm{sgn}(s), s \in (-\infty, +\infty), \tag{39}$$

and

$$0 \leq g_n'(s) \leq \frac{c}{s},\ 0 < s < \frac{1}{n}, \tag{40}$$

where c is independent of n.

By a process of limit, we can choose $g_n(\phi(u-v))$ as the test function, then

$$\begin{aligned} &\int_\Omega g_n(\phi(u-v)) \frac{\partial(u-v)}{\partial t} dx + \int_\Omega \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \phi \nabla(u-v) g_n' dx \\ &+ \int_\Omega \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla\phi (u-v) g_n' dx \\ &+ \sum_{i=1}^N \int_\Omega (b_i(u) - b_i(v)) \cdot (u-v)_{x_i} g_n' \phi dx + \sum_{i=1}^N \int_\Omega (b_i(u) - b_i(v)) \cdot \phi_{x_i} (u-v) g_n' dx = 0. \end{aligned} \tag{41}$$

Thus

$$\lim_{n\to\infty} \lim_{\lambda \to 0} \int_\Omega g_n(\phi(u-v)) \frac{\partial(u-v)}{\partial t} dx = \frac{d}{dt} \|u-v\|_1, \tag{42}$$

$$\int_\Omega \rho^\alpha (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla(u-v) g_n' \phi(x) dx \geq 0. \tag{43}$$

By L'Hospital rule,

$$\begin{aligned}\lim_{\lambda\to 0}\frac{\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} g_n'(\phi(u-v))(u-v)dx}{\lambda} &= \lim_{\lambda\to 0}\frac{\int_\lambda^{2\lambda}\int_{\rho=\zeta} g_n'(\phi(u-v))(u-v)d\Sigma d\zeta}{\lambda}\\ &= \lim_{\lambda\to 0}\int_{\rho=2\lambda} g_n'(u-v)(u-v)d\Sigma = \int_{\partial\Omega} g_n'(2(u-v))(u-v)d\Sigma\\ &= \int_{\Sigma_p'} g_n'(u-v)(u-v)d\Sigma.\end{aligned} \tag{44}$$

Since we assume that $\rho|\nabla u|^p < \infty, \rho|\nabla v|^p < \infty$, we have

$$\begin{aligned}&\lim_{\lambda\to 0}|\int_\Omega \rho^\alpha(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi(u-v)g_n'(\phi(u-v))dx|\\ =&\lim_{\lambda\to 0}|\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\cdot\nabla\phi|u-v|g_n'(\phi(u-v))dx|\\ \le& c\lim_{\lambda\to 0}\frac{\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} g_n'(\phi(u-v))(u-v)dx}{\lambda} = \int_{\Sigma_p'} g_n'(u-v)(u-v)d\Sigma.\end{aligned} \tag{45}$$

While,

$$|\int_\Omega (b_i(u)-b_i(v))g_n{}'(\phi(u-v))(u-v)\phi_{x_i}(x)dx| \le \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} |b_i(u)-b_i(v)||u-v|g_n{}'(\phi(u-v))\frac{c}{\lambda}dx. \tag{46}$$

By $|b_i(u)-b_i(v)| \le c|u-v|$, and by (40), according to the definition of the trace, we have

$$\begin{aligned}&\lim_{\lambda\to 0}|\int_\Omega (b_i(u)-b_i(v))g_n{}'(\phi(u-v))(u-v)\phi_{x_i}(x)dx| \le \lim_{\lambda\to 0}\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} |u-v|^2 g_n{}'(\phi(u-v))\frac{c}{\lambda}dx\\ =&c\int_{\partial\Omega}|u-v|^2 g_n{}'((u-v))d\Sigma \le c\int_{\partial\Omega}|u-v|d\Sigma = c\int_{\Sigma_p'}|u-v|d\Sigma.\end{aligned} \tag{47}$$

Moreover,

$$\begin{aligned}&\lim_{\lambda\to 0}|\int_\Omega (b_i(u)-b_i(v))g_n{}'(\phi(u-v))(u-v)_{x_i}\phi(x)dx|\\ =&|\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}[b_i(u)-b_i(v)]g_n{}'(u-v)(u-v)_{x_i}dx|\\ \leqslant& c\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}|\frac{b_i(u)-b_i(v)}{u-v}||(u-v)_{x_i}|dx\\ =&c\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}||\rho^{\frac{\alpha}{p}}(u-v)_{x_i}|dx\\ \leqslant& c[\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}(|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}|)^{\frac{p}{p-1}}dx]^{\frac{p-1}{p}}\cdot[\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}|\rho^\alpha\nabla(u-v)|^p dx]^{\frac{1}{p}}.\end{aligned} \tag{48}$$

Since $\alpha < p-1$,

$$\int_{\{x\in\Omega:|u-v|<\frac{1}{n}\}}(|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}|)^{\frac{p}{p-1}}dx \leqslant c\int_\Omega \rho^{-\frac{\alpha}{p-1}}dx \leqslant c. \tag{49}$$

In (47), let $n \to \infty$. If $\{x \in \Omega : |u-v| = 0\}$ is a set with 0 measure, then

$$\lim_{n\to\infty} \int\limits_{\{x\in\Omega:|u-v|<\frac{1}{n}\}} \rho^{-\frac{\alpha}{p-1}} dx = \int\limits_{\{x\in\Omega:|u-v|=0\}} \rho^{-\frac{\alpha}{p-1}} dx = 0. \tag{50}$$

If the set $\{x \in \Omega : |u-v| = 0\}$ has a positive measure, then,

$$\lim_{n\to\infty} \int\limits_{\{x\in\Omega:|u-v|<\frac{1}{n}\}} \rho^{\alpha}|\nabla(u-v)|^p dx = \int\limits_{\{x\in\Omega:|u-v|=0\}} \rho^{\alpha}|\nabla(u-v)|^p dx = 0. \tag{51}$$

Therefore, in both cases, we have

$$\lim_{n\to\infty}\lim_{\lambda\to 0}\int\limits_{\Omega}(b_i(u)-b_i(v))g_n{}'(\phi(u-v))(u-v)_{x_i}\phi(x)dx = 0. \tag{52}$$

Now, after letting $\lambda \to 0$, let $n \to \infty$ in (41). Then

$$\frac{d}{dt}\|u-v\|_1 \leqslant c\int\limits_{\Sigma_p'}|u-v|d\Sigma + \lim\sup_{n\to\infty}\int\limits_{\Sigma_p'} g_n(u-v)|u-v|d\Sigma.$$

It implies that

$$\int\limits_{\Omega}|u(x,t)-v(x,t)|\,\mathrm{d}x \leqslant \int\limits_{\Omega}|u_0-v_0|\,\mathrm{d}x + c\int\limits_{\Sigma_p'}|u-v|d\Sigma + \lim\sup_{n\to\infty}\int\limits_{\Sigma_p'} g_n(u-v)|u-v|d\Sigma, \quad \forall t \in [0,T).$$

Theorem 1.6 is proved. □

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