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A fixed point approach to the Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation

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Abstract: In this paper, we have presented and studied two types of the Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation. We prove that the fractional order delay integral equation is Mittag-Leffler-Hyers-Ulam stable on a compact interval with respect to the Chebyshev and Bielecki norms by two notions.

Keywords: Fractional order delay integral equation, Mittag-Leffler-Hyers-Ulam stability, Chebyshev norm, Bielecki norm

MSC: 26A33, 34D10, 45N05

1 Introduction

Fractional differential and integral equations can serve as excellent tools for description of mathematical modelling of systems and processes in the fields of economics, physics, chemistry, aerodynamics, and polymerrheology. It also serves as an excellent tool for description of hereditary properties of various materials and processes. For more details on fractional calculus theory, one can see the monographs of Kilbas et al. [1], Miller and Ross [2] and Podlubny [3]. The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The stability problem posed by Ulam was the following: Under what conditions there exists an additive mapping near an approximately additive mapping? (for more details see [4]). The first answer to the question of Ulam was given by Hyers [5] in 1941 in the case of Banach spaces: Let X_1, X_2 be two Banach spaces and $\varepsilon > 0$. Then for every mapping $f : X_1 \rightarrow X_2$ satisfying $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in X_1$, there exists a unique additive mapping $g : X_1 \rightarrow X_2$ with the property $\|f(x) - g(x)\| \leq \varepsilon, \forall x \in X_1$.

This type of stability is called Hyers-Ulam stability. In 1978, Th. M. Rassias [6] provided a remarkable generalization of the Hyers-Ulam stability by considering variables on the right-hand side of the inequalities. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the initial equation (see [7–10]). Recently some authors ([11–18]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation. For more results on Ulam type stability of fractional differential equations see [19–23].

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In this paper we present both Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability for the following fractional Volterra type integral equations with delay of the form

$$y(x) = I_{c+}^q f(x, x, y(x), y(\alpha(x))) = \frac{1}{\Gamma(q)} \int_c^x (x-\tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau, \quad (1)$$

where $q \in (0, 1)$, I_{c+}^q is the fractional integral of the order q , $\Gamma(\cdot)$ is the Gamma function, a, b and c are fixed real numbers such that $-\infty < a \leq x \leq b < +\infty$, and $c \in (a, b)$. Also $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $\alpha : [a, b] \rightarrow [a, b]$ is a continuous delay function which fulfils $\alpha(x) \leq x$, for all $x \in [a, b]$.

2 Preliminaries

In this section, we introduce notations, definitions and preliminaries which are used throughout this paper.

Definition 2.1. Given an interval $[a, b]$ of \mathbb{R} , the fractional order integral of a function $h \in L^1([a, b], \mathbb{R})$ of order $\gamma \in \mathbb{R}_+$ is defined by

$$I_{a+}^\gamma h(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} h(s) ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

In the sequel, we will use a *Banach's* fixed point theorem in a framework of a generalized complete metric space. For a nonempty set X , we introduce the definition of the generalized metric on X .

Definition 2.2. For a function h given on the interval $[a, b]$, the α th Riemann-Liouville fractional order derivative of h , is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{(n-\alpha-1)} h(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

Definition 2.3. For a function h given on the interval $[a, b]$, the Caputo fractional order derivative of h , is defined by

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Definition 2.4. A function $d : X \times X \rightarrow [0, +\infty]$ is called a generalized metric on X if and only if it satisfies the following three properties:

- 1) $d(x, y) = 0$ if and only if $x = y$;
- 2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- 3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The above concept differs from the usual concept of a complete metric space by the fact that not every two points in X have necessarily a finite distance. One might call such a space a generalized complete metric space.

We now introduce one of the fundamental results of the *Banach's* fixed point theorem in a generalized complete metric space.

Theorem 2.5. Let (X, d) be a generalized complete metric space. Assume that $\Lambda : X \longrightarrow X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a non negative integer k such that $d(\Lambda^{k+1}x, \Lambda^k x) < \infty$ for some $x \in X$, then, the following properties are true:

- (a) The sequence $\Lambda^n x$ converges to a fixed point x^* of Λ ;
- (b) x^* is the unique fixed point of Λ in $X^* = \{y \in X | d(\Lambda^k x, y) < \infty\}$;
- (c) If $y \in X^*$, then $d(y, x^*) \leq \frac{1}{1-L} d(\Lambda y, y)$.

