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Research Article

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Bounds for the Z-eigenpair of general nonnegative tensors

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Abstract: In this paper, we consider the Z-eigenpair of a tensor. A lower bound and an upper bound for the Z-spectral radius of a weakly symmetric nonnegative irreducible tensor are presented. Furthermore, upper bounds of Z-spectral radius of nonnegative tensors and general tensors are given. The proposed bounds improve some existing ones. Numerical examples are reported to show the effectiveness of the proposed bounds.

Keywords: Z-eigenvalues, Z-spectral radius, Weakly symmetric

MSC: 15A15, 15A69, 65F25

1 Introduction

We start with some preliminaries. First, denote $[n] = \{1, 2, \cdots, n\}$. A real mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ consists of n^m real entries:

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R},$$

where $i_j = 1, 2, \cdots, n$ for $j \in [m]$ [1–5]. It is obvious that a vector is an order 1 tensor and a matrix is an order 2 tensor. Moreover, a tensor $\mathcal{A} = (a_{i_1 \cdots i_m})$ is called nonnegative (positive) if each entry is nonnegative (positive). A tensor $\mathcal{A}$ is said to be symmetric [6, 7] if its entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of the indices. We shall denote the set of all real mth order n-dimensional tensors by $\mathbb{R}^{[m,n]}$, and the set of all nonnegative mth order n-dimensional tensors by $\mathbb{R}_+^{[m,n]}$. For an n-dimensional vector $x = (x_1, x_2, \cdots, x_n)$, real or complex, we define the n-dimensional vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2, \ldots, i_m \in [n]} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1 \le i \le n},$$

and the n-dimensional vector:

$$x^{[m-1]} := \left(x_i^{m-1} \right)_{1 \le i \le n}.$$

The following two definitions were first introduced and studied by Qi and Lim [7, 8].

Definition 1.1 ([7, 8]). *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

We call (λ, x) an H-eigenpair if they are both real.

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Definition 1.2 ([7, 8])**.** *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1. \end{cases} \tag{2}$$

We call (λ, x) a Z-eigenpair if they are both real.

The mth degree homogeneous polynomial of n variables $f_{\mathcal{A}}(x)$ associated with an mth order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ can be represented as

$$f_{\mathcal{A}}(x) \equiv \mathcal{A}x^m := \sum_{i_1, i_2, \ldots, i_m \in [n]} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m},$$

where x^m can be regarded as an mth order n-dimensional rank-one tensor with entries $x_{i_1} \cdots x_{i_m}$ [2, 5, 9], and $\mathcal{A}x^m$ is the inner product of $\mathcal{A}$ and x^m.

Following concept about weakly symmetric of tensors was first introduced and used by Chang, Pearson, and Zhang [6] for studying the properties of Z-eigenvalue of nonnegative tensor.

Definition 1.3 ([6])**.** *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called weakly symmetric, if the associated homogenous of polynomial $f_{\mathcal{A}}(x)$ satisfy*

$$\nabla f_{\mathcal{A}}(x) = m\mathcal{A}x^{m-1}, \;\; \forall x \in \mathbb{R}^n,$$

and the right-hand side is not identical to zero.

It should be noted for $m = 2$, symmetric matrices and weakly symmetric matrices are the same. However, it is shown in [6] that a symmetric tensor is necessarily weakly symmetric for $m > 2$, but the converse is not true in general. Thus, the results of this paper derived for weakly symmetric tensors, apply also for symmetric tensors.

In [8], the notion of irreducible tensors was introduced.

Definition 1.4 ([8])**.** *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $I \subset [n]$, such that*

$$a_{i_1 i_2 \cdots i_m} = 0, \;\; \forall i_1 \in I, \;\; \forall i_2, \cdots, i_m \notin I,$$

otherwise, we say $\mathcal{A}$ is irreducible.

The Z-spectral radius of a tensor is defined as follows in [10].

Definition 1.5 ([10])**.** *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. We define the Z-spectrum of $\mathcal{A}$, denoted $\sigma(\mathcal{A})$ to be the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as*

$$\varrho(\mathcal{A}) := max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Chang, Pearson and Zhang [10] studied the Z-eigenpair problem for nonnegative tensors and presented the following Perron-Frobenius type theorem.

Lemma 1.6 ([10])**.** *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, then there exists a Z-eigenvalue $\lambda_0 \geq 0$ and a nonnegative Z-eigenvector $x_0 \neq 0$ of $\mathcal{A}$ such that $\mathcal{A}x_0^{m-1} = \lambda_0 x_0$, in particular, if $\mathcal{A}$ is irreducible, then the eigenvalue λ_0 and the eigenvector x_0 are positive. Furthermore, if $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is weakly symmetric irreducible, then the spectral radius $\varrho(\mathcal{A})$ is a positive Z-eigenvalue with a positive Z-eigenvector.*

Z-eigenvalues play a fundamental role in the symmetric best rank-one approximation which has numerous applications in engineering and higher order statistics, such as Statistical Data Analysis [2, 5, 9]. The symmetric best rank-one approximation of $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a rank-one tensor $\nu x^m = (\nu x_{i_1} x_{i_2} \cdots x_{i_m})$, where $\nu \in \mathbb{R}$,

$x \in \mathbb{R}^n$, $\|x\|_2 = 1$ and $\|x\|_2$ is the Euclidean norm of x in $\mathbb{R}^n$, such that the Frobenius norm $\|\mathcal{A} - \nu x^m\|_F$ is minimized. The Frobenius norm of the tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ has the form

$$\|\mathcal{A}\|_F := \sqrt{\sum_{i_1, i_2, \ldots, i_m \in [n]} a^2_{i_1 i_2 \cdots i_m}}.$$

