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Research Article

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On the weakly (α, ψ, ξ)-contractive condition for multi-valued operators in metric spaces and related fixed point results

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Abstract: The aim of this paper is to introduce the concept of a new nonlinear multi-valued mapping so called weakly (α, ψ, ξ)-contractive mapping and prove fixed point results for such mappings in metric spaces. Our results unify, generalize and complement various results from the literature. We give some examples which support our main results while previous results in literature are not applicable. Also, we analyze the existence of fixed points for mappings satisfying a general contractive inequality of integral type. Many fixed point results for multi-valued mappings in metric spaces endowed with an arbitrary binary relation and metric spaces endowed with graph are given here to illustrate the results in this paper.

Keywords: α-admissible multi-valued mappings, α_*-admissible multi-valued mappings, Weakly (α, ψ, ξ)-contractive multivalued mappings

MSC: 47H10, 54H25

1 Introduction and preliminaries

First, in this section, we recollect some essential notations, required definitions and primary results coherent with the literature.

For a nonempty set X, we denote by $N(X)$ the class of all nonempty subsets of X. Let (X, d) be a metric space. We denote by $CL(X)$ the class of all nonempty closed subsets of X and by $CB(X)$ the class of all nonempty closed bounded subsets of X. The Pompeiu-Hausdorff metric on $CL(X)$ induced by d is the function H defined by

$$H(A, B) = \begin{cases} \max\left\{\sup\limits_{a\in A} d(a, B), \sup\limits_{b\in B} d(b, A)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise} \end{cases}$$

for all $A, B \in CL(X)$, where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from a to $B \subseteq X$.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, non-negative real numbers and real numbers, respectively. Also, we denote Ψ by the class of all nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ satisfying $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the n^{th} iterate of ψ. For each $\psi \in \Psi$, we can easily see that $\psi(t) < t$ for each $t > 0$ and $\psi(t) = 0$ if and only if $t = 0$ (see more detail in [1]).

In 2012, Samet et al. [1] introduced the concepts of α-ψ-contractive mapping and α-admissible mapping as follows:

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Definition 1.1 ([1]). *Let (X, d) be a metric space and $t : X \to X$ be a mapping. A mapping t is called an α-ψ-contractive mapping if there exist two functions $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(x, y)d(tx, ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

Definition 1.2 ([1]). *Let t be a self-mapping on a nonempty set X and $\alpha : X \times X \to [0, \infty)$ be a mapping. A mapping t is said to be α-*admissible *if the following condition holds:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \implies \alpha(tx, ty) \geq 1.$$

They studied fixed point results for such mappings in metric spaces and also showed that the results can be utilized to derive some fixed point results in partially ordered metric spaces.

On the other hand, von Neumann [2] first studied fixed point results for a multi-valued (set-valued) mappings in game theory. Afterward, investigations on the existence of fixed points for multi-valued contraction mappings in the setting of metric spaces were initiated by Nadler [3] in 1969. Using the concept of the Pompeiu-Hausdorff metric, he established multi-valued version of Banach's contraction principle [4], which is usually referred as Nadler's contraction principle. Many modifications and generalizations of Nadler's contraction principle have been developed in successive years.

In 2012, Asl et al. [5] extended the concept of the α-admissible mapping from single-valued mappings to multi-valued mappings so called α_*–admissible multi-valued mapping and introduced the concept of the α_*-ψ-contractive multi-valued mapping as follows:

Definition 1.3 ([5]). *Let X be a nonempty set, $T : X \to N(X)$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. A mapping T is said to be α_*-admissible if the following condition holds:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \implies \alpha_*(Tx, Ty) \geq 1,$$

where $\alpha_*(Tx, Ty) := \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$.

Definition 1.4 ([5]). *Let (X, d) be a metric space. A multi-valued mapping $T : X \to CL(X)$ is called an α_*-ψ-contractive mapping if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to [0, \infty)$ such that*

$$\alpha_*(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in X$.

They also proved fixed point results for α_*-ψ-contractive multi-valued mapping by using the concept of α_*–admissible mapping.

Throughout this paper, we denote by Ξ the family of all functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(ξ_1) ξ is continuous;
(ξ_2) ξ is nondecreasing on $[0, \infty)$;
(ξ_3) $\xi(t) = 0$ if and only if $t = 0$;
(ξ_4) $\xi(t) > 0$ for all $t \in (0, \infty)$;
(ξ_5) ξ is subadditive.

