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Research Article

Jianxing Zhao* and Caili Sang

Some new bounds of the minimum eigenvalue for the Hadamard product of an M-matrix and an inverse M-matrix

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Abstract: Some convergent sequences of the lower bounds of the minimum eigenvalue for the Hadamard product of a nonsingular M-matrix B and the inverse of a nonsingular M-matrix A are given by using Brauer's theorem. It is proved that these sequences are monotone increasing, and numerical examples are given to show that these sequences could reach the true value of the minimum eigenvalue in some cases. These results in this paper improve some known results.

Keywords: M-matrix, Hadamard product, Minimum eigenvalue, Lower bounds, Sequence

MSC: 15A06, 15A18, 15A42

1 Introduction

For a positive integer n, N denotes the set $\{1, 2, \cdots, n\}$, and $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) denotes the set of all $n \times n$ real (complex) matrices throughout.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix if $a_{ij} \le 0, i, j \in N, i \ne j$, A is nonsingular and $A^{-1} \ge 0$ (see [1, p.133]). Denote by M_n the set of all $n \times n$ nonsingular M-matrices.

If A is a nonsingular M-matrix, then there exists a positive eigenvalue of A equal to $\tau(A) \equiv [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the perron eigenvalue of the nonnegative matrix A^{-1}. It is easy to prove that $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A (see [2, p.357]).

A matrix A is called reducible if there exists a nonempty proper subset $I \subset N$ such that $a_{ij} = 0, \forall\, i \in I$, $\forall\, j \notin I$. If A is not reducible, then we call A irreducible (see [3, p.128]).

For two real matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, the Hadamard product of A and B is defined as the matrix $A \circ B = [a_{ij} b_{ij}]$. If A and B are two nonsingular M-matrices, then it was proved in [4, Proposition 3] that $A \circ B^{-1}$ is also a nonsingular M-matrix.

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}, a_{ii} \ne 0$. For $i, j, k \in N, j \ne i$, denote

$$d_j = \frac{\sum\limits_{k\ne j} |a_{jk}|}{|a_{jj}|},\ s_{ji} = \frac{|a_{ji}| + \sum\limits_{k\ne j,i} |a_{jk}| d_k}{|a_{jj}|},\ s_i = \max_{j\ne i}\{s_{ij}\};\ u_{ji} = \frac{|a_{ji}| + \sum\limits_{k\ne j,i} |a_{jk}| s_{ki}}{|a_{jj}|},\ u_i = \max_{j\ne i}\{u_{ij}\}.$$

Recently, some lower bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse have been proposed. Let $A = [a_{ij}] \in M_n$, it was proved in [5] that

$$0 < \tau(A \circ A^{-1}) \le 1.$$

***Corresponding Author: Jianxing Zhao:** College of Science, Guizhou Minzu University, Guiyang 550025, P.R. China, E-mail: zjx810204@163.com

Caili Sang: College of Science, Guizhou Minzu University, Guiyang 550025, P.R. China, E-mail: sangcl@126.com

Subsequently, Fiedler and Markham [4] gave a lower bound on $\tau(A \circ A^{-1})$,

$$\tau(A \circ A^{-1}) \geq \frac{1}{n},$$

and conjectured that

$$\tau(A \circ A^{-1}) \geq \frac{2}{n}.$$

Chen [6], Song [7] and Yong [8] have independently proved this conjecture.

In 2007, Li et al. improved the conjecture $\tau(A \circ A^{-1}) \geq \frac{2}{n}$ when A^{-1} is a doubly stochastic matrix and gave the following result (i.e., [9, Theorem 3.1]): Let $A = [a_{ij}] \in M_n$ and A^{-1} be a doubly stochastic matrix. Then

$$\tau(A \circ A^{-1}) \geq \min_{i \in N} \left\{ \frac{a_{ii} - s_i \sum\limits_{j \neq i} |a_{ij}|}{1 + \sum\limits_{j \neq i} s_{ji}} \right\}. \tag{1}$$

In 2013, Zhou et al. [10] also obtained (1) under the same condition.