According to [11], νx^m is a symmetric best rank-one approximation of $\mathcal{A}$ if and only if ν is a Z-eigenvalue of $\mathcal{A}$ with the largest absolute value, while x is a Z-eigenvector of $\mathcal{A}$ associated with the Z-eigenvalue ν. In particular, when $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is weakly symmetric irreducible, $\varrho(\mathcal{A}) x_0^m$ is a symmetric best rank-one approximation of $\mathcal{A}$, where x_0 is a Z-eigenvector of $\mathcal{A}$ associated with Z-spectral radius $\varrho(\mathcal{A})$, i.e.,

$$\min_{\nu \in \mathbb{R}, x \in \mathbb{R}^n, \|x\|_2 = 1} \|\mathcal{A} - \nu x^m\|_F = \|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \varrho(\mathcal{A})^2}. \tag{3}$$

Thus, we obtain the quotient of the residual of a symmetric best rank-one approximation of tensor $\mathcal{A}$ and the Frobenius norm of tensor $\mathcal{A}$ as follows:

$$\frac{\|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F}{\|\mathcal{A}\|_F} = \sqrt{1 - \frac{\varrho(\mathcal{A})^2}{\|\mathcal{A}\|_F^2}}. \tag{4}$$

By Equalities (3) and (4), if we give a bound of Z-spectral radius of $\mathcal{A}$, then a bound of $\min_{\nu \in \mathbb{R}, x \in \mathbb{R}^n, \|x\|_2 = 1} \|\mathcal{A} - \nu x^m\|_F$ and $\frac{\|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F}{\|\mathcal{A}\|_F}$ will be obtained. It follows from [12–16] that the bound of $\frac{\|\mathcal{A} - \varrho(\mathcal{A}) x_0^m\|_F}{\|\mathcal{A}\|_F}$ gives a convergence rate for the greedy rank-one update algorithm.

Recently, some H-spectral of matrices have been successfully extended to higher order tensors [17–19]. For the Z-eigenpair case, Chang, Pearson and Zhang [10] discussed the variation principles of Z-eigenvalues of nonnegative tensors, as a corollary of the main results, they presented a lower bound of the Z-spectral radius for weakly symmetric nonnegative irreducible tensors as follows:

$$\max\left\{ \max_{i \in [n]} a_{i i \cdots i},\ (\frac{1}{\sqrt{n}})^{m-2} \min_{i \in [n]} \sum_{i_2, \cdots, i_m \in [n]} a_{i i_2 \cdots i_m} \right\} \le \varrho(\mathcal{A}). \tag{5}$$

For a general tensor case, they also provided an upper bound for the Z-spectral radius:

$$\varrho(\mathcal{A}) \le \sqrt{n} \max_{i \in [n]} \sum_{i_2, \cdots, i_m \in [n]} |a_{i i_2 \cdots i_m}|. \tag{6}$$

Song and Qi [20] obtained the following upper bound for a general mth order n-dimensional tensor:

$$\varrho(\mathcal{A}) \le \max_{i \in [n]} \sum_{i_2, \cdots, i_m \in [n]} |a_{i i_2 \cdots i_m}|. \tag{7}$$

He and Huang [21] gave a bound for a weakly symmetric positive tensor:

$$\varrho(\mathcal{A}) \le R(\mathcal{A}) - l(\mathcal{A})(1 - \theta(\mathcal{A})), \tag{8}$$

where $r_i(\mathcal{A}) = \sum_{i_2, \cdots, i_m \in [n]} a_{i i_2 \cdots i_m}$, for all $i \in [n]$,

$$R(\mathcal{A}) = \max_{i \in [n]} r_i(\mathcal{A}), r(\mathcal{A}) = \min_{i \in [n]} r_i(\mathcal{A}),$$

$$l(\mathcal{A}) = \min_{i_1, \cdots, i_m \in [n]} a_{i_1 i_2 \cdots i_m},\ and\ \theta(\mathcal{A}) = \left(\frac{r(\mathcal{A})}{R(\mathcal{A})}\right)^{\frac{1}{m}}. \tag{9}$$

Since $\theta(\mathcal{A}) \le 1$, it is easy to see that the bound (8) is smaller than those in (6) and (7) if the tensor is weakly symmetric positive. Recently, Li, Liu and Vong [22] have given a lower bound and an upper bound for a weakly symmetric nonnegative irreducible tensor:

$$d_{m,n} \le \varrho(\mathcal{A}) \le \max_{i, j \in [n]} \{r_i(\mathcal{A}) + a_{i j \cdots j}(\delta(\mathcal{A})^{-\frac{m-1}{m}} - 1)\} \tag{10}$$

where

$$\delta(\mathcal{A}) = \frac{\min\limits_{i,j\in[n]} a_{ij\cdots j}}{r(\mathcal{A}) - \min\limits_{i,j\in[n]} a_{ij\cdots j}} \left(\gamma(\mathcal{A})^{\frac{m-1}{m}} - \gamma(\mathcal{A})^{\frac{1}{m}}\right) + \gamma(\mathcal{A}), \tag{11}$$

$$\gamma(\mathcal{A}) = \frac{R(\mathcal{A}) - \min\limits_{i,j\in[n]} a_{ij\cdots j}}{r(\mathcal{A}) - \min\limits_{i,j\in[n]} a_{ij\cdots j}}, \tag{12}$$

and

$$\begin{aligned} d_{m,n} = \max_{k\in[m]\setminus\{1\}} \min_{i_1\in[n]} \Bigg[& (\delta(\mathcal{A})^{\frac{1}{m}} - 1) \min_{i_t,t\in[m]\setminus\{1\}} a_{i_1 i_2\cdots i_{k-1} i_k i_{k+1}\cdots i_m} \\ & + \min_{i_t,t\in[m]\setminus\{1,k\}} \sum_{i_k\in[n]} a_{i_1 i_2\cdots i_{k-1} i_k i_{k+1}\cdots i_m} \Bigg]. \end{aligned} \tag{13}$$