Recently, Ali et al. [6] introduced the concept of (α, ψ, ξ)-contractive multi-valued mappings, where $\psi \in \Psi$ and $\xi \in \Xi$, as follows:

Definition 1.5 ([6]). *Let (X, d) be a metric space. A multi-valued mapping $T : X \to CL(X)$ is called (α, ψ, ξ)-contractive mapping if there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that the following condition holds:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \implies \xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))), \tag{1}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$. *If* ψ *is strictly increasing, then* T *is called a strictly* (α, ψ, ξ)*-contractive mapping.*

They also obtained fixed point results for (α, ψ, ξ)-contractive multi-valued mappings in complete metric spaces by using the following lemma:

Lemma 1.6 ([6]). *Let* (X, d) *be a metric space,* $\xi \in \Xi$ *and* $B \in CL(X)$. *If there exists* $x \in X$ *such that* $\xi(d(x, B)) > 0$, *then there exists* $y \in B$ *such that*

$$\xi(d(x, y)) < q\xi(d(x, B)),$$

where $q > 1$.

In 2013, Mohammadi et al. [7] extended the concept of an α_*–admissible mapping to an α-admissible mapping as follows:

Definition 1.7 ([7]). *Let* X *be a nonempty set,* $T : X \to N(X)$ *and* $\alpha : X \times X \to [0, \infty)$. *A mapping* T *is said to be* α*-admissible whenever, for each* $x \in X$ *and* $y \in Tx$ *with* $\alpha(x, y) \geq 1$, *we have* $\alpha(y, z) \geq 1$ *for all* $z \in Ty$.

Remark 1.8. *If* T *is an* α_**-admissible mapping, then* T *is also an* α*-admissible mapping.*

For more details of fixed point results and recently related results by using the concepts of α-admissible single valued and α-admissible multi-valued mappings, one can refer to [8–16] and references therein.

In this paper, we introduce the new nonlinear multi-valued mapping so called a weakly (α, ψ, ξ)-contractive mapping which is a generalization of an (α, ψ, ξ)-contractive multi-valued mapping and prove some fixed point results for weakly (α, ψ, ξ)-contractive mappings by using the properties of α-admissible multi-valued mappings. The main results in this paper generalize the main results of Ali et al. [6] and many other results in literature. Also, we give some examples to illustrate our main theorems and show that, from our examples, the results of Ali et al. [6] are not applicable.

Finally, from our main results, we also obtain several fixed point results such as fixed point results for mappings satisfying contractive conditions of integral type, fixed point results in ordinary metric spaces, fixed point results in metric spaces endowed with the arbitrary relation and fixed point results in metric spaces endowed with graph.

2 Main results

In this section, we introduce the concept of a weakly (α, ψ, ξ)-contractive mapping and give fixed point result for such mapping.

Definition 2.1. *Let* (X, d) *be a metric space.*

(1) *A multi-valued mapping* $T : X \to CL(X)$ *is called a weakly* (α, ψ, ξ)*-contractive mapping if there exist three functions* $\psi \in \Psi$, $\xi \in \Xi$ *and* $\alpha : X \times X \to [0, \infty)$ *such that the following condition holds:*

$$x \in X, y \in Tx \text{ with } \alpha(x, y) \geq 1 \implies \xi(d(y, Ty)) \leq \psi(\xi(M(x, y))), \tag{2}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

(2) *If* ψ *is strictly increasing, then the weakly* (α, ψ, ξ)*-contractive mapping is called a strictly weakly* (α, ψ, ξ)*-contractive mapping.*

Next, we give first main result in this paper.

Theorem 2.2. *Let* (X, d) *be a complete metric space and* $T : X \to CL(X)$ *be a strictly weakly* (α, ψ, ξ)*-contractive mapping satisfying the following conditions:*

(S_1) *T is an α-admissible multi-valued mapping;*
(S_2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

Proof. For x_0, x_1 in (S_2), if $x_0 = x_1$ or $x_1 \in Tx_1$, then x_1 is a fixed point of T. We have nothing to prove.