In 2015, Chen gave the following result (i.e., [11, Theorem 3.2]): Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$ be a doubly stochastic matrix. Then

$$\begin{aligned} \tau(A \circ A^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & a_{ii}\alpha_{ii} + a_{jj}\alpha_{jj} \\ & - \Big[(a_{ii}\alpha_{ii} - a_{jj}\alpha_{jj})^2 + 4u_i u_j \alpha_{ii}\alpha_{jj} \Big(\sum_{k \neq i} |a_{ki}| \Big) \Big(\sum_{k \neq j} |a_{kj}| \Big) \Big]^{\frac{1}{2}} \Big\} \end{aligned} \tag{2}$$

and they have obtained

$$\begin{aligned} & \min_{i \neq j} \tfrac{1}{2} \Big\{ a_{ii}\alpha_{ii} + a_{jj}\alpha_{jj} - \Big[(a_{ii}\alpha_{ii} - a_{jj}\alpha_{jj})^2 + 4u_i u_j \alpha_{ii}\alpha_{jj} \Big(\sum_{k \neq i} |a_{ki}| \Big) \Big(\sum_{k \neq j} |a_{kj}| \Big) \Big]^{\frac{1}{2}} \Big\} \\ & \geq \min_{i \in N} \left\{ \frac{a_{ii} - s_i \sum_{j \neq i} |a_{ij}|}{1 + \sum_{j \neq i} s_{ij}} \right\}, \end{aligned}$$

i.e., under this condition, the bound of (2) is better than the one of (1).

In this paper, we present some convergent sequences of the lower bounds of $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$, which improve (2). Numerical examples show that these sequences could reach the true value of $\tau(A \circ A^{-1})$ in some cases.

2 Some notations and lemmas

In this section, we first give the following notations, which will be useful in the following proofs.

Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}, a_{ii} \neq 0$. For $i, j, k \in N, j \neq i, t = 1, 2, \ldots$, denote

$$h_i = \max_{j \neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}| s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}| s_{ki}} \right\},\ v_{ji}^{(0)} = \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| s_{ki} h_i}{|a_{jj}|},\ p_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| v_{ki}^{(t-1)}}{|a_{jj}|},$$

$$p_i^{(t)} = \max_{j \neq i} \{ p_{ij}^{(t)} \},\ h_i^{(t)} = \max_{j \neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}| p_{ji}^{(t)} - \sum\limits_{k \neq j,i} |a_{jk}| p_{ki}^{(t)}} \right\},\ v_{ji}^{(t)} = \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}| p_{ki}^{(t)} h_i^{(t)}}{|a_{jj}|}.$$

Lemma 2.1. *If $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is strictly row diagonally dominant, then, for all $i, j \in N, j \neq i, t = 1, 2, \ldots$,*

(a) $1 > d_j \geq s_{ji} \geq u_{ji} \geq v_{ji}^{(0)} \geq p_{ji}^{(1)} \geq v_{ji}^{(1)} \geq p_{ji}^{(2)} \geq v_{ji}^{(2)} \geq \cdots \geq p_{ji}^{(t)} \geq v_{ji}^{(t)} \geq \cdots \geq 0$;

(b) $1 \geq h_i \geq 0,\ 1 \geq h_i^{(t)} \geq 0$.

Proof. Since A is a strictly row diagonally dominant matrix, that is, $|a_{jj}| > \sum_{k \neq j} |a_{jk}|, j \in N$, obviously, $1 > d_j \geq 0,\ j \in N$. By the definitions of d_j, s_{ji}, we have $1 > d_j \geq s_{ji} \geq 0, j, i \in N, j \neq i$. And then, by the definitions of s_{ji}, u_{ji}, we have $s_{ji} \geq u_{ji}, j, i \in N, j \neq i$. Hence,

$$\frac{|a_{ji}|}{|a_{jj}|s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}|s_{ki}} = \frac{|a_{jj}|s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}|d_k}{|a_{jj}|s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}|s_{ki}} \leq 1,$$

from the definition of h_i, we have $0 \leq h_i \leq 1, i \in N$. Furthermore, by the definitions of $u_{ji}, v_{ji}^{(0)}$, we have $u_{ji} \geq v_{ji}^{(0)}, j, i \in N, j \neq i$.

Since $h_i = \max\limits_{j \neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}|s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}|s_{ki}} \right\}, i \in N$, we have

$$h_i \geq \frac{|a_{ji}|}{|a_{jj}|s_{ji} - \sum\limits_{k \neq j,i} |a_{jk}|s_{ki}},\ i.e.,\ s_{ji}h_i \geq \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}|s_{ki}h_i}{|a_{jj}|} = v_{ji}^{(0)}, j, i \in N, j \neq i.$$

From the definitions of $v_{ji}^{(0)}, p_{ji}^{(1)}$, we have $v_{ji}^{(0)} \geq p_{ji}^{(1)} \geq 0, j, i \in N, j \neq i$.