They also proved that the upper bound (10) is smaller than that in (8). Furthermore, Li, Liu and Vong [22] obtained the following upper bound for a general mth order n-dimensional tensor:

$$\varrho(\mathcal{A}) \le \min_{k\in[m]} \max_{i_k\in[n]} \sum_{i_t=1,t\in[m]\setminus\{k\}}^{n} |a_{i_1\cdots i_k\cdots i_m}|. \tag{14}$$

In this paper, we continue this research on the Z-eigenpair and present some bounds as follows: for a weakly symmetric nonnegative irreducible tensor, we present a bound for Z-spectral radius, which improves the bound in (10). For a weakly symmetric nonnegative tensor, we give an upper bound for Z-spectral radius. Furthermore, for a nonnegative tensor and a general tensor, an upper bound for Z-spectral radius is also provided, which is tighter than the bound in (14) in some sense.

Our paper is organized as follows. In Section 2, an upper bound for the ratio of the largest and smallest values of a Z-eigenvector is given. Also, a lower bound and an upper bound for the Z-spectral radius of a weakly symmetric nonnegative irreducible tensor are presented. Moreover, an upper bound for the Z-spectral radius of a weakly symmetric nonnegative tensor is provided. An upper bound for the Z-spectral radius of a nonnegative tensor and an upper bound for the Z-spectral radius of a general tensor are obtained in section 3. Numerical examples are presented in the final section.

We first add a comment on the notation that is used. For a tensor $\mathcal{A}$, let $|\mathcal{A}|$ denote the tensor whose $(i_1,\cdots,i_m)$-th entry is defined as $|a_{i_1\cdots i_m}|$. For a set S, $|S|$ denotes the number of elements of S. The function $\lfloor x \rfloor$ indicates the integer round-down of x. Denote

$$\Delta(j;k) = \bigcup_{\substack{S\subseteq\{2,\cdots,m\},\\ |S|=k}} \{(i_2,\cdots,i_m) : i_t = j,\ for\ all\ t \in S\ and\ i_t \ne j,\ for\ all\ t \notin S\}$$

where $j \in [n]$, $k = 0, 1, \cdots, m-1$.

2 Bounds for weakly symmetric nonnegative tensors

In this section, a lower bound and an upper bound for the Z-spectral radius of a weakly symmetric nonnegative irreducible tensor are provided, which improves the bound (10). We first establish a lemma to estimate the ratio of the largest and smallest values of a Z-eigenvector.

Lemma 2.1. *Suppose that $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ is an irreducible tensor. Then for any Z-eigenpair (λ_0, x) of $\mathcal{A}$ with a positive eigenvector x, we have*

$$\frac{x_{max}}{x_{min}} \ge \tau(\mathcal{A})^{\frac{1}{m}}, \tag{15}$$

where $x_{min} = \min_{i\in[n]} x_i$, $x_{max} = \max_{i\in[n]} x_i$,

$$\tau(\mathcal{A}) = \frac{\sum\limits_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})(\alpha(\mathcal{A})^{\frac{m-k-1}{m}} - \alpha(\mathcal{A})^{\frac{k+1}{m}})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} + \alpha(\mathcal{A}),$$

$$\beta_k(\mathcal{A}) = \min_{i,j\in[n]} \{a_{ii_2\cdots i_m} : (i_2,\cdots,i_m) \in \Delta(j; m-k-1)\},$$

$$\alpha(\mathcal{A}) = \frac{R(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})}. \tag{16}$$

Proof. Since (λ_0, x) is a Z-eigenpair of $\mathcal{A}$ with x being positive, then

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda_0 x, \\ x^T x = 1. \end{cases} \tag{17}$$

For simplicity, let $x_l = x_{max}$, $x_s = x_{min}$, $r_p(\mathcal{A}) = \max_{i\in[n]} r_i(\mathcal{A}) = R(\mathcal{A})$, $r_q(\mathcal{A}) = \min_{i\in[n]} r_i(\mathcal{A}) = r(\mathcal{A})$. Consider the ith equation of (17), we obtain

$$\begin{aligned}
\lambda_0 x_i &= \sum_{i_2,\cdots,i_m\in[n]} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m} \geq a_{il\cdots l} x_l^{m-1} \\
&+ \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-2)} a_{ii_2\cdots i_m} x_l^{m-2} x_s + \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-3)} a_{ii_2\cdots i_m} x_l^{m-3} x_s^2 + \cdots \\
&+ \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-\lfloor \frac{m}{2}\rfloor)} a_{ii_2\cdots i_m} x_l^{m-\lfloor \frac{m}{2}\rfloor} x_s^{\lfloor \frac{m}{2}\rfloor-1} + \sum_{(i_2,\cdots,i_m)\in \bigcup\limits_{k=0}^{m-\lfloor \frac{m}{2}\rfloor-1} \Delta(l;k)} a_{ii_2\cdots i_m} x_s^{m-1} \\
&= a_{il\cdots l}(x_l^{m-1} - x_s^{m-1}) + \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-2)} a_{ii_2\cdots i_m}(x_l^{m-2}x_s - x_s^{m-1}) \\
&+ \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-3)} a_{ii_2\cdots i_m}(x_l^{m-3}x_s^2 - x_s^{m-1}) + \cdots \\
&+ \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-\lfloor \frac{m}{2}\rfloor)} a_{ii_2\cdots i_m}(x_l^{m-\lfloor \frac{m}{2}\rfloor} x_s^{\lfloor \frac{m}{2}\rfloor-1} - x_s^{m-1}) + r_i(\mathcal{A})x_s^{m-1} \\
&= \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-k-1)} a_{ii_2\cdots i_m}(x_l^{m-k-1}x_s^k - x_s^{m-1}) + r_i(\mathcal{A})x_s^{m-1}.
\end{aligned} \tag{18}$$