So, we assume that $x_0 \neq x_1$ and $x_1 \notin Tx_1$. Since $x_1 \in Tx_0$ and $\alpha(x_0, x_1) \geq 1$, by the strictly weakly (α, ψ, ξ)-contractive condition of T, we get

$$\begin{aligned} 0 &< \xi(d(x_1, Tx_1)) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right)\right) \\ &= \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\}\right)\right) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2}\right\}\right)\right) \\ &= \psi(\xi(\max\{d(x_0, x_1), d(x_1, Tx_1)\})). \end{aligned} \tag{3}$$

By the property of ψ, the above relation is impossible if $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$. Hence, from (3), it follows that

$$0 < \xi(d(x_1, Tx_1)) \leq \psi(\xi(d(x_0, x_1))). \tag{4}$$

Using Lemma 1.6, there exists $x_2 \in Tx_1$ such that

$$\xi(d(x_1, x_2)) < q\xi(d(x_1, Tx_1)), \tag{5}$$

where q is a fixed real number such that $q > 1$. If $x_1 = x_2$ or $x_2 \in Tx_2$, then x_2 is a fixed point of T and hence we have noting to prove. Now, we may assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. From (4) and (5), we obtain that

$$0 < \xi(d(x_1, x_2)) < q\psi(\xi(d(x_0, x_1))). \tag{6}$$

Applying ψ in the above inequality, we get

$$0 < \psi(\xi(d(x_1, x_2))) < \psi(q\psi(\xi(d(x_0, x_1)))) \tag{7}$$

and so

$$q_1 := \frac{\psi(q\psi(\xi(d(x_0, x_1))))}{\psi(\xi(d(x_1, x_2)))} > 1. \tag{8}$$

Since T is an α-admissible multi-valued mapping, we get $\alpha(x_1, x_2) \geq 1$. From the weakly (α, ψ, ξ)-contractive condition of T, we have

$$\begin{aligned} 0 &< \xi(d(x_2, Tx_2)) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right\}\right)\right) \\ &= \psi\left(\xi\left(\max\left\{d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, Tx_2)}{2}\right\}\right)\right) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, x_2) + d(x_2, Tx_2)}{2}\right\}\right)\right) \\ &= \psi(\xi(\max\{d(x_1, x_2), d(x_2, Tx_2)\})). \end{aligned} \tag{9}$$

From the above inequality, it follows that

$$\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2) \tag{10}$$

and thus

$$0 < \xi(d(x_2, Tx_2)) \leq \psi(\xi(d(x_1, x_2))). \tag{11}$$

For $q_1 > 1$, by using Lemma 1.6, there exists $x_3 \in Tx_2$ such that

$$\xi(d(x_2, x_3)) < q_1 \xi(d(x_2, Tx_2)). \tag{12}$$

If $x_2 = x_3$ or $x_3 \in Tx_3$, then x_3 is a fixed point of T and hence we have noting to prove. Now, we may assume that $x_2 \neq x_3$ and $x_3 \notin Tx_3$. From (11) and (12), we get

$$0 < \xi(d(x_2, x_3)) < q_1 \psi(\xi(d(x_1, x_2))) = \psi(q\psi(\xi(d(x_0, x_1)))). \tag{13}$$

Since ψ is a strictly increasing function, we obtain

$$0 < \psi(\xi(d(x_2, x_3))) < \psi^2(q\psi(\xi(d(x_0, x_1)))). \tag{14}$$

By continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_n \neq x_{n+1} \in Tx_n$,

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{15}$$

and

$$0 < \xi(d(x_{n+1}, x_{n+2})) < \psi^n(q\psi(\xi(d(x_0, x_1)))) \tag{16}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Next, we prove that $\{x_n\}$ is a Cauchy sequence in X. Let $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $m > n$. From the triangle inequality and (16), we obtain

$$\xi(d(x_m, x_n)) \leq \sum_{i=n}^{m-1} \xi(d(x_i, x_{i+1})) < \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(\xi(d(x_0, x_1)))).$$

Using the property of ψ, we get $\lim_{n,m\to\infty} \xi(d(x_m, x_n)) = 0$. By the continuity of ξ and (ξ_3), we have

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0 \tag{17}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$, that is, $\lim_{n\to\infty} d(x_n, x^*) = 0$. From the continuity of T, we obtain

$$\lim_{n\to\infty} H(Tx_n, Tx^*) = 0. \tag{18}$$

Further, we have

$$d(x^*, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) \leq \lim_{n\to\infty} H(Tx_n, Tx^*) = 0.$$

By the closedness of Tx^*, we get $x^* \in Tx^*$. Therefore, x^* is a fixed point of T. This completes the proof. □

Next, we give second main result in this paper.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a strictly weakly (α, ψ, ξ)-contractive mapping satisfying the following conditions:*

(S_1) T is an α-admissible multi-valued mapping;