Theorem 2.6 ([24, Theorem 1]). Suppose that \tilde{a} is a nonnegative function locally integrable on $[0, \infty)$ and $\tilde{g}(t)$ is a nonnegative, nondecreasing continuous function defined on $\tilde{g}(t) \leq M, t \in [0, \infty)$, and suppose $u(t)$ is nonnegative and locally integrable on $[0, \infty)$ with

$$\tilde{a}(t) + \tilde{g}(t) \int_0^t (t-s)^{q-1} u(s) ds, \quad t \in [0, \infty).$$

Then

$$u(t) \leq \tilde{a}(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(\tilde{g}(t)\gamma(q))^n}{\Gamma(nq)} (t-s)^{nq-1} \tilde{a}(s) \right] ds, \quad t \in [0, \infty).$$

Remark 2.7 ([24]). Under the hypothesis of Theorem 2.6, let $\tilde{a}(t)$ be a nondecreasing function on $[0, \infty)$. Then we have $u(t) \leq \tilde{a}(t) E_q[\tilde{g}(t)\Gamma(q)t^q]$, where E_q is the Mittag-Leffler function defined by $E_q[z] = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kq+1)}$, $z \in \mathbb{C}$.

Remark 2.8 ([16]). Let $y \in C(I, \mathbb{R})$ be a solution of the following inequality

$$|{}^C D_t^\alpha y(t) - f(t, y(t), y(g(t)))| \leq \varepsilon.$$

Then y is a solution of the following integral inequality:

$$|y(t) - y(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), y(g(s))) ds| \leq \varepsilon.$$

3 Mittag-Leffler-Hyers-Ulam stability of the first type

Definition 3.1. Equation (1) is Mittag-Leffler-Ulam-Hyers stable of first type, with respect to E_q , if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution y of the inequality

$$|y(x) - \frac{1}{\Gamma(q)} \int_c^x (x-\tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau| \leq \varepsilon E_q(\tau^q), \quad (2)$$

there exists a unique solution y_0 of equation (1) satisfying the following inequality:

$$|y(x) - y_0(x)| \leq c\varepsilon E_q(x^q).$$

Theorem 3.2. Suppose that $\alpha : [a, b] \longrightarrow [a, b]$ is a continuous function such that $\alpha(x) \leq x$, for all $x \in [a, b]$ and $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which satisfies the following Lipschitz condition

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \leq L|y_1 - y_2|, \quad (3)$$

for any $x, \tau \in [a, b]$ and $y_1, y_2 \in \mathbb{R}$ and equation (2). Then, the equation (1) is Mittag-Leffler-Hyers-Ulam stable of the first type.

Proof. Let us consider the space of continuous functions

$$X = \{g : [a, b] \longrightarrow \mathbb{R} \mid g \text{ is continuous}\}.$$

Similar to the well-known Theorem 3.1 of [25], endowed with the generalized metric defined by

$$d(g, h) = \inf\{K \in [0, \infty] \mid |g(x) - h(x)| \leq K\varepsilon E_q(x^q) \forall x \in [a, b]\}, \quad (4)$$

it is known that (X, d) is a complete generalized metric space. Define an operator $\Lambda : X \longrightarrow X$ by

$$(\Lambda g)(x) = \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} g(x, \tau, g(\tau), g(\alpha(\tau))) d\tau, \quad (5)$$

for all $g \in X$ and $x \in [a, b]$. Since g is a continuous function, it follows that Λg is also continuous and this ensures that Λ is a well-defined operator. For any $g, h \in X$, let $K_{gh} \in [0, \infty]$ such that