Taking $i = p$ in (18) and multiplying by x_p^{-1} on the both sides of (18) gives

$$\begin{aligned}
\lambda_0 &\geq \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \sum_{(i_2,\cdots,i_m)\in\Delta(l;m-k-1)} a_{pi_2\cdots i_m}\left(\frac{x_l^{m-k-1}x_s^k - x_s^{m-1}}{x_p}\right) + R(\mathcal{A})\frac{x_s^{m-1}}{x_p} \\
&\geq \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})\left(x_l^{m-k-2}x_s^k - \frac{x_s^{m-1}}{x_l}\right) + R(\mathcal{A})\frac{x_s^{m-1}}{x_l} \\
&= \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) x_l^{m-k-2}x_s^k + \left(R(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2}\rfloor-1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})\right)\frac{x_s^{m-1}}{x_l}.
\end{aligned} \tag{19}$$

Similarly, we have

$$\lambda_0 x_i \le \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{i i_2 \cdots i_m}(x_s^{m-k-1}x_l^k - x_l^{m-1}) + r_i(\mathcal{A})x_l^{m-1}. \quad (20)$$

Taking $i = q$. By (20) and the similar technique to (19) we have

$$\lambda_0 \le \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) x_s^{m-k-2}x_l^k + \left(r(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \right) \frac{x_l^{m-1}}{x_s}. \quad (21)$$

Combining (19) with (21) together gives

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) x_l^{m-k-2}x_s^k + \left(R(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \right) \frac{x_s^{m-1}}{x_l} \\ \le &\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) x_s^{m-k-2}x_l^k + \left(r(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \right) \frac{x_l^{m-1}}{x_s}. \end{aligned}$$

Multiplying $\frac{x_l}{x_s^{m-1}}$ on the both sides of the above inequality yields

$$\begin{aligned} &\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \left(\frac{x_l}{x_s}\right)^{m-k-1} + R(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \\ \le &\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \left(\frac{x_l}{x_s}\right)^{k+1} + \left(r(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}) \right) \left(\frac{x_l}{x_s}\right)^{m}. \quad (22) \end{aligned}$$

Note that $\left(\frac{x_l}{x_s}\right)^{m-k-1} \ge \left(\frac{x_l}{x_s}\right)^{k+1}$, for all $k = 0, 1, \cdots, \lfloor \frac{m}{2} \rfloor - 1$. Hence, by (22), we have

$$\left(\frac{x_l}{x_s}\right)^m \ge \frac{R(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})}{r(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} := \alpha(\mathcal{A}),$$

i.e. $\frac{x_l}{x_s} \ge \alpha(\mathcal{A})^{\frac{1}{m}}$, which together with (22) yields

$$\left(\frac{x_l}{x_s}\right)^m \ge \frac{\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})(\alpha(\mathcal{A})^{\frac{m-k-1}{m}} - \alpha(\mathcal{A})^{\frac{k+1}{m}})}{r(\mathcal{A}) - \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} + \alpha(\mathcal{A}) = \tau(\mathcal{A}).$$

So the desired conclusion follows. □

We next compare the bounds (15) in Lemma 2.1 with the corresponding bounds in Theorem 3.1 of [22], in which the authors presented the following bounds:

$$\frac{x_{max}}{x_{min}} \ge \delta(\mathcal{A})^{\frac{1}{m}}, \quad (23)$$

where $\delta(\mathcal{A})$ given as (11).

Lemma 2.2. *Suppose that $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ is an irreducible tensor. Then*

$$\tau(\mathcal{A}) \ge \alpha(\mathcal{A}) \ge \gamma(\mathcal{A}) \ge 1, \quad (24)$$

and

$$\tau(\mathcal{A}) \ge \delta(\mathcal{A}) \ge \gamma(\mathcal{A}) \ge 1. \quad (25)$$

Proof. It follows from Inequality (3.10) of Li, Liu and Vong [22] that $\delta(\mathcal{A}) \geq \gamma(\mathcal{A}) \geq 1$. Hence, we only prove

$$\tau(\mathcal{A}) \geq \alpha(\mathcal{A}) \geq \gamma(\mathcal{A}) \tag{26}$$

and

$$\tau(\mathcal{A}) \geq \delta(\mathcal{A}). \tag{27}$$

We have from Equality (16) that

$$\begin{aligned}\alpha(\mathcal{A}) &= \frac{R(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} \\ &\geq \frac{R(\mathcal{A}) - \beta_0(\mathcal{A})}{r(\mathcal{A}) - \beta_0(\mathcal{A})} \\ &= \frac{R(\mathcal{A}) - \min\limits_{i,j \in [n]} a_{ij\cdots j}}{r(\mathcal{A}) - \min\limits_{i,j \in [n]} a_{ij\cdots j}} \\ &= \gamma(\mathcal{A}).\end{aligned} \tag{28}$$

Note that $\alpha(\mathcal{A})^{\frac{m-k-1}{m}} \geq \alpha(\mathcal{A})^{\frac{k+1}{m}}$, for all $k = 0, 1, \cdots, \lfloor \frac{m}{2} \rfloor - 1$, and

$$r(\mathcal{A}) > \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A}),$$

hence

$$\tau(\mathcal{A}) \geq \alpha(\mathcal{A}),$$

which together with (28), yields Inequality (26).