(S_2) there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

(S_3') if $\{x_n\}$ is a sequence in X with $x_{n+1} \in Tx_n$, $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have $\xi(d(x_{n+1}, Tx)) \leq \psi(\xi(M(x_n, x)))$ for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof. Following the proof of Theorem 2.2, we can construct a Cauchy sequence $\{x_n\}$ in X such that $x_n \to x^*$ as $n \to \infty$ and

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{19}$$

for all $n \in \mathbb{N}$. From the condition (S_3'), we get

$$\xi(d(x_{n+1}, Tx^*)) \leq \psi\left(\xi\left(\max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right\}\right)\right) \tag{20}$$

for all $n \in \mathbb{N}$. Suppose that $d(x^*, Tx^*) > 0$ and let $\epsilon := \frac{d(x^*, Tx^*)}{2}$. Since $x_n \to x^*$ as $n \to \infty$, we can find $N_1 \in \mathbb{N}$ such that

$$d(x^*, x_n) < \frac{d(x^*, Tx^*)}{2} \tag{21}$$

for all $n \geq N_1$. Furthermore, we obtain

$$d(x^*, Tx_n) \leq d(x^*, x_{n+1}) < \frac{d(x^*, Tx^*)}{2} \tag{22}$$

for all $n \geq N_1$. Also, since $\{x_n\}$ is a Cauchy sequence, there exists $N_2 \in \mathbb{N}$ such that

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) < \frac{d(x^*, Tx^*)}{2} \tag{23}$$

for all $n \geq N_2$. Since $d(x_n, Tx^*) \to d(x^*, Tx^*)$ as $n \to \infty$, it follows that there exists $N_3 \in \mathbb{N}$ such that

$$d(x_n, Tx^*) < \frac{3d(x^*, Tx^*)}{2} \tag{24}$$

for all $n \geq N_3$. Using (21)-(24), it follows that

$$\max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right\} = d(x^*, Tx^*) \tag{25}$$

for all $n \geq N := \max\{N_1, N_2, N_3\}$. For each $n \geq N$, from (20) and the triangle inequality, it follows that

$$\begin{aligned}
&\xi(d(x^*, Tx^*)) \\
&\leq \xi(d(x^*, x_{n+1})) + \xi(d(x_{n+1}, Tx^*)) \\
&\leq \xi(d(x^*, x_{n+1})) + \psi\left(\xi\left(\max\left\{d(x_n, x^*), d(x_n, Tx_n), d(x^*, Tx^*), \frac{d(x_n, Tx^*) + d(x^*, Tx_n)}{2}\right\}\right)\right) \\
&= \xi(d(x^*, x_{n+1})) + \psi(\xi(d(x^*, Tx^*))).
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\xi(d(x^*, Tx^*)) \leq \psi(\xi(d(x^*, Tx^*))).$$

This implies that $\xi(d(x^*, Tx^*)) = 0$, which is a contradiction. Therefore, $d(x^*, Tx^*)) = 0$, that is, $x^* \in Tx^*$. This completes the proof. □

Next, we give an example to show that our result is more general than the results of Ali et al. [6] and many known results in literature.

Example 2.4. *Let* $X = [0, 100]$ *and the metric* $d : X \times X \to \mathbb{R}$ *defined by* $d(x, y) = |x - y|$ *for all* $x, y \in X$. *Define* $T : X \to CL(X)$ *and* $\alpha : X \times X \to [0, \infty)$ *by*

$$Tx = \begin{cases} \left[0, \frac{x}{100}\right], & x \in [0, 1], \\ [x^2 + 5, 50], & x \in (1, 5], \\ \left[\frac{x-1}{2}, 100\right], & x \in (5, 100], \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1, & x, y \in [0, 5], \\ 0, & otherwise. \end{cases}$$

Here we show that the contractive condition (1) does not hold for all functions $\psi \in \Psi$ *and* $\xi \in \Xi$. *Let* $x = 0$ *and* $y = 2$. *We observe that*

$$\alpha(x, y) = \alpha(0, 2) = 1,$$

but

$$\xi(H(Tx, Ty)) = \xi(H(T0, T2)) = \xi(9) > \xi(7) > \psi(\xi(7)) = \psi(\xi(M(0, 2))) = \psi(\xi(M(x, y))).$$

Therefore, the results of Ali et al. [6] are not applicable here.