Hence,

$$\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum\limits_{k \neq j,i} |a_{jk}|p_{ki}^{(1)}} = \frac{|a_{jj}|p_{ji}^{(1)} - \sum\limits_{k \neq j,i} |a_{jk}|v_{ki}^{(0)}}{|a_{jj}|p_{ji}^{(1)} - \sum\limits_{k \neq j,i} |a_{jk}|p_{ki}^{(1)}} \leq 1,$$

by the definition of $h_i^{(1)}$, we have $0 \leq h_i^{(1)} \leq 1, i \in N$.

Since $h_i^{(1)} = \max\limits_{j \neq i} \left\{ \frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum\limits_{k \neq j,i} |a_{jk}|p_{ki}^{(1)}} \right\}, i \in N$, we have

$$h_i^{(1)} \geq \frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum\limits_{k \neq j,i} |a_{jk}|p_{ki}^{(1)}},\ i.e.,\ p_{ji}^{(1)}h_i^{(1)} \geq \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}|p_{ki}^{(1)}h_i^{(1)}}{|a_{jj}|} = v_{ji}^{(1)}, j, i \in N, j \neq i.$$

By $0 \leq h_i^{(1)} \leq 1, i \in N$, we have $p_{ji}^{(1)} \geq v_{ji}^{(1)} \geq 0, j, i \in N, j \neq i$. From the definitions of $v_{ji}^{(1)}, p_{ji}^{(2)}$, we obtain $v_{ji}^{(1)} \geq p_{ji}^{(2)} \geq 0, j, i \in N, j \neq i$.

In the same way as above, we can also prove that

$$p_{ji}^{(2)} \geq v_{ji}^{(2)} \geq \cdots p_{ji}^{(t)} \geq v_{ji}^{(t)} \geq \cdots \geq 0,\ 1 \geq h_i^{(t)} \geq 0, j, i \in N, j \neq i, t = 2, 3, \ldots.$$

The proof is completed. □

Using the same technique as the proof of Lemma 2.2 in [11], we can obtain the following lemma.

Lemma 2.2. *If $A = [a_{ij}] \in M_n$ is a strictly row diagonally dominant matrix, then $A^{-1} = [\alpha_{ij}]$ exists, and*

$$\alpha_{ji} \leq \frac{|a_{ji}| + \sum\limits_{k \neq j,i} |a_{jk}|v_{ki}^{(t)}}{a_{jj}} \alpha_{ii} = p_{ji}^{(t+1)}\alpha_{ii},\ j, i \in N,\ j \neq i,\ t = 0, 1, 2, \ldots.$$

Lemma 2.3. [12] *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and $x_1, x_2, \cdots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \Big(x_i \sum_{k \neq i} \frac{1}{x_k} |a_{ki}| \Big) \Big(x_j \sum_{k \neq j} \frac{1}{x_k} |a_{kj}| \Big) \right\}.$$

3 Main results

In this section, we give several convergent sequences for $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$.

Theorem 3.1. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots,$*

$$\tau(B \circ A^{-1}) \geq \min_{i \neq j} \frac{1}{2}\Big\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Big[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}\Big(\sum_{k\neq i}|b_{ki}|\Big)\Big(\sum_{k \neq j}|b_{kj}|\Big)\Big]^{\frac{1}{2}}\Big\} = \Omega_t. \tag{3}$$

Proof. It is evident that the result holds with equality for $n = 1$.

We next assume that $n \geq 2$. Since $A \in M_n$, there exists a positive diagonal matrix D such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix, and

$$\tau(B \circ A^{-1}) = \tau(D^{-1}(B \circ A^{-1})D) = \tau(B \circ (D^{-1}AD)^{-1}).$$

Therefore, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix.