$$|g(x) - h(x)| \leq K_{gh}\varepsilon E_q(x^q) \quad (6)$$

for any $x \in [a, b]$. From the definition of Λ , (3) and (6) we have

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x - \tau)^{q-1} (f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))) d\tau \right| \\ &\leq \frac{L}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} |g(\tau) - h(\tau)| d\tau \leq \frac{LK_{gh}\varepsilon}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} E_q(\tau^q) d\tau \\ &\leq \frac{LK_{gh}\varepsilon}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} \sum_{k=0}^{\infty} \frac{\tau^{kq}}{\Gamma(kq + 1)} d\tau \\ &= \frac{LK_{gh}\varepsilon}{\Gamma(q)} \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq + 1)} \int_0^x (x - \tau)^{q-1} \tau^{kq} d\tau \\ &= \frac{LK_{gh}\varepsilon}{\Gamma(q)} \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq + 1)} \int_0^x (x - xt)^{q-1} (xt)^{kq} x dt \\ &= \frac{LK_{gh}\varepsilon}{\Gamma(q)} \sum_{k=0}^{\infty} \frac{x^{(k+1)q}}{\Gamma(kq + 1)} \int_0^x (1 - t)^{q-1} t^{kq} dt \\ &= \frac{LK_{gh}\varepsilon}{\Gamma(q)} \sum_{k=0}^{\infty} \frac{x^{(k+1)q}}{\Gamma(kq + 1)} \frac{\Gamma(q)\Gamma(kq + 1)}{\Gamma(q + (kq + 1))} \\ &= LK_{gh}\varepsilon \sum_{k=0}^{\infty} \frac{x^{(k+1)q}}{\Gamma((k+1)q + 1)} \leq LK_{gh}\varepsilon \sum_{n=0}^{\infty} \frac{x^{nq}}{\Gamma(nq + 1)} = LK_{gh}\varepsilon E_q(x^q) \end{aligned}$$

for all $x \in [a, b]$; that is, $d(\Lambda g, \Lambda h) \leq LK_{gh}\varepsilon E_q(x^q)$. Hence, we can conclude that $d(\Lambda g, \Lambda h) \leq Ld(g, h)$ for any $g, h \in X$, and since $0 < L < 1$, the strictly continuous property is verified.

Let us take $g_0 \in X$. From the continuous property of g_0 and Λg_0 , it follows that there exists a constant $0 < K_1 < \infty$ such that

$$|(\Lambda g_0)(x) - g_0(x)| = \left| \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, g_0(\tau), g_0(\alpha(\tau))) d\tau - g_0(x) \right| \leq K_1 E_q(x^q),$$

for all $x \in [a, b]$, since f and g_0 are bounded on $[a, b]$ and $\min_{x \in [a, b]} E_q(x^q) > 0$. Thus, (4) implies that $d(\Lambda g_0, g_0) < \infty$.

Therefore, according to Theorem 2.5 (a), there exists a continuous function $y_0 : [a, b] \rightarrow \mathbb{R}$ such that $\Lambda^n g_0 \rightarrow y_0$ in (X, d) as $n \rightarrow \infty$ and $\Lambda y_0 = y_0$; that is, y_0 satisfies the equation (1) for every $x \in [a, b]$.

We will now prove that $\{g \in X \mid d(g_0, g) < \infty\} = X$. For any $g \in X$, since g and g_0 are bounded on $[a, b]$ and $\min_{x \in [a, b]} E_q(x^q) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(x) - g(x)| \leq C_g E_q(x^q),$$

for any $x \in [a, b]$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$; that is,

$$\{g \in X \mid d(g_0, g) < \infty\} = X.$$

Hence, in view of Theorem 2.5 (b), we conclude that y_0 is the unique continuous function which satisfies the equation (1). Now we have $d(y, \Lambda y) \leq \varepsilon E_q(x^q)$. Finally, Theorem 2.5 (c) together with the above inequality imply that

$$d(y, y_0) \leq \frac{1}{1-L} d(\Lambda y, y) \leq \frac{1}{1-L} \varepsilon E_q(x^q).$$