Next, we show that Theorem 2.3 can be used to guarantee the existence of fixed point of T. *Define the functions* $\psi, \xi : [0, \infty) \to [0, \infty)$ *by* $\psi(t) = \frac{t}{10}$ *and* $\xi(t) = \sqrt{t}$ *for all* $t \in [0, \infty)$. *It is easy to see that* $\psi \in \Psi$ *and* $\xi \in \Xi$. *First, we show that* T *is a strictly weakly* (α, ψ, ξ)*-contractive mapping. Suppose that* $x \in X$, $y \in Tx$ *and* $\alpha(x, y) \geq 1$ *and hence* $x \in [0, 1]$ *and* $y \in [0, 0.01]$. *Also, we obtain*

$$\begin{aligned} \xi(d(y, Ty)) &= \sqrt{d(y, Ty)} \\ &\leq \sqrt{H(Tx, Ty)} \\ &= \sqrt{\frac{|x - y|}{100}} \\ &= \frac{1}{10}\sqrt{|x - y|} \\ &\leq \frac{1}{10}\sqrt{M(x, y)} \\ &= \psi(\xi(M(x, y))). \end{aligned}$$

It is easy to observe that ψ *is a strictly increasing function. Therefore,* T *is a strictly weakly* (α, ψ, ξ)*-contractive mapping. It is easy to see that* T *is an* α*-admissible multi-valued mapping. Moreover, there exists* $x_0 = 1 \in X$ *and* $x_1 = 0.0001 \in Tx_0$ *such that*

$$\alpha(x_0, x_1) = \alpha(1, 0.0001) \geq 1.$$

Finally, for each sequence $\{x_n\}$ *in* X *with* $x_{n+1} \in Tx_n$, $x_n \to x \in X$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$, *we get* $x_n, x \in [0, 1]$ *for all* $n \in \mathbb{N}$ *and so*

$$\begin{aligned} \xi(d(x_{n+1}, Tx)) &= \sqrt{d(x_{n+1}, Tx)} \\ &\leq \sqrt{H(Tx_n, Tx)} \\ &= \sqrt{\frac{|x_n - x|}{100}} \\ &= \frac{1}{10}\sqrt{|x_n - x|} \\ &\leq \frac{1}{10}\sqrt{M(x_n, x)} \\ &= \psi(\xi(M(x_n, x))). \end{aligned}$$

Therefore, the condition (S_3') *in Theorem 2.3 holds. By using Theorem 2.3, we can conclude that* T *has a fixed point in* X. *In this case,* T *has infinitely fixed points such as* 0, 6 *and* 7.

From Remark 1.8, Theorems 2.2 and 2.3 apply particularly in the case when T is an α_*-admissible multi-valued mapping.

It is easy to see that the contractive condition (1) implies the contractive condition (2). Therefore, we give the following results without the proof:

Corollary 2.5 (Theorem 2.5 in [6]). *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping satisfying the following conditions:*
(S_1) *T is an α-admissible (or α_*-admissible) multi-valued mapping;*
(S_2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

Corollary 2.6 (Theorem 2.6 in [6]). *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a strictly (α, ψ, ξ)-contractive mapping satisfying the following conditions:*
(S_1) *T is an α-admissible (or α_*-admissible) multi-valued mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(S_3'') *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point in X.

3 Further fixed point results

Theorem 2.2 and Theorem 2.3 generalize, extend and improve Theorems 2.5 and 2.6 of Ali et al. [6]. Also, our results improve several results, for example, Theorem 2.1 and Theorem 2.2 of Samet et al. [1], Theorem 2.3 of Asl et al. [5], Theorem 2.1 and Theorem 2.2 of Amiri et al. [17], Theorem 2.1 of Salimi et al. [18], Theorem 3.1 and Theorem 3.4 of Mohammadi et al. [7], Banach's contraction principle [4], the main results of Kannan [19], the main results Chaterjea [20], Nadler's contraction principle [3] and many others.

In this section, we give the related fixed point results which are obtained by Theorem 2.2 and Theorem 2.3 as fixed point results for mapping satisfying contractive conditions of integral type, fixed point results in ordinary metric spaces, fixed point results in metric spaces endowed with arbitrary relation and fixed point results in metric spaces endowed with graph.