(a) First, we assume that A and B are irreducible matrices. Since A is irreducible, then $0 < p_i^{(t)} < 1$, for any $i \in N$. Let $\tau(B \circ A^{-1}) = \lambda$. Since λ is an eigenvalue of $B \circ A^{-1}$, then $0 < \lambda < b_{ii}\alpha_{ii}$. By Lemma 2.2 and Lemma 2.3, there is a pair (i, j) of positive integers with $i \neq j$ such that

$$\begin{aligned}
|\lambda - b_{ii}\alpha_{ii}||\lambda - b_{jj}\alpha_{jj}| &\leq \Big(p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}|b_{ki}\alpha_{ki}|\Big)\Big(p_j^{(t)}\sum_{k\neq j}\frac{1}{p_k^{(t)}}|b_{kj}\alpha_{kj}|\Big) \\
&\leq \Big(p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}|b_{ki}p_{ki}^{(t)}\alpha_{ii}|\Big)\Big(p_j^{(t)}\sum_{k\neq j}\frac{1}{p_k^{(t)}}|b_{kj}p_{kj}^{(t)}\alpha_{jj}|\Big) \\
&\leq \Big(p_i^{(t)}\sum_{k\neq i}\frac{1}{p_k^{(t)}}|b_{ki}p_k^{(t)}\alpha_{ii}|\Big)\Big(p_j^{(t)}\sum_{k\neq j}\frac{1}{p_k^{(t)}}|b_{kj}p_k^{(t)}\alpha_{jj}|\Big) \\
&= \Big(p_i^{(t)}\alpha_{ii}\sum_{k\neq i}|b_{ki}|\Big)\Big(p_j^{(t)}\alpha_{jj}\sum_{k\neq j}|b_{kj}|\Big).
\end{aligned} \tag{4}$$

From inequality (4), we have

$$(\lambda - b_{ii}\alpha_{ii})(\lambda - b_{jj}\alpha_{jj}) \leq p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}\Big(\sum_{k\neq i}|b_{ki}|\Big)\Big(\sum_{k\neq j}|b_{kj}|\Big). \tag{5}$$

Thus, (5) is equivalent to

$$\lambda \geq \frac{1}{2}\Big\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Big[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}\Big(\sum_{k\neq i}|b_{ki}|\Big)\Big(\sum_{k\neq j}|b_{kj}|\Big)\Big]^{\frac{1}{2}}\Big\},$$

that is

$$\begin{aligned}
\tau(B \circ A^{-1}) &\geq \frac{1}{2}\Big\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Big[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}\Big(\sum_{k\neq i}|b_{ki}|\Big)\Big(\sum_{k\neq j}|b_{kj}|\Big)\Big]^{\frac{1}{2}}\Big\} \\
&\geq \min_{i\neq j}\frac{1}{2}\Big\{\alpha_{ii}b_{ii} + \alpha_{jj}b_{jj} - \Big[(\alpha_{ii}b_{ii} - \alpha_{jj}b_{jj})^2 + 4p_i^{(t)}p_j^{(t)}\alpha_{ii}\alpha_{jj}\Big(\sum_{k\neq i}|b_{ki}|\Big)\Big(\sum_{k\neq j}|b_{kj}|\Big)\Big]^{\frac{1}{2}}\Big\}.
\end{aligned}$$

(b) Now, assume that one of A and B is reducible. It is well known that a matrix in $Z_n = \{A = [a_{ij}] \in \mathbb{R}^{n\times n} : a_{ij} \leq 0, i \neq j\}$ is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $C = [c_{ij}]$ the $n \times n$ permutation matrix with $c_{12} = c_{23} = \cdots =$

$c_{n-1,n} = c_{n1} = 1$, the remaining c_{ij} zero, then both $A - \varepsilon C$ and $B - \varepsilon C$ are irreducible nonsingular M-matrices for any chosen positive real number ε, sufficiently small such that all the leading principal minors of both $A - \varepsilon C$ and $B - \varepsilon C$ are positive. Now we substitute $A - \varepsilon C$ and $B - \varepsilon C$ for A and B, in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. □

Theorem 3.2. *The sequence $\{\Omega_t\}, t = 1, 2, \ldots$ obtained from Theorem 3.1 is monotone increasing with an upper bound $\tau(B \circ A^{-1})$ and, consequently, is convergent.*

Proof. By Lemma 2.1, we have $p_{ji}^{(t)} \geq p_{ji}^{(t+1)} \geq 0, j, i \in N, j \neq i, t = 1, 2, \ldots$, so by the definiton of $p_i^{(t)}$, it is easy to see that the sequence $\{p_i^{(t)}\}$ is monotone decreasing. Then $\{\Omega_t\}$ is a monotonically increasing sequence with an upper bound $\tau(B \circ A^{-1})$. Hence, the sequence is convergent. □

If $B = A$, according to Theorem 3.1, the following corollary is established.