From Equality (16), we obtain

$$\begin{aligned}\tau(\mathcal{A}) &= \frac{\sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})(\alpha(\mathcal{A})^{\frac{m-k-1}{m}} - \alpha(\mathcal{A})^{\frac{k+1}{m}})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} + \alpha(\mathcal{A}) \\ &\geq \frac{\beta_0(\mathcal{A})(\alpha(\mathcal{A})^{\frac{m-1}{m}} - \alpha(\mathcal{A})^{\frac{1}{m}})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} + \alpha(\mathcal{A}) \\ &= \frac{\min\limits_{i,j \in [n]} a_{ij\cdots j}(\alpha(\mathcal{A})^{\frac{m-1}{m}} - \alpha(\mathcal{A})^{\frac{1}{m}})}{r(\mathcal{A}) - \sum\limits_{k=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-1}{k}(n-1)^k \beta_k(\mathcal{A})} + \alpha(\mathcal{A}).\end{aligned} \tag{29}$$

Since $\alpha(\mathcal{A}) \geq \gamma(\mathcal{A}) \geq 1$ and $m \geq 2$,

$$\alpha(\mathcal{A})^{\frac{m-1}{m}} - \alpha(\mathcal{A})^{\frac{1}{m}} \geq \gamma(\mathcal{A})^{\frac{m-1}{m}} - \gamma(\mathcal{A})^{\frac{1}{m}},$$

which together with Inequality (29), yields

$$\tau(\mathcal{A}) \geq \frac{\min\limits_{i,j \in [n]} a_{ij\cdots j}}{r(\mathcal{A}) - \min\limits_{i,j \in [n]} a_{ij\cdots j}} \left(\gamma(\mathcal{A})^{\frac{m-1}{m}} - \gamma(\mathcal{A})^{\frac{1}{m}}\right) + \gamma(\mathcal{A}) = \delta(\mathcal{A}),$$

which implies Inequality (27) holds. The proof is completed. □

Remark 2.3. *If $\mathcal{A}$ is a nonnegative irreducible tensor, Inequality* (25) *implies the bounds* (15) *are always larger than the bounds* (23).

Now we establish an upper and a lower bound for Z-spectral radius of a weakly symmetric nonnegative irreducible tensor.

Theorem 2.4. *Suppose that $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$ is an irreducible weakly symmetric tensor. Then*

$$\mu(\mathcal{A}) \le \varrho(\mathcal{A}) \le \eta(\mathcal{A}), \tag{30}$$

where

$$\eta(\mathcal{A}) = \max_{i,j \in [n]} \left\{ \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(j;m-k-1)} a_{i i_2 \cdots i_m} \left(\tau(\mathcal{A})^{-\frac{m-k-1}{m}} - 1 \right) + r_i(\mathcal{A}) \right\},$$

$$\begin{aligned} \mu(\mathcal{A}) = \max_{k\in[m]\setminus\{1\}} \min_{i_1\in[n]} \Bigg[& \left(\tau(\mathcal{A})^{\frac{1}{m}} - 1 \right) \min_{i_t,t\in[m]\setminus\{1\}} a_{i_1 i_2 \cdots i_{k-1} i_k i_{k+1} \cdots i_m} \\ & + \min_{i_t,t\in[m]\setminus\{1,k\}} \sum_{i_k\in[n]} a_{i_1 i_2 \cdots i_{k-1} i_k i_{k+1} \cdots i_m} \Bigg], \end{aligned} \tag{31}$$

and $\tau(\mathcal{A})$ is given by Lemma 2.1.

Proof. It follows from Lemma 1.6 that there exists a positive Z-eigenvector x corresponding to $\varrho(\mathcal{A})$. Taking the similar technique of (18) we obtain

$$\varrho(\mathcal{A}) x_i \le \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{i i_2 \cdots i_m} (x_s^{m-k-1} x_l^k - x_l^{m-1}) + r_i(\mathcal{A}) x_l^{m-1}, \tag{32}$$

Since $x^T x = 1$, we have $x_i^{m-1} \le x_i$, which together with (32) yields

$$\varrho(\mathcal{A}) x_i^{m-1} \le \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{i i_2 \cdots i_m} (x_s^{m-k-1} x_l^k - x_l^{m-1}) + r_i(\mathcal{A}) x_l^{m-1}.$$

Taking $i = l$ and multiplying x_l^{1-m} on the both sides of the above inequality gives

$$\begin{aligned} \varrho(\mathcal{A}) &\le \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{l i_2 \cdots i_m} \left(\frac{x_s^{m-k-1} x_l^k - x_l^{m-1}}{x_l^{m-1}} \right) + r_l(\mathcal{A}) \\ &= \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{l i_2 \cdots i_m} \left(\left(\frac{x_s}{x_l} \right)^{m-k-1} - 1 \right) + r_l(\mathcal{A}) \\ &\le \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(s;m-k-1)} a_{l i_2 \cdots i_m} \left(\tau(\mathcal{A})^{-\frac{m-k-1}{m}} - 1 \right) + r_l(\mathcal{A}) \\ &\le \max_{i,j\in[n]} \left\{ \sum_{k=0}^{m-2} \sum_{(i_2,\cdots,i_m)\in\Delta(j;m-k-1)} a_{i i_2 \cdots i_m} \left(\tau(\mathcal{A})^{-\frac{m-k-1}{m}} - 1 \right) + r_i(\mathcal{A}) \right\}. \end{aligned}$$

This proves the second Inequality (30). On the other hand, it is known from Theorem 3.3 of Li, Liu and Vong [22] that for a weakly symmetric nonnegative irreducible tensor $\mathcal{A}$, we have

$$\varrho(\mathcal{A}) \ge \left(\frac{x_l}{x_s} - 1 \right) \min_{i_t,t\in[m]\setminus\{1,k\}} a_{s i_2 \cdots i_{k-1} l i_{k+1} \cdots i_m} + \min_{i_t,t\in[m]\setminus\{1,k\}} \sum_{i_k\in[n]} a_{s i_2 \cdots i_{k-1} i_k i_k \cdots i_m},$$

which together with (15) yields

$$\varrho(\mathcal{A}) \geq \left(\tau(\mathcal{A})^{\frac{1}{m}} - 1\right) \min_{i_t, t \in [m] \setminus \{1,k\}} a_{si_2 \cdots i_{k-1} l i_{k+1} \cdots i_m} + \min_{i_t, t \in [m] \setminus \{1,k\}} \sum_{i_k \in [n]} a_{si_2 \cdots i_{k-1} i_k i_k \cdots i_m}$$