3.1 Fixed point results for mapping satisfying contractive conditions of integral type

In this subsection, we prove some fixed point results for the mappings satisfying the contractive conditions of integral type by using the fixed point results related with the function ξ in Section 2.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multi-valued mapping. Suppose that there exist three functions $\psi \in \Psi$, $\phi : [0, \infty) \to [0, \infty)$ and $\alpha : X \times X \to [0, \infty)$ satisfying the following conditions:*
(Int_1) *$\phi : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, \infty)$;*
(Int_2) *for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$;*
(Int_3) *for each $a, b > 0$, we have*

$$\int_0^{a+b} \phi(t)dt \leq \int_0^{a} \phi(t)dt + \int_0^{b} \phi(t)dt;$$

(Int_4) *the following condition holds:*

$$x \in X, y \in Tx \text{ with } \alpha(x, y) \geq 1 \implies \int_0^{d(y,Ty)} \phi(w)dw \leq \psi\Big(\int_0^{M(x,y)} \phi(w)dw \Big),$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$;
(Int_5) *ψ is strictly increasing function.*

Further, if the following assertions hold:
(S_1) *T is an α-admissible multi-valued mapping;*
(S_2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(S_3) *T is a continuous multi-valued mapping,*

then T has a fixed point in X.

Proof. Define a mapping $\xi : [0,\infty) \to [0,\infty)$ by $\xi(t) = \int_0^t \phi(s)ds$ for each $t \in [0,\infty)$. From the conditions (Int_1)-(Int_3), we get $\xi \in \Xi$. By the definition of mapping ξ and the condition (Int_4), it follows that the condition (2) holds. Since ψ is a strictly increasing function, it follows that T is a strictly weakly (α, ψ, ξ)-contractive mapping. Hence the conclusion of Theorem 3.1 follows from Theorem 2.2. □

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multi-valued mapping. Suppose that there exist three functions $\psi \in \Psi$, $\phi : [0,\infty) \to [0,\infty)$ and $\alpha : X \times X \to [0,\infty)$ satisfying the following conditions:*

(Int_1) *$\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0,\infty)$;*
(Int_2) *for each $\epsilon > 0$, $\int_0^\epsilon \phi(t)dt > 0$;*
(Int_3) *for each $a, b > 0$, we have*

$$\int_0^{a+b} \phi(t)dt \leq \int_0^{a} \phi(t)dt + \int_0^{b} \phi(t)dt;$$

(Int_4) *the following condition holds:*

$$x, y \in X \text{ with } \alpha(x, y) \geq 1 \implies \int_0^{H(Tx,Ty)} \phi(w)dw \leq \psi\left(\int_0^{M(x,y)} \phi(w)dw\right),$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$;
(Int_5) *ψ is strictly increasing function.*

Further, if the following assertions hold:
(S_1) *T is an α-admissible multi-valued mapping;*
(S_2) *there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;*
(S_3) *T is a continuous multi-valued mapping,*

then T has a fixed point in X.

Remark 3.3. *Theorem 3.1 and Corollary 3.2 are still valid if we replace condition (S_3) by the following condition:*

– *if $\{x_n\}$ is a sequence in X with $x_{n+1} \in Tx_n$, $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have*

$$\int_0^{d(x_{n+1},Tx)} \phi(w)dw \leq \psi\left(\int_0^{M(x_n,x)} \phi(w)dw\right)$$

for all $n \in \mathbb{N}$.

3.2 Fixed point results in ordinary metric spaces

In this subsection, we can derive some fixed point results in ordinary metric spaces from the fixed point results related with the contractive condition (2).

Theorem 3.4. *Let (X, d) be a complete metric space and $T : X \to CL(X)$ be a multi-valued mapping. Suppose that there exist three functions $\psi \in \Psi$, $\xi \in \Xi$ and $\alpha : X \times X \to [0, \infty)$ such that ψ is strictly increasing and*

$$\alpha(x, y)\xi(d(y, Ty)) \le \psi(\xi(M(x, y))) \tag{26}$$

for all $x \in X$ and $y \in Tx$, where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Further, if the following conditions hold:

(S_1) *T is an α-admissible multi-valued mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$;*
(S_3) *T is a continuous multi-valued mapping,*

then T has a fixed point in X.