Corollary 3.3. *Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots$,*

$$\begin{aligned} \tau(A \circ A^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & a_{ii}\alpha_{ii} + a_{jj}\alpha_{jj} \\ & - \Big[(a_{ii}\alpha_{ii} - a_{jj}\alpha_{jj})^2 + 4 p_i^{(t)} p_j^{(t)} \alpha_{ii}\alpha_{jj} \Big(\sum_{k \neq i} |a_{ki}| \Big) \Big(\sum_{k \neq j} |a_{kj}| \Big) \Big]^{\frac{1}{2}} \Big\} = \Upsilon_t. \end{aligned} \tag{6}$$

Remark 3.4. *We give a simple comparison between* (2) *and* (6). *According to Lemma 2.1, we know that $u_{ji} \geq p_{ji}^{(t)}, j, i \in N, j \neq i, t = 1, 2, \ldots$. Furthermore, by the definitions of $u_i, p_i^{(t)}$, we have $u_i \geq p_i^{(t)}, i \in N$, $t = 1, 2, \ldots$. Obviously, for $t = 1, 2, \ldots$, the bounds in* (6) *are bigger than the bound in* (2).

Similar to the proof of Theorem 3.1, Theorem 3.2 and Corollary 3.3, we can obtain Theorem 3.5, Theorem 3.6 and Corollary 3.7, respectively.

Theorem 3.5. *Let $A = [a_{ij}], B = [b_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots$,*

$$\begin{aligned} \tau(B \circ A^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & \alpha_{ii} b_{ii} + \alpha_{jj} b_{jj} \\ & - \Big[(\alpha_{ii} b_{ii} - \alpha_{jj} b_{jj})^2 + 4 s_i s_j \alpha_{ii}\alpha_{jj} \Big(\sum_{k \neq i} \frac{|b_{ki}| p_{ki}^{(t)}}{s_k} \Big) \Big(\sum_{k \neq j} \frac{|b_{kj}| p_{kj}^{(t)}}{s_k} \Big) \Big]^{\frac{1}{2}} \Big\} = \Delta_t. \end{aligned}$$

Theorem 3.6. *The sequence $\{\Delta_t\}, t = 1, 2, \ldots$ obtained from Theorem 3.5 is monotone increasing with an upper bound $\tau(B \circ A^{-1})$ and, consequently, is convergent.*

Corollary 3.7. *Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots$,*

$$\begin{aligned} \tau(A \circ A^{-1}) \geq \min_{i \neq j} \frac{1}{2} \Big\{ & \alpha_{ii} a_{ii} + \alpha_{jj} a_{jj} \\ & - \Big[(\alpha_{ii} a_{ii} - \alpha_{jj} a_{jj})^2 + 4 s_i s_j \alpha_{ii}\alpha_{jj} \Big(\sum_{k \neq i} \frac{|a_{ki}| p_{ki}^{(t)}}{s_k} \Big) \Big(\sum_{k \neq j} \frac{|a_{kj}| p_{kj}^{(t)}}{s_k} \Big) \Big]^{\frac{1}{2}} \Big\} = \Gamma_t. \end{aligned}$$

Let $\mathrm{L}_t = \max\{\Upsilon_t, \Gamma_t\}$. By Corollary 3.3 and Corollary 3.7, the following theorem is easily found.

Theorem 3.8. *Let $A = [a_{ij}] \in M_n$ and $A^{-1} = [\alpha_{ij}]$. Then, for $t = 1, 2, \ldots$,*

$$\tau(A \circ A^{-1}) \geq \mathrm{L}_t.$$

4 Numerical examples

In this section, several numerical examples are given to verify the theoretical results.

Example 4.1. *Consider the following M-matrix:*

$$A = \begin{bmatrix} 20 & -1 & -2 & -3 & -4 & -1 & -1 & -3 & -2 & -2 \\ -1 & 18 & -3 & -1 & -1 & -4 & -2 & -1 & -3 & -1 \\ -2 & -1 & 10 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -3 & -1 & 0 & 16 & -4 & -2 & -1 & -1 & -1 & -2 \\ -1 & -3 & 0 & -2 & 15 & -1 & -1 & -1 & -2 & -3 \\ -3 & -2 & -1 & -1 & -1 & 12 & -2 & 0 & -1 & 0 \\ -1 & -3 & -1 & -1 & 0 & -1 & 9 & 0 & -1 & 0 \\ -3 & -1 & -1 & -4 & -1 & 0 & 0 & 12 & 0 & -1 \\ -2 & -4 & -1 & -1 & -1 & 0 & -1 & -3 & 14 & 0 \\ -3 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 11 \end{bmatrix}.$$

Since $Ae = e$ and $A^T e = e$, $e = [1, 1, \cdots, 1]^T$, A^{-1} is doubly stochastic. Numerical results are given in Table 1 for the total number of iterations $T = 10$. In fact, $\tau(A \circ A^{-1}) = 0.9678$.