This means that the equation (1) is Mittag-Leffler-Hyers-Ulam stable. \square

Example 3.3. Consider the following fractional order system

$${}^C D_t^{\frac{1}{2}} x(t) = \frac{1}{5} \frac{x^2(t-1)}{1+x^2(t-1)} + \frac{1}{5} \sin(2x(t)), t \in [0, 1]$$

and set $x(0) = 0$. The following inequality holds:

$$|{}^C D_t^{\frac{1}{2}} y(t) - f(t, y(t), y(t-1))| \leq \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

By Remark 15, $x(0) = 0$, $L = \frac{2}{5}$ and above inequality, all assumptions in Theorem 3.2 are satisfied. So our fractional integral is Mittag-Leffler-Hyers-Ulam stable of the first type and

$$|y(t) - x(t)| \leq C \varepsilon E_{\frac{1}{2}}(t^{\frac{1}{2}}).$$

Next, we use the Chebyshev norm $\|\cdot\|_c$ to derive the above similar result for the equation (1).

Theorem 3.4. Suppose that $\alpha : [a, b] \rightarrow [a, b]$ is a continuous function such that $\alpha(x) \leq x$, for all $x \in [a, b]$ and $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which additionally satisfies the following Lipschitz condition

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \leq L|y_1 - y_2|$$

for any $x, \tau \in [a, b]$ and $y_1, y_2 \in \mathbb{R}$ and equation (2). Also suppose that $0 < 2LE_q(b) < 1$. Then, the initial integral equation (1) is Mittag-Leffler-Hyers-Ulam stable of the first type via the Chebyshev norm.

Proof. Just like the discussion in Theorem 3.2, we only prove that Λ defined in (5) is a contraction mapping on X with respect to the Chebyshev norm:

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x-\tau)^{q-1} (f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))) d\tau \right| \\ &\leq \frac{L}{\Gamma(q)} \int_0^x (x-\tau)^{q-1} \left(\max_{x \in [a, b]} |g(\tau) - h(\tau)| + \max_{x \in [a, b]} |g(\alpha(\tau)) - h(\alpha(\tau))| \right) d\tau \\ &\leq \frac{2L}{\Gamma(q)} \|g - h\|_c \int_0^x (x-\tau)^{q-1} d\tau \leq \frac{2Lb^q}{\Gamma(q+1)} \|g - h\|_c \leq 2L \|g - h\|_c E_q(b) \end{aligned}$$

for all $x \in [a, b]$; that is, $d(\Lambda g, \Lambda h) \leq 2L\|g - h\|_c E_q(b)$. Hence, we can conclude that $d(\Lambda g, \Lambda h) \leq 2LE_q(b)d(g, h)$ for any $g, h \in X$. By letting $0 < 2LE_q(b) < 1$, the strictly continuous property is verified. Now by proceeding a proof similar to the proof of Theorem 3.2, we have

$$d(y, y_0) \leq \frac{1}{1 - 2LE_q(b)} d(\Lambda y, y) \leq \frac{1}{1 - 2LE_q(b)} \varepsilon E_q(x^q) \leq C \varepsilon E_q(x^q),$$

which means that the equation (1) is Mittag-Leffler-Hyers-Ulam stable of the first type via the Chebyshev norm. \square

In the following Theorem we have used the Bielecki norm

$$\|x\|_B := \max_{t \in J} |x(t)| e^{-\theta t}, \quad \theta > 0, \quad J \subset \mathbb{R}_+$$

to derive the similar Theorem 3.2 for the fundamental equation (1) via the Bielecki norm.

Theorem 3.5. Suppose that $\alpha : [a, b] \rightarrow [a, b]$ is a continuous function such that $\alpha(x) \leq x$, for all $x \in [a, b]$ and $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which additionally satisfies the Lipschitz condition

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \leq L|y_1 - y_2|$$

for any $x, \tau \in [a, b]$ and $y_1, y_2 \in \mathbb{R}$ and equation (2). Also suppose that $0 < \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}} < 1$. Then, equation (1) is Mittag-Leffler-Hyers-Ulam stable of the first type via the Bielecki norm.