$$\geq \min_{i_1 \in [n]} \left[\left(\tau(\mathcal{A})^{\frac{1}{m}} - 1\right) \min_{i_t, t \in [m] \setminus \{1\}} a_{i_1 i_2 \cdots i_{k-1} i_k i_{k+1} \cdots i_m} + \min_{i_t, t \in [m] \setminus \{1,k\}} \sum_{i_k \in [n]} a_{i_1 i_2 \cdots i_{k-1} i_k i_k \cdots i_m} \right],$$

which proves the first Inequality of (30). This proves the theorem. □

Remark 2.5. *It follows from Inequality* (25) *that for the parameters* $\eta(\mathcal{A})$ *and* $\mu(\mathcal{A})$ *given by* (31) *we have*

$$\eta(\mathcal{A}) \leq \max_{i,j \in [n]} \{r_i(\mathcal{A}) + a_{ij\cdots j}(\delta(\mathcal{A})^{-\frac{m-1}{m}} - 1)\},$$

and

$$\mu(\mathcal{A}) \geq d_{m,n}.$$

This implies the bounds (30) *are always better than the corresponding bounds* (10)*.*

Remark 2.6. *For an irreducible weakly symmetric tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$*, when m and n are very large, the bounds in* (30) *need more computations than the bounds in* (10)*. As stated in Section 1, by the bounds in* (30)*, we can obtain a more sharp bound of* $\min_{\nu \in \mathbb{R}, x \in \mathbb{R}^n, \|x\|_2 = 1} \|\mathcal{A} - \nu x^m\|_F$*, which plays an important role in the symmetric best rank-one approximation [12–16]. This can be seen in the following example.*

Example 2.7. *Let* $\mathcal{A} = (a_{i_1 i_2 i_3 i_4}) \in \mathbb{R}_+^{[4,2]}$ *with entries defined as follows:*

$$a_{1111} = 10, \ \ a_{2222} = 20, a_{1122} = a_{1212} = a_{2112} = a_{2121} = 0.25, a_{1221} = a_{2211} = 0.3$$
$$\text{and } a_{i_1 i_2 i_3 i_4} = 0.4, \ \text{elsewhere}.$$

It is not difficult see that $\mathcal{A}$ *is a weakly symmetric nonnegative irreducible tensor, and*

$$\|\mathcal{A}\|_F = 22.3989.$$

By the bound (10)*, we have*

$$0.6914 \leq \varrho(\mathcal{A}) \leq 22.2525,$$

which implies

$$2.5569 \leq \min_{\nu \in \mathbb{R}, x \in \mathbb{R}^n, \|x\|_2 = 1} \|\mathcal{A} - \nu x^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \varrho(\mathcal{A})^2} \leq 22.3882.$$

By the bound (30)*, we have*

$$0.6937 \leq \varrho(\mathcal{A}) \leq 21.8479,$$

which means

$$4.9374 \leq \min_{\nu \in \mathbb{R}, x \in \mathbb{R}^n, \|x\|_2 = 1} \|\mathcal{A} - \nu x^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \varrho(\mathcal{A})^2} \leq 22.3881.$$

Before generalizing the upper bound (30) of Theorem 2.4 to weakly symmetric nonnegative tensor, which will be used in Section 4, we first give a lemma in [20].

Lemma 2.8 ([20])**.** *Suppose that* $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ *is weakly symmetric. If* $0 \leq \mathcal{A} \leq \mathcal{B}$*, then* $\varrho(\mathcal{A}) \leq \varrho(\mathcal{B})$*.*

Theorem 2.9. *Suppose that* $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ *is weakly symmetric. Then*

$$\varrho(\mathcal{A}) \leq \eta(\mathcal{A}), \tag{33}$$

where $\eta(\mathcal{A})$ *given by Theorem 2.4.*

Proof. If $\mathcal{A}$ is irreducible. Inequality (33) follows from Theorem 2.4. If $\mathcal{A}$ is reducible. Let $\mathcal{A}_t = \mathcal{A} + \frac{1}{t}\mathcal{E}$, where $t = 1, 2, \cdots$ and $\mathcal{E}$ is a tensor with all entries being 1. Then $\{\mathcal{A}_t\}$ is a sequence of weakly symmetric and positive tensors satisfying $0 \le \mathcal{A} < \mathcal{A}_{t+1} < \mathcal{A}_t$. By Lemma 2.8, $\varrho(\mathcal{A}_t)$ is a monotone decreasing sequence with lower bound $\varrho(\mathcal{A})$ so that $\varrho(\mathcal{A}_t)$ has a limit. Thus, by Theorem 2.4, we have

$$\begin{aligned}\varrho(\mathcal{A}) &\le \varrho(\mathcal{A}_t)\\ &\le \max_{i,j\in[n]}\left\{\sum_{k=0}^{m-2}\sum_{(i_2,\cdots,i_m)\in\Delta(j;m-k-1)}\left(a_{ii_2\cdots i_m}+\frac{1}{t}\right)\left(\tau(\mathcal{A}_t)^{-\frac{m-k-1}{m}}-1\right)+r_i(\mathcal{A})+\frac{n^{m-1}}{t}\right\},\end{aligned}$$

letting $t \to \infty$, note that $\tau(\mathcal{A}_t) \to \tau(\mathcal{A})$, we obtain $\varrho(\mathcal{A}) \le \eta(\mathcal{A})$. This yields the desired conclusion. □

3 Upper bound for nonnegative tensors

In this section, we present an upper bound for the Z-eigenvalue of a nonnegative tensor and an upper bound for the Z-eigenvalue of a general tensor, which improves the bound (14). The following two lemmas will be used.