Table 1. The lower upper of $\tau(A \circ A^{-1})$

Method	t	L_t
Corollary 2.5 of [13]		0.1401
Conjecture of [4]		0.2000
Theorem 3.1 of [9]		0.2519
Theorem 3.2 of [15]		0.4125
Theorem 3.1 of [14]		0.4471
Theorem 3.2 of [11]		0.4732
Corollary 3 of [16]		0.6064
Theorem 3.8	$t = 1$	0.7388
	$t = 2$	0.8553
	$t = 3$	0.9059
	$t = 4$	0.9261
	$t = 5$	0.9346
	$t = 6$	0.9383
	$t = 7$	0.9400
	$t = 8$	0.9407
	$t = 9$	0.9409
	$t = 10$	0.9409

Remark 4.2. *Numerical results in Table 1 show that:*
(a) *Lower bounds obtained from Theorem 3.8 are bigger than these corresponding bounds in [4,9,11,13-16].*
(b) *Sequence obtained from Theorem 3.8 is monotone increasing.*
(c) *The sequence obtained from Theorem 3.8 can approximate effectively to the true value of $\tau(A \circ A^{-1})$.*

Example 4.3. *A nonsingular M-matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ whose inverse is doubly stochastic, is randomly generated by Matlab 7.1 (with 0-1 average distribution).*

The numerical results obtained for $T = 500$ are listed in Table 2, where T are defined in Example 4.1.

Remark 4.4. *Numerical results in Table 2 show that it is effective by Theorem 3.8 to estimate $\tau(A \circ A^{-1})$ for large order matrices.*

Table 2. The lower upper of $\tau(A \circ A^{-1})$

Method	t	$\mathrm{L}_t(n = 200)$	$\mathrm{L}_t(n = 500)$
Theorem 3.8	$t = 1$	0.0319	0.0133
	$t = 30$	0.3953	0.1972
	$t = 60$	0.6065	0.3452
	$t = 90$	0.7293	0.4647
	$t = 120$	0.8016	0.5609
	$t = 150$	0.8428	0.6384
	$t = 180$	0.8647	0.7011
	$t = 210$	0.8773	0.7519
	$t = 240$	0.8844	0.7928
	$t = 270$	0.8885	0.8255
	$t = 300$	0.8909	0.8520
	$t = 330$	0.8923	0.8734
	$t = 360$	0.8930	0.8908
	$t = 390$	0.8935	0.9049
	$t = 420$	0.8937	0.9163
	$t = 450$	0.8938	0.9249
	$t = 480$	0.8939	0.9316
	$t = 500$	0.8940	0.9352

Example 4.5. *Let* $A = [a_{ij}] \in \mathbb{R}^{n \times n}$*, where* $a_{11} = a_{22} = \cdots = a_{n,n} = 2,\ a_{12} = a_{23} = \cdots = a_{n-1,n} = a_{n,1} = -1$*, and* $a_{ij} = 0$ *elsewhere.*

It is easy to know that A is a nonsingular M-matrix. If we apply Theorem 3.8 for $n = 10$ and $n = 100$, we have $\tau(A \circ A^{-1}) \geq 0.7507$ and $\tau(A \circ A^{-1}) \geq 0.7500$ when $t = 1$, respectively. In fact, $\tau(A \circ A^{-1}) = 0.7507$ for $n = 10$ and $\tau(A \circ A^{-1}) = 0.7500$ for $n = 100$.

Remark 4.6. *Numerical results in Example 4.5 show that the lower bound obtained from Theorem 3.8 could reach the true value of* $\tau(A \circ A^{-1})$ *in some cases.*

5 Conclusions

In this paper, we present a convergent sequence $\{\mathrm{L}_t\}, t = 1, 2, \ldots$ to approximate $\tau(A \circ A^{-1})$. Although we do not give the error analysis, i.e., how accurately these bounds can be computed, from the numerical experiments of Example 4.3, we are pleased to see that the bounds are still on the increase with the increase of the number of iterations. At present, it is very difficult for the authors to give the error analysis. Next, we will study this problem.

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