Proof. Just like the discussion in Theorem 3.4, we only prove that Λ defined in (5) is a contraction mapping on X with respect to the Bielecki norm:

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x - \tau)^{q-1} (f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))) d\tau \right| \\ &\leq \frac{L}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} e^{\theta \tau} \left(\max_{x \in [a, b]} |g(\tau) - h(\tau)| e^{-\theta \tau} + \max_{x \in [a, b]} |g(\alpha(\tau)) - h(\alpha(\tau))| e^{-\theta \tau} \right) d\tau \\ &\leq \frac{L}{\Gamma(q)} \|g - h\|_B \int_0^x (x - \tau)^{q-1} e^{\theta \tau} d\tau \leq \frac{2L}{\Gamma(q)} \|g - h\|_B \left[\left(\int_0^x (x - \tau)^{2(q-1)} d\tau \right)^{\frac{1}{2}} \left(\int_0^x e^{2\theta \tau} d\tau \right)^{\frac{1}{2}} \right] \\ &\leq \frac{2L}{\Gamma(q)} \|g - h\|_B \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}} \end{aligned}$$

for all $x \in [a, b]$; that is, $d(\Lambda g, \Lambda h) \leq \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}} \|g - h\|_B$. Hence, we can conclude that $d(\Lambda g, \Lambda h) \leq \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}} d(g, h)$ for any $g, h \in X$. By letting $0 < \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}} < 1$, the strictly continuous property is verified. Now by a similar process with Theorem 3.4, we have

$$d(y, y_0) \leq \frac{1}{1 - \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}}} d(\Lambda y, y) \leq \frac{1}{1 - \frac{2L}{\Gamma(q)} \frac{b^q e^{\theta b}}{\sqrt{2\theta(2q-1)}}} \varepsilon E_q(x^q) \leq C \varepsilon E_q(x^q),$$

which means that equation (1) is Mittag-Leffler-Hyers-Ulam stable of the first type via the Bielecki norm. \square

4 Mittag-Leffler-Hyers-Ulam-Rassias stability of the first type

Definition 4.1. Equation (1) is Mittag-Leffler-Hyers-Ulam-Rassias stable of the first type, with respect to E_q , if there exists a real number $C > 0$ such that for each $\varepsilon > 0$ and for each solution y of the following inequality

$$|y(x) - \frac{1}{\Gamma(q)} \int_c^x (x-\tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau| \leq \varphi(x) \varepsilon E_q(\tau^q), \quad (7)$$

there exists a unique solution y_0 of equation (1) satisfying

$$|y(x) - y_0(x)| \leq C \varphi(x) \varepsilon E_q(x^q),$$

where $\varphi : X \rightarrow [0, \infty)$ is a continuous function.

Theorem 4.2. Set $M = \frac{1}{\Gamma(q)} (\frac{1-p}{q-p})^{1-p} (b-a)^{q-p}$ with $0 < p < q$. Let K and L be positive constants with $0 < KLM < 1$. Assume that $\alpha : [a, b] \rightarrow [a, b]$ is a continuous function such that $\alpha(x) \leq x$ for all $x \in [a, b]$ and $f : [a, b] \times [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which additionally satisfies the Lipschitz condition:

$$|f(x, \tau, y_1(\tau), y_1(\alpha(\tau))) - f(x, \tau, y_2(\tau), y_2(\alpha(\tau)))| \leq L|y_1 - y_2| \quad (8)$$

for any $x, \tau \in [a, b]$ and $y_1, y_2 \in \mathbb{R}$. If a continuous function $y : [a, b] \rightarrow \mathbb{R}$ satisfies (7) for all $x \in [a, b]$, where $\varphi : [a, b] \rightarrow (0, \infty)$ is a $L^{\frac{1}{p}}$ -integrable function satisfying

$$\left(\int_0^x (\varphi(\tau))^{\frac{1}{p}} d\tau \right)^p \leq K\varphi(x), \quad (9)$$

then there exists a unique continuous function $y_0 : [a, b] \rightarrow \mathbb{R}$ such that y_0 satisfies equation (1) and

$$|y(x) - y_0(x)| \leq C \varphi(x) \varepsilon E_q(x^q), \quad (10)$$

for all $x \in [a, b]$.