Lemma 3.1 ([6, 18])**.** *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $\mathcal{B} = (b_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be given as follows:*

$$b_{i_1i_2\cdots i_m} = \sum_{\pi\in Sym_m} a_{i_{\pi(1)}i_{\pi(2)}\cdots i_{\pi(m)}}, \tag{34}$$

then $\mathcal{B}$ is symmetric and

$$\mathcal{A}x^m = \frac{1}{m!}\mathcal{B}x^m, \tag{35}$$

where Sym_m is the set of all permutation in [m].

Lemma 3.2 ([10])**.** *If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is weakly symmetric, then $\sigma(\mathcal{A})$ consists precisely of all critical values of $f_{\mathcal{A}}(x) = \mathcal{A}x^m$ on S^{n-1}, where S^{n-1} is the standard unit sphere in $\mathbb{R}^n$.*

Based on Lemma 3.2, we have the following lemma.

Lemma 3.3. *Suppose that $\mathcal{A}$ is weakly symmetric. Then*

$$\varrho(\mathcal{A}) = \max\{|\mathcal{A}x^m| : x^Tx = 1, x \in \mathbb{R}^n\}. \tag{36}$$

Proof. Assume that $\mathcal{A}$ is a weakly symmetric tensor. By Lemma 3.2, we have

$$\max\{|\mathcal{A}x^m| : x^Tx = 1, x \in \mathbb{R}^n\} \le \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\} = \varrho(\mathcal{A}). \tag{37}$$

On the other hand, assume that λ is a Z-eigenvalue of $\mathcal{A}$ with Z-eigenvector x. It follows from Equation (1.2) that $\lambda = \mathcal{A}x^m$, thus

$$\begin{aligned}\varrho(\mathcal{A}) &= \max\{|\lambda| : \mathcal{A}x^{m-1} = \lambda x,\ x^Tx = 1, x \in \mathbb{R}^n\}\\ &= \max\{|\mathcal{A}x^m| : \mathcal{A}x^{m-1} = \lambda x,\ x^Tx = 1, x \in \mathbb{R}^n\}\\ &\le \max\{|\mathcal{A}x^m| :\ x^Tx = 1, x \in \mathbb{R}^n\},\end{aligned}$$

which together with (37), yields (36). The proof is completed. □

In the next theorem we have given an upper bound for the Z-eigenvalues of a nonnegative mth order n-dimensional tensor.

Theorem 3.4. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_+^{[m,n]}$. *Then for any Z-eigenvalue* λ*, we have*

$$|\lambda| \leq \frac{1}{m!}\varrho(\mathcal{B}) \leq \frac{1}{m!}\eta(\mathcal{B}),$$

where $\mathcal{B}$ *given by (34) and* $\eta(\mathcal{B})$ *defined as Theorem 2.4.*

Proof. Assume that λ is a Z-eigenvalue of $\mathcal{A}$ with Z-eigenvector x. It follows from Equation (1.2) that $\lambda = \mathcal{A}x^m$, thus

$$\begin{aligned} |\lambda| &= \{|\mathcal{A}x^m| : \mathcal{A}x^{m-1} = \lambda x, x^T x = 1, x \in \mathbb{R}^n\} \\ &\leq \max\{|\mathcal{A}x^m| : x^T x = 1, x \in \mathbb{R}^n\} \\ &= \frac{1}{m!}\max\{|\mathcal{B}x^m| : x^T x = 1, x \in \mathbb{R}^n\}. \end{aligned} \tag{38}$$

Note that $\mathcal{B}$ is a symmetric tensor, by Lemma 3.3, we have

$$\max\{|\mathcal{B}x^m| : x^T x = 1, x \in \mathbb{R}^n\} = \varrho(\mathcal{B}),$$

which together with Inequality (38), yields

$$|\lambda| \leq \frac{1}{m!}\varrho(\mathcal{B}). \tag{39}$$

From $\mathcal{A}$ being nonnegative it follows that $\mathcal{B}$ is also nonnegative, furthermore, $\mathcal{B}$ is symmetric. By Theorem 2.9, we obtain

$$\varrho(\mathcal{B}) \leq \eta(\mathcal{B}),$$

which together with Inequality (39) , yields

$$|\lambda| \leq \frac{1}{m!}\varrho(\mathcal{B}) \leq \frac{1}{m!}\eta(\mathcal{B}).$$

The proof is completed. □

The following result gives an upper bound for the Z-eigenvalue of a general mth order n-dimensional tensor.

Theorem 3.5. *Let* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$. *Then for any Z-eigenvalue* λ*, we have*

$$|\lambda| \leq \frac{1}{m!}\varrho(|\mathcal{B}|) \leq \frac{1}{m!}\eta(|\mathcal{B}|), \tag{40}$$

where $\mathcal{B}$ *given by (34),* $|\mathcal{B}| = (|b_{i_1 i_2 \cdots i_m}|) \in \mathbb{R}_+^{[m,n]}$ *and* $\eta(|\mathcal{B}|)$ *defined as Theorem 2.4.*

Proof. Assume that λ is a Z-eigenvalue of $\mathcal{A}$ with Z-eigenvector x. It follows from Equation (1.2) that $\lambda = \mathcal{A}x^m$, thus

$$\begin{aligned} |\lambda| &= \{|\mathcal{A}x^m| : \mathcal{A}x^{m-1} = \lambda x, x^T x = 1, x \in \mathbb{R}^n\} \\ &\leq \max\{|\mathcal{A}x^m| : x^T x = 1, x \in \mathbb{R}^n\} \\ &= \frac{1}{m!}\max\{|\mathcal{B}x^m| : x^T x = 1, x \in \mathbb{R}^n\} \\ &\leq \frac{1}{m!}\max\{|\mathcal{B}||x|^m : x^T x = 1, x \in \mathbb{R}^n\} \\ &\leq \frac{1}{m!}\max\{||\mathcal{B}|x^m| : x^T x = 1, x \in \mathbb{R}^n\}. \end{aligned} \tag{41}$$

