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Research Article

Dumitru Baleanu, Sunil Dutt Purohit*, and Jyotindra C. Prajapati

Integral inequalities involving generalized Erdélyi-Kober fractional integral operators

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Abstract: Using the generalized Erdélyi-Kober fractional integrals, an attempt is made to establish certain new fractional integral inequalities, related to the weighted version of the Chebyshev functional. The results given earlier by Purohit and Raina (2013) and Dahmani *et al.* (2011) are special cases of results obtained in present paper.

Keywords: Fractional integral inequalities, Chebyshev functional, Erdélyi-Kober operators

MSC: 26A33, 26D10, 26D15

1 Introduction

Fractional integral inequalities have many applications in numerical quadrature, transform theory, probability, and statistical problems, but the most useful ones are in establishing uniqueness of solutions in fractional boundary value problems. Moreover, they also provide upper and lower bounds to the solutions of the above equations. Therefore, a significant development in the classical and fractional integral inequalities, particularly in analysis, has been witnessed; see, for instance, the papers [1]-[5] and the references cited therein. Moreover, these inequalities are also playing an important role to interpret the stability of a class of fractional oscillators (see [6]).

The following inequality is well known in the literature as Chebyshev inequality [7]:

If $f, g : [a, b] \to \mathbb{R}^+$ are absolutely continuous functions, whose first derivatives are bounded and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)dt - \left(\frac{1}{b-a}\int_a^b f(t)dt\right)\left(\frac{1}{b-a}\int_a^b g(t)dt\right), \tag{1}$$

then

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \,||f'||_\infty \,||g'||_\infty\,,$$

where $||.||_\infty$ denotes the norm in $L_\infty[a,b]$.

Under suitable assumptions (Chebyshev inequality, Grüss inequality, Minkowski inequality, Hermite-Hadamard inequality, Ostrowski inequality etc.), inequalities are playing a significant role in the field of mathematical sciences, particularly, in the theory of approximations. Therefore, in the literature we found several extensions and

Dumitru Baleanu: Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Cankaya University-06530, Ankara, Turkey, and Institute of Space Sciences, Magurele-Bucharest, Romania, E-mail: dumitru@cankaya.edu.tr
***Corresponding Author: Sunil Dutt Purohit:** Department of HEAS (Mathematics), Rajasthan Technical University, Kota-324010, India, E-mail: sunil_a_purohit@yahoo.com
Jyotindra C. Prajapati: Department of Mathematics, Faculty of Technology and Engineering, Marwadi Education Foundation Group of Institutions (MEFGI), Rajkot-360003, Gujarat, India, E-mail: jyotindra18@rediffmail.com

generalizations of these classical integral inequalities, including fractional calculus and q-calculus operators also (see [8]-[26]).

Further, a weighted version of the Chebyshev functional (see [7]) is defined as:

$$T(f, g, p) = \int_a^b p(t)dt \int_a^b f(t)g(t)p(t)dt - \int_a^b f(t)p(t)dt \int_a^b g(t)p(t)dt, \tag{2}$$

where f and g are integrable functions on $[a, b]$ and $p(t)$ is a positive and integrable function on $[a, b]$. In 2000, Dragomir [27] derived the following inequality, related to the weighted Chebyshev functional (2):

$$2\,|T(f, g, p)| \le \left\|f'\right\|_r \left\|g'\right\|_s \left[\int_a^b\int_a^b |x-y|\, p(x)p(y)dxdy\right], \tag{3}$$

where f, g are differentiable functions and $f' \in L_r(a, b), g' \in L_s(a, b), r > 1, r^{-1} + s^{-1} = 1$. The several extensions of inequality (3) are studied by many authors. Recently, Dahmani *et al.* [28], Purohit and Raina [29], Baleanu *et al.* [30] and Ntouyas *et al.* [31] obtained certain generalized Chebyshev type integral inequalities involving various type of fractional integral operators.

In present paper, we add one more dimension to this study by establishing certain integral inequalities for the functional (2) associated with the differentiable functions whose derivatives belong to the space $L_p([0, \infty))$, involving the generalized Erdélyi-Kober fractional integrals. Further, we also derive certain known and new integral inequalities for the fractional integrals by suitably choosing the special cases of our main results.