Proof. We show that T satisfies the condition (2). For any $x \in X$ and $y \in Tx$, assume that $\alpha(x, y) \ge 1$. Using (26), we get

$$\xi(d(y, Ty)) \le \alpha(x, y)\xi(d(y, Ty)) \le \psi(\xi(M(x, y))).$$

This implies that the condition (2) holds. Therefore, the conclusions of this theorem follows from Theorem 2.2. □

Remark 3.5. *Theorem 3.4 is still valid if we replace the contractive condition (26) by the following contractive conditions:*

- $[\tau + \xi(d(y, Ty))]^{\alpha(x,y)} \le \tau + \psi(\xi(M(x, y)))$ *for all $x \in X$ and $y \in Tx$, where $\tau \ge 1$, or*
- $[\tau + \alpha(x, y) - 1]^{\xi(d(y,Ty))} \le \tau^{\psi(\xi(M(x,y)))}$ *for all $x \in X$ and $y \in Tx$, where $\tau > 1$.*

Also, Theorem 3.4 is still valid if we replace condition (S_3) by condition (S_3').

3.3 Fixed point results in metric spaces endowed with an arbitrary binary relation

In this section, we give some fixed point results on metric spaces endowed with the arbitrary binary relation which can be regarded as consequences of the results presented in the previous section.

The following notions and definitions are needed.

Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X. Denote

$$\mathcal{S} := \mathcal{R} \cup \mathcal{R}^{-1}.$$

It is easy to see that, for all $x, y \in X$,

$$x\mathcal{S}y \iff x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Definition 3.6. *Let X be a nonempty set and $\mathcal{R}$ be a binary relation over X. A multi-valued mapping $T : X \to N(X)$ is said to be $\mathcal{R}$-weakly comparative if, for each $x \in X$ and $y \in Tx$ with $x\mathcal{S}y$, we have $y\mathcal{S}z$ for all $z \in Ty$.*

Definition 3.7. *Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation over X.*

(1) *A mapping $T : X \to CL(X)$ is said to be weakly $(\mathcal{S}, \psi, \xi)$-contractive if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that the following condition holds:*

$$x \in X, y \in Tx \text{ with } x\mathcal{S}y \implies \xi(d(y, Ty)) \le \psi(\xi(M(x, y))), \tag{27}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

(2) *If ψ is strictly increasing, then the the weakly $(\mathcal{S}, \psi, \xi)$-contractive mapping is said to be strictly weakly $(\mathcal{S}, \psi, \xi)$-contractive.*

Theorem 3.8. *Let (X, d) be a complete metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CL(X)$ be a strictly weakly $(\mathcal{S}, \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T is a $\mathcal{R}$-weakly comparative mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

Proof. Define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in x\mathcal{S}y, \\ 0, & \text{otherwise.} \end{cases}$$

The conclusion of Theorem 3.8 follows from Theorem 2.2. □

Using Theorem 2.3, we get the following result:

Theorem 3.9. *Let (X, d) be a complete metric space, $\mathcal{R}$ be a binary relation over X and $T : X \to CL(X)$ be a strictly weakly $(\mathcal{S}, \psi, \xi)$-contractive mapping satisfying the following conditions:*
(S_1) *T is a $\mathcal{R}$-weakly comparative mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0 \mathcal{S} x_1$;*
(S_3') *if $\{x_n\}$ is a sequence in X with $x_{n+1} \in Tx_n$, $x_n \to x \in X$ as $n \to \infty$ and $x_n \mathcal{S} x_{n+1}$ for all $n \in \mathbb{N}$, then we have*

$$\xi(d(x_{n+1}, Tx)) \leq \psi(\xi(M(x_n, x)))$$

for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Next, we give some fixed point results on partially ordered metric spaces from Theorem 3.8 and Theorem 3.9.

We need the following notions and definitions:

Definition 3.10. *$(X, d, \preceq)$ is called a partially ordered metric space when (X, d) is a metric space and $(X, \preceq)$ is a partially ordered set.*

Let $(X, d, \preceq)$ be a partially ordered metric space. Denotes $\asymp := \preceq \cup \preceq^{-1}$. This implies that, for all $x, y \in X$,

$$x \asymp y \iff x \preceq y \text{ or } y \preceq x.$$

Definition 3.11. *Let $(X, \preceq)$ be a partially ordered set. A multi-valued mapping $T : X \to N(X)$ is said to be $\preceq$-weakly comparative if, for each $x \in X$ and $y \in Tx$ with $x \preceq y$, we have $y \preceq z$ for all $z \in Ty$.*

Definition 3.12. *Let $(X, d, \preceq)$ be a partially ordered metric spaces.*
1. *A mapping $T : X \to CL(X)$ is said to be weakly $(\asymp, \psi, \xi)$-contractive if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that the following condition holds:*

$$x \in X, y \in Tx \text{ with } x \asymp y \implies \xi(d(y, Ty)) \leq \psi(\xi(M(x, y))), \tag{28}$$

   where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.
2. *If ψ is strictly increasing, then the the weakly $(\asymp, \psi, \xi)$-contractive mapping is said to be strictly weakly $(\asymp, \psi, \xi)$-contractive.*