Since $|\mathcal{B}|$ is a symmetric tensor, by Lemma 3.3, we obtain

$$\max\{||\mathcal{B}|x^m| : x^T x = 1, x \in \mathbb{R}^n\} = \varrho(|\mathcal{B}|),$$

which together with Inequality (41), yields

$$|\lambda| \leq \frac{1}{m!}\varrho(|\mathcal{B}|). \tag{42}$$

Obviously, $|\mathcal{B}|$ is a nonnegative symmetric tensor, by Theorem 2.9, we have

$$\varrho(|\mathcal{B}|) \leq \eta(|\mathcal{B}|),$$

which together with Inequality (42) , yields

$$|\lambda| \leq \frac{1}{m!}\varrho(|\mathcal{B}|) \leq \frac{1}{m!}\eta(|\mathcal{B}|).$$

The proof is completed. □

By means of the proof technique of Theorem 3.5, the following conclusions are obtained.

Corollary 3.6. *Suppose that $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a symmetric tensor. Then for any Z-eigenvalue λ, we have*

$$|\lambda| \leq \varrho(|\mathcal{A}|) \leq \eta(|\mathcal{A}|),$$

where $|\mathcal{A}| = (|a_{i_1 i_2 \cdots i_m}|) \in \mathbb{R}_+^{[m,n]}$, and $\eta(|\mathcal{A}|)$ defined as Theorem 2.4.

Proof. Assume that λ is a Z-eigenvalue of $\mathcal{A}$ with Z-eigenvector x. It follows from Equation (1.2) that $\lambda = \mathcal{A}x^m$, thus

$$\begin{aligned}
|\lambda| &= \{|\mathcal{A}x^m| : \mathcal{A}x^{m-1} = \lambda x, x^T x = 1, x \in \mathbb{R}^n\} \\
&\leq \max\{|\mathcal{A}x^m| : x^T x = 1, x \in \mathbb{R}^n\} \\
&\leq \max\{|\mathcal{A}||x|^m : x^T x = 1, x \in \mathbb{R}^n\} \\
&= \max\{||\mathcal{A}|x^m| : x^T x = 1, x \in \mathbb{R}^n\} \\
&= \varrho(|\mathcal{A}|) \\
&\leq \eta(|\mathcal{A}|).
\end{aligned}$$

The proof is completed. □

4 Comparisons with existing bounds

In this section, we will give some comparisons between our bounds and existing bounds for the Z-spectral radius for tensors. For a weakly symmetric nonnegative irreducible tensor, from Lemma 2.2, we obtain the bounds in (30), which are always better than the ones in (10). In particular, for a weakly symmetric positive tensor, the upper bound in (30) is always smaller than ones in (8).

The following example given in [10, 22] shows the efficiency of the new bound (30).

Example 4.1. *Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be a symmetric tensor defined by*

$$a_{1111} = \frac{1}{2}, a_{2222} = 3, \ and \ a_{i_1 \cdots i_4} = \frac{1}{3} \ elsewhere.$$

It follows from [10] that

$$\varrho(\mathcal{A}) \approx 3.1092.$$

By the bound (7)*, we have*

$$\varrho(\mathcal{A}) \leq 5.3333.$$

By the bound (8)*, we have*

$$\varrho(\mathcal{A}) \leq 5.2846.$$

By the bound (10)*, we have*

$$0.7330 \leq \varrho(\mathcal{A}) \leq 5.1935.$$

By the bound (30)*, we have*

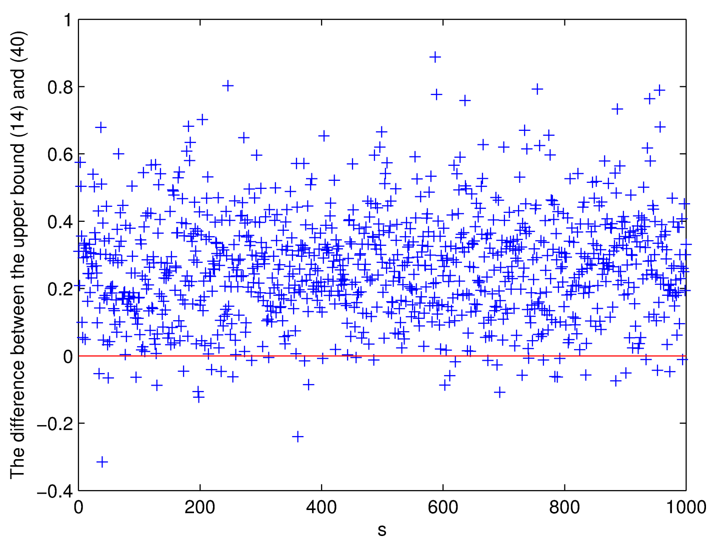
$$0.7663 \leq \varrho(\mathcal{A}) \leq 4.5147.$$

This example shows that the bound (30) *is the best.*

For a general tensor, the following example shows that our bound in (40) is better than the bound in (14) for some tensors.

Example 4.2. *Randomly generate* 1000 *tensors of* 4*th order* 3*-dimensional tensor such that the elements of each tensor are generated by uniform distribution* $(-0.05, 0.40)$*. We compare the upper bounds of the Z-spectral radius of general tensor in* (14) *and* (40)*. The numerical results are showed in Fig. 1. The* x*-axis of Fig. 1 refers to the sth random generated tensor. The plus symbol in blue color denotes the difference between the upper bound* (14) *and* (40)*. As observed from Fig. 1, there are* 96.2% *of cases above the* x*-axis.*

Fig. 1. The randomly generated results for Example 4.2



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