2 Fractional calculus

Authors mention the preliminaries and notations of some well-known operators of fractional calculus, which shall be used in the sequel.

The generalized Erdélyi-Kober fractional integral operator $I_\beta^{\eta,\alpha}$ of order α for a real-valued continuous function $f(t)$ is defined as (see [32, p. 14, eqn. (1.1.17)]):

$$I_\beta^{\eta,\alpha}\{f(t)\} = \frac{t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta\eta}(t^\beta - \tau^\beta)^{\alpha-1} f(\tau)d(\tau^\beta) = \frac{\beta\, t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta(\eta+1)-1}(t^\beta - \tau^\beta)^{\alpha-1} f(\tau)d\tau, \tag{4}$$

where $\alpha > 0$, $\beta > 0$ and $\eta \in \mathbb{R}$.

Following Kiryakova [32], the generalized Erdélyi-Kober fractional integral operators (4), possess the advantage that a number of generalized integration and differentiation operators happen to be the particular cases of this operator. Some important special cases of the integral operator $I_\beta^{\eta,\alpha}$ are mentioned below:

(i) For $\eta = 0, \alpha = n$ $(integer > 0)$ and $\beta = 1$, the operator (4) yields the following ordinary n-fold integrations:

$$l^n\{f(t)\} = t^n I_1^{0,n}\{f(t)\} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)d\tau. \tag{5}$$

(ii) If we set $\eta = 0$ and $\beta = 1$, the operator (4) reduces to the Riemann-Liouville fractional integral operators with the following relationship:

$$R^\alpha\{f(t)\} = t^\alpha I_1^{0,\alpha}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau. \tag{6}$$

(iii) Again, for $\eta = 0$, $\alpha = 1$ and $\beta = 1$, the operator (4) leads to the Hardy-Littlewood (Cesaro) integration operator:

$$L_{1,0}\{f(t)\} = I_1^{0,1}\{f(t)\} = \frac{1}{t}\int_0^t f(\tau)d\tau, \tag{7}$$

and its generalization for integers $n > m - 1$ (when $\eta = n$, $\alpha = 1$ and $\beta = 1$), we have

$$L_{m,n}\{f(t)\} = t^{n-m+1} I_1^{n,1}\{f(t)\} = t^{-m}\int_0^t \tau^n f(\tau)d\tau. \tag{8}$$

(iv) When $\beta = 1$, operator (4) reduces to the fractional integral operator, which was originally considered by Kober [33] and Erdélyi [34]:

$$I^{\alpha,\eta}\{f(t)\} = I_1^{\eta,\alpha}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\tau^\eta f(\tau)d\tau \quad (\alpha > 0, \eta \in \mathbb{R}). \tag{9}$$

(v) Also for $\beta = 2$, the operator (4) yields the Erdélyi-Kober fractional integral operator $I_{\eta,\alpha}$ (Sneddon [35]):

$$I_{\eta,\alpha} = I_2^{\eta,\alpha}\{f(t)\} = \frac{2\,t^{-2(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{2\eta+1}(t^2-\tau^2)^{\alpha-1} f(\tau)d\tau. \tag{10}$$

(vi) Further, if we set $\eta = -\frac{1}{2}$, $\beta = 2$ and α is replaced by $\alpha + \frac{1}{2}$, the Uspensky integral transform ([36]) can easily be obtained as under:

$$P^\alpha\{f(t)\} = \frac{1}{2} I_2^{-\frac{1}{2},\alpha+\frac{1}{2}}\{f(t)\} = \frac{1}{\Gamma(\alpha+\frac{1}{2})}\int_0^1 (1-\tau^2)^{\alpha-\frac{1}{2}} f(t\tau)d\tau. \tag{11}$$

For a detailed information about fractional integral operator (4) and its more special cases one may refer the book [32, pp. 15-17].