It is easy to see that $\preceq$ is a binary relation for a partially ordered metric spaces $(X, d, \preceq)$. Therefore, we obtain the following results from Theorem 3.8 and Theorem 3.9:

Corollary 3.13. *Let $(X, d, \preceq)$ be a complete partially ordered metric spaces and $T : X \to CL(X)$ be a strictly weakly $(\asymp, \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T is a $\preceq$-weakly comparative mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0 \asymp x_1$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

Corollary 3.14. *Let $(X, d, \preceq)$ be a complete partially ordered metric spaces and $T : X \to CL(X)$ be a strictly weakly $(\asymp, \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T is a $\preceq$-weakly comparative mapping;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $x_0 \asymp x_1$;*
(S_3') *if $\{x_n\}$ is a sequence in X with $x_{n+1} \in Tx_n$, $x_n \to x \in X$ as $n \to \infty$ and $x_n \asymp x_{n+1}$ for all $n \in \mathbb{N}$, then we have*

$$\xi(d(x_{n+1}, Tx)) \leq \psi(\xi(M(x_n, x)))$$

for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

3.4 Fixed point results in metric spaces endowed with graph

Throughout this subsection, let (X, d) be a metric space. A set $\{(x, x) : x \in X\}$ is called a *diagonal* of the Cartesian product $X \times X$, which is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, i.e., $\Delta \subseteq E(G)$. We assume G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. In this subsection, we prove fixed point results on a metric space endowed with graph.

Before presenting our results, we give the following notions and definitions:

Definition 3.15. *Let X be a nonempty set endowed with a graph G and $T : X \to N(X)$ be a multi-valued mapping, where X is a nonempty set X. A mapping T is said to have* weakly preserve edge *if, for each $x \in X$ and $y \in Tx$ with $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for all $z \in Ty$.*

Definition 3.16. *Let (X, d) be a metric space endowed with a graph G. A mapping $T : X \to CL(X)$ is said to be weakly $(E(G), \psi, \xi)$-contractive if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that the following condition holds:*

$$x \in X, y \in Tx \text{ with } (x, y) \in E(G) \implies \xi(d(y, Ty)) \leq \psi(\xi(M(x, y))), \tag{29}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Definition 3.17. *Let (X, d) be a metric space endowed with a graph G.*

(1) *A mapping $T : X \to CL(X)$ is said to be $(E(G), \psi, \xi)$-contractive if there exist two functions $\psi \in \Psi$ and $\xi \in \Xi$ such that the following condition holds:*

$$x \in X, y \in Tx \text{ with } (x, y) \in E(G) \implies \xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))), \tag{30}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

(2) *If $\psi \in \Psi$ is strictly increasing, then the (weakly) $(E(G), \psi, \xi)$-contractive mapping is said to be strictly* (*weakly*) *$(E(G), \psi, \xi)$-contractive.*

Remark 3.18. *The $(E(G), \psi, \xi)$-contractive condition implies the weakly $(E(G), \psi, \xi)$-contractive condition.*

Theorem 3.19. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CL(X)$ be a strictly weakly $(E(G), \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T has weakly preserve edge;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

Proof. The conclusion of this theorem can be obtain from Theorem 2.2 if we define a mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in (x, y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

□

Corollary 3.20. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CL(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T has weakly preserve edge;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
(S_3) *T is a continuous multi-valued mapping.*

Then T has a fixed point in X.

By using Theorem 2.3, we get the following result:

Theorem 3.21. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CL(X)$ be a strictly weakly $(E(G), \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T has weakly preserve edge;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
(S_3') *if $\{x_n\}$ is a sequence in X with $x_{n+1} \in Tx_n$, $x_n \to x \in X$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then we have*

$$\xi(d(x_n, Tx)) \leq \psi(\xi(M(x_n, x)))$$

for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Corollary 3.22. *Let (X, d) be a complete metric space endowed with a graph G and $T : X \to CL(X)$ be a strictly $(E(G), \psi, \xi)$-contractive mapping satisfying the following conditions:*

(S_1) *T has weakly preserve edge;*
(S_2) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$;*
(S_3'') *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then we have $(x_n, x) \in E(G)$ for all $n \in \mathbb{N}$.*

Then T has a fixed point in X.

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