Proof. Let us consider the space of continuous functions

$$X = \{g : [a, b] \rightarrow \mathbb{R} \mid g \text{ is continuous}\}. \quad (11)$$

Similar to Theorem 3.1 of [25], endowed with the generalized metric defined by

$$d(g, h) = \inf\{K \in [0, \infty] \mid |g(x) - h(x)| \leq K\varphi(x) \forall x \in [a, b]\},$$

it is known that (X, d) is a complete generalized metric space. Define an operator $\Lambda : X \rightarrow X$ by the formula

$$(\Lambda g)(x) = \frac{1}{\Gamma(q)} \int_0^x (x-\tau)^{q-1} g(x, \tau, g(\tau), g(\alpha(\tau))) d\tau,$$

for all $g \in X$ and $x \in [a, b]$. Since g is a continuous function, it follows that Λg is also continuous and this ensures that Λ is a well-defined operator. For any $g, h \in X$, let $K_{gh} \in [0, \infty]$ such that inequality

$$|g(x) - h(x)| \leq K_{gh} \varphi(x) \quad (12)$$

holds for any $x \in [a, b]$. From the definition of Λ and inequalities (8), (9) and (12), we have

$$\begin{aligned} |(\Lambda g)(x) - (\Lambda h)(x)| &= \frac{1}{\Gamma(q)} \left| \int_0^x (x-\tau)^{q-1} (f(x, \tau, g(\tau), g(\alpha(\tau))) - f(x, \tau, h(\tau), h(\alpha(\tau)))) d\tau \right| \\ &\leq \frac{L}{\Gamma(q)} \int_0^x (x-\tau)^{q-1} |g(\tau) - h(\tau)| d\tau \leq \frac{L}{\Gamma(q)} K_{gh} \int_0^x (x-\tau)^{q-1} \varphi(\tau) d\tau \\ &\leq \frac{L}{\Gamma(q)} K_{gh} \left(\int_0^x (x-\tau)^{\frac{q-1}{1-p}} d\tau \right)^{1-p} \left(\int_0^x (\varphi(\tau))^{\frac{1}{p}} d\tau \right)^p \leq L K_{gh} K M \varphi(x), \end{aligned}$$

for all $x \in [a, b]$; that is, $d(\Lambda g, \Lambda h) \leq KLMK_{gh}\varphi(x)$. Hence, we conclude that $d(\Lambda g, \Lambda h) \leq KLMd(g, h)$ for any $g, h \in X$, and since $0 < KLM < 1$, the strictly continuous property is verified.

Let us take $g_0 \in X$. There exists a constant $0 < K_1 < \infty$ such that

$$|(\Lambda g_0)(x) - g_0(x)| = \left| \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, g_0(\tau), g_0(\alpha(\tau))) d\tau - g_0(x) \right| \leq K_1 \varphi(x),$$

for all $x \in [a, b]$, since f and g_0 are bounded on $[a, b]$ and $\min_{x \in [a, b]} \varphi(x) > 0$. Thus, (11) implies that $d(\Lambda g_0, g_0) < \infty$. Therefore, according to Theorem 2.5 (a), there exists a continuous function $y_0 : [a, b] \rightarrow \mathbb{R}$ such that $\Lambda^n g_0 \rightarrow y_0$ in (X, d) as $n \rightarrow \infty$ and $\Lambda y_0 = y_0$; that is, y_0 satisfies equation (1) for every $x \in [a, b]$.

We will now verify that $\{g \in X \mid d(g_0, g) < \infty\} = X$. For any $g \in X$, since g and g_0 are bounded on $[a, b]$ and $\min_{x \in [a, b]} E_q(x^q) > 0$, there exists a constant $0 < C_g < \infty$ such that

$$|g_0(x) - g(x)| \leq C_g \varphi(x),$$

for any $x \in [a, b]$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$; that is,

$$\{g \in X \mid d(g_0, g) < \infty\} = X.$$

So, in view of Theorem 2.5 (b), we conclude that y_0 is the unique continuous function such that it satisfies equation (1).