Next, we recall a composition formula of fractional integral (4) with a power function (see also as special case of image formula [32, p. 29, eqn. (1.2.26)]).

$$I_\beta^{\eta,\alpha}\left\{t^\lambda\right\} = \frac{\Gamma(1+\eta+\frac{\lambda}{\beta})}{\Gamma(1+\alpha+\eta+\frac{\lambda}{\beta})}\, t^\lambda \quad (\lambda > -\beta(\eta+1)). \tag{12}$$

3 Fractional integral inequalities

In this section, we obtain certain integral inequality which gives an estimation for the fractional integral of a product in terms of the product of the individual function fractional integrals, involving generalized Erdélyi-Kober fractioal integral operators. We give our results related to the Chebyshev's functional (2) in the case of differentiable mappings whose derivatives belong to the space $L_p([0,\infty))$ and satisfy the Holder's inequality.

Theorem 3.1. *Suppose that p be a positive function, f and g be differentiable functions on $[0,\infty)$, $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$ such that $r^{-1} + s^{-1} = 1$ with $r > 1$. Then for all $t > 0$, $\alpha > 0$, $\beta > 0$, $\eta \in \mathbb{R}$ and $\eta > -1$:*

$$\begin{aligned} &2\left|I_\beta^{\eta,\alpha}\{p(t)\}\, I_\beta^{\eta,\alpha}\{p(t)f(t)g(t)\} - I_\beta^{\eta,\alpha}\{p(t)f(t)\}\; I_\beta^{\eta,\alpha}\{p(t)g(t)\}\right| \\ &\leq \frac{\beta^2\, t^{-2\beta(\eta+\alpha)}\,\|f'\|_r\,\|g'\|_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\beta-\rho^\beta)^{\alpha-1}\times \\ &\qquad \times p(\tau)p(\rho)\,|\tau-\rho|\;d\tau d\rho \\ &\leq \left\|f'\right\|_r\,\left\|g'\right\|_s t\left(I_\beta^{\eta,\alpha}\{p(t)\}\right)^2. \end{aligned} \tag{13}$$

Proof. Let us define

$$\mathcal{H}(\tau,\rho)=\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right), \tag{14}$$

and

$$F(t,\tau)=\frac{\beta\, t^{-\beta(\eta+\alpha)}\tau^{\beta(\eta+1)-1}}{\Gamma(\alpha)}(t^{\beta}-\tau^{\beta})^{\alpha-1},\quad \tau\in(0,t),\quad t>0. \tag{15}$$

Under the conditions stated in the theorem, we observed that the function $F(t,\tau)>0$, for all $\tau\in(0,t)$ $(t>0)$. Upon multiplying (14) by $F(t,\tau)p(\tau)$ and integrating with respect to τ from 0 to t, we get

$$\begin{aligned}\frac{\beta\, t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)}\int_0^t \tau^{\beta(\eta+1)-1}\left(t^{\beta}-\tau^{\beta}\right)^{\alpha-1}p(\tau)\mathcal{H}(\tau,\rho)d\tau &= I_{\beta}^{\eta,\alpha}\{p(t)f(t)g(t)\}\\ -f(\rho)I_{\beta}^{\eta,\alpha}\{p(t)g(t)\}-g(\rho)I_{\beta}^{\eta,\alpha}\{p(t)f(t)\}&+f(\rho)g(\rho)I_{\beta}^{\eta,\alpha}\{p(t)\}.\end{aligned} \tag{16}$$

Next, on multiplying above relation (16) by $F(t,\rho)p(\rho)$, and then integrating with respect to ρ from 0 to t, we obtain

$$\begin{aligned}&\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^{\beta}-\tau^{\beta}\right)^{\alpha-1}\left(t^{\beta}-\rho^{\beta}\right)^{\alpha-1}p(\tau)p(\rho)\mathcal{H}(\tau,\rho)d\tau d\rho\\ &=2\left(I_{\beta}^{\eta,\alpha}\{p(t)\}I_{\beta}^{\eta,\alpha}\{p(t)f(t)g(t)\}-I_{\beta}^{\eta,\alpha}\{p(t)f(t)\}I_{\beta}^{\eta,\alpha}\{p(t)g(t)\}\right).\end{aligned} \tag{17}$$