On the other hand, from (7) it follows that

$$d(y, \Lambda y) \leq \varepsilon E_q(x^q) \varphi(x). \quad (13)$$

Finally, Theorem 2.5 (c) together with (13) imply that

$$d(y, y_0) \leq \frac{1}{1 - KML} d(\Lambda y, y) \leq \frac{1}{1 - KML} \varphi(x) \varepsilon E_q(x^q),$$

which means that the inequality (10) holds for all $x \in [a, b]$. \square

5 Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second type

Let us consider equation (1) in the case $I = [0, b]$.

Definition 5.1. Equation (1) is Mittag-Leffler-Hyers-Ulam stable of the second type, with respect to E_q , if for every $\varepsilon > 0$ and solution $y \in C([a, b], \mathbb{R})$ of the following equation

$$|y(x) - \frac{1}{\Gamma(q)} \int_c^x (x - \tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau| \leq \varepsilon, \quad (14)$$

there exists a solution $x \in (C([a, b], \mathbb{R}))$ of equation (1) with

$$|y(t) - x(t)| \leq \varepsilon E_q(cx^q),$$

for all $x \in [a, b]$ and $c \in \mathbb{R}$.

Theorem 5.2. Let B be a Banach algebra. We suppose that: (i) $f \in C([0, b] \times B, B)$; (ii) f satisfies the following Lipschitz condition

$$|f(t, w_1) - f(t, w_2)| \leq L|w_1 - w_2|,$$

for all $t \in [0, b]$, $w_1, w_2 \in B$ and equation (14). Then equation (1) is Mittag-Leffler-Hyers-Ulam stable of the second type.

Proof. Let $y \in C(I, B)$ satisfy the inequality (14). Let us denote by $x \in C([0, b], B)$ the unique solution of the (1). We have

$$\begin{aligned} |y(t) - x(t)| &\leq |y(t) - \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau| \\ &+ |\frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau - \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, x(\tau), x(\alpha(\tau))) d\tau| \\ &+ |\frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} f(x, \tau, x(\tau), x(\alpha(\tau))) d\tau - x(t)| \\ &\leq \varepsilon + |\frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} (f(x, \tau, y(\tau), y(\alpha(\tau))) - f(x, \tau, x(\tau), x(\alpha(\tau)))) d\tau| \\ &\leq \varepsilon + \frac{1}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} |f(x, \tau, y(\tau), y(\alpha(\tau))) - f(x, \tau, x(\tau), x(\alpha(\tau)))| d\tau \\ &\leq \varepsilon + \frac{L}{\Gamma(q)} \int_0^x (x - \tau)^{q-1} |y - x| d\tau. \end{aligned}$$

Now by Remark 2.7, we have

$$u(t) \leq \varepsilon E_q(Lx^q).$$

Thus, the conclusion of our theorem holds. \square

Definition 5.3. Equation (1) is Mittag-Leffler-Hyers-Ulam-Rassias stable of the second type, with respect to E_q , if for every $\varepsilon > 0$ and solution $y \in C([a, b], \mathbb{R})$ of the following inequality

$$|y(x) - \frac{1}{\Gamma(q)} \int_c^x (x - \tau)^{q-1} f(x, \tau, y(\tau), y(\alpha(\tau))) d\tau| \leq \varepsilon \varphi(x), \quad (15)$$

there exists a solution $x \in (C([a, b], \mathbb{R})$ of equation (1) with

$$|y(t) - x(t)| \leq \lambda E_q(cx^q),$$

for all $x \in [a, b]$ such that the function $\varphi : X \rightarrow [0, \infty)$ is a non-negative non-decreasing locally integrable function on $[0, \infty)$ and $c \in \mathbb{R}$.

Theorem 5.4. Let B be a Banach algebra. We suppose that: (i) $f \in C([0, b] \times B, B)$; (ii) f satisfies the following Lipschitz condition

$$|f(t, w_1) - f(t, w_2)| \leq L|w_1 - w_2|,$$

for all $t \in [0, b]$, $w_1, w_2 \in B$ and inequality (15). Then the initial equation (1) is Mittag-Leffler-Hyers-Ulam-Rassias stable of the second type.

Proof. By putting $\varepsilon\varphi(x)$ instead of ε in the proof of Theorem 5.2, the proof is complete. \square

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