Now, in view of (14), we have

$$\mathcal{H}(\tau,\rho)=\int_{\tau}^{\rho}\int_{\tau}^{\rho}f'(y)g'(z)dydz.$$

On making use of the Hölder's inequality, namely

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}f(y)g(z)dydz\right|\le\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|f(y)|^{r}dydz\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}|g(z)|^{s}dydz\right|^{s^{-1}},\qquad (r^{-1}+s^{-1}=1, r>1),$$

we obtain

$$|\mathcal{H}(\tau,\rho)|\le\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dydz\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dydz\right|^{s^{-1}}. \tag{18}$$

Since

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dydz\right|^{r^{-1}}=|\tau-\rho|^{r^{-1}}\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dy\right|^{r^{-1}}$$

and

$$\left|\int_{\tau}^{\rho}\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dydz\right|^{s^{-1}}=|\tau-\rho|^{s^{-1}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dz\right|^{s^{-1}},$$

therefore, inequality (18) reduces to

$$|\mathcal{H}(\tau,\rho)|\le|\tau-\rho|\left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r}dy\right|^{r^{-1}}\left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s}dz\right|^{s^{-1}}. \tag{19}$$

It follows from (17) that

$$\begin{aligned}
&\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} p(\tau) p(\rho) \left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho \\
&\leq \frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} p(\tau) p(\rho) \left|\tau-\rho\right| \times \qquad (20) \\
&\quad \times \left| \int_\tau^\rho \left|f'(y)\right|^r dy \right|^{r^{-1}} \left| \int_\tau^\rho \left|g'(z)\right|^s dz \right|^{s^{-1}} d\tau d\rho.
\end{aligned}$$

Again using Hölder's inequality on the right-hand side of (20), we get

$$\begin{aligned}
&\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} p(\tau) p(\rho) \left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho \\
&\leq \left[\frac{\beta^r\, t^{-r\beta(\eta+\alpha)}}{\Gamma^r(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} \times \right. \\
&\qquad \left. \times p(\tau) p(\rho) \left|\tau-\rho\right| \left| \int_\tau^\rho \left|f'(y)\right|^r dy \right| d\tau d\rho \right]^{r^{-1}} \times \\
&\times \left[\frac{\beta^s\, t^{-s\beta(\eta+\alpha)}}{\Gamma^s(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} \times \right. \\
&\qquad \left. \times p(\tau) p(\rho) \left|\tau-\rho\right| \left| \int_\tau^\rho \left|g'(z)\right|^s dz \right| d\tau d\rho \right]^{s^{-1}}.
\end{aligned}$$

Using the fact that

$$\left| \int_\tau^\rho \left|f(y)\right|^p dy \right| \leq ||f||_p^p,$$

we get

$$\begin{aligned}
&\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \rho^\beta\right)^{\alpha-1} p(\tau) p(\rho) \left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho \\
&\leq \left[\frac{\beta^r\, t^{-r\beta(\eta+\alpha)}\, ||f'||_r^r}{\Gamma^r(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \times \right. \\
&\qquad \left. \times p(\tau) p(\rho) \left|\tau-\rho\right| d\tau d\rho \right]^{r^{-1}} \times \\
&\times \left[\frac{\beta^s\, t^{-s\beta(\eta+\alpha)}\, ||g'||_s^s}{\Gamma^s(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} \times \right. \\
&\qquad \left. \times p(\tau) p(\rho) \left|\tau-\rho\right| d\tau d\rho \right]^{s^{-1}}. \qquad (21)
\end{aligned}$$

From (21), we arrived at

$$\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\rho^\beta\right)^{\alpha-1} p(\tau)p(\rho)\left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho$$

$$\leq \frac{\beta^2\, t^{-2\beta(\eta+\alpha)}\,||f'||_r\,||g'||_s}{\Gamma^2(\alpha)}\left[\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\times\right.$$
$$\left.\times p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\right]^{r^{-1}}\times$$

$$\times\left[\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1} p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\right]^{s^{-1}}. \tag{22}$$

Using the relation $r^{-1}+s^{-1}=1$, the above inequality yields to

$$\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\rho^\beta\right)^{\alpha-1} p(\tau)p(\rho)\left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho$$

$$\leq \frac{\beta^2\, t^{-2\beta(\eta+\alpha)}\,||f'||_r\,||g'||_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\times$$
$$\times p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho. \tag{23}$$

On the other hand (17) gives

$$2\left| I_\beta^{\eta,\alpha}\{p(t)\}\; I_\beta^{\eta,\alpha}\{p(t)f(t)g(t)\} - I_\beta^{\eta,\alpha}\{p(t)f(t)\}\; I_\beta^{\eta,\alpha}\{p(t)g(t)\}\right|$$
$$\leq \frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\rho^\beta\right)^{\alpha-1}\times \tag{24}$$
$$\times p(\tau)p(\rho)\left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho.$$

On making use of (23) and (24), one can easily arrive at the left-hand side of the inequality (13).

Now, to derive the right-hand side of the inequality (13), we have $0\leq\tau\leq t,\;\; 0\leq\rho\leq t$, and therefore,

$$0\leq|\tau-\rho|\leq t.$$

Evidently, from (23), we obtain

$$\frac{\beta^2\, t^{-2\beta(\eta+\alpha)}}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\rho^\beta\right)^{\alpha-1}\times p(\tau)p(\rho)\left|\mathcal{H}(\tau,\rho)\right| d\tau d\rho$$

$$\leq \frac{\beta^2\, t^{1-2\beta(\eta+\alpha)}\,||f'||_r\,||g'||_s}{\Gamma^2(\alpha)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\beta(\eta+1)-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\left(t^\beta-\tau^\beta\right)^{\alpha-1}\times p(\tau)p(\rho)d\tau d\rho$$

$$= \left|\left|f'\right|\right|_r\left|\left|g'\right|\right|_s t\left(I_\beta^{\eta,\alpha}\{p(t)\}\right)^2.$$

This leads to the proof of Theorem 3.1. □

Next, we establish a further generalization of Theorem 3.1.

Theorem 3.2. *Suppose that p is a positive function, f and g are differentiable functions on $[0,\infty)$ and $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$ such that $r > 1$ and $r^{-1}+s^{-1} = 1$. Then for all $t > 0$ the following inequality holds:*

$$\begin{aligned}&\Big| I_\beta^{\eta,\alpha}\{p(t)\}\ I_\delta^{\zeta,\gamma}\{p(t)f(t)g(t)\} + I_\delta^{\zeta,\gamma}\{p(t)\}\ I_\beta^{\eta,\alpha}\{p(t)f(t)g(t)\}\\ &- I_\beta^{\eta,\alpha}\{p(t)f(t)\}\ I_\delta^{\zeta,\gamma}\{p(t)g(t)\} - I_\delta^{\zeta,\gamma}\{p(t)f(t)\}\ I_\beta^{\eta,\alpha}\{p(t)g(t)\}\Big|\\ &\le \frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}\,||f'||_r\,||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times\\ &\qquad\times p(\tau)p(\rho)\,|\tau-\rho|\ d\tau d\rho\\ &\le ||f'||_r\,||g'||_s\ t\ I_\beta^{\eta,\alpha}\{p(t)\}\ I_\delta^{\zeta,\gamma}\{p(t)\},\end{aligned}$$

where $\alpha,\beta,\gamma,\delta > 0$, $\eta,\zeta \in \mathbb{R}$ such that $\eta > -1$ and $\zeta > -1$.

Proof. To obtain the desired results, we multiply (16) by

$$\frac{\delta\, t^{-\delta(\zeta+\gamma)}\rho^{\delta(\zeta+1)-1}\ p(\rho)}{\Gamma(\gamma)}(t^\delta-\tau^\delta)^{\gamma-1},\ \rho\in(0,t),\quad t>0,$$

and make integration from 0 to t (with respect to ρ), to obtain

$$\begin{aligned}&\frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times p(\tau)p(\rho)\mathcal{H}(\tau,\rho)\ d\tau d\rho\\ &= I_\beta^{\eta,\alpha}\{p(t)\}\ I_\delta^{\zeta,\gamma}\{p(t)f(t)g(t)\} + I_\delta^{\zeta,\gamma}\{p(t)\}\ I_\beta^{\eta,\alpha}\{p(t)f(t)g(t)\}\\ &\quad -I_\beta^{\eta,\alpha}\{p(t)f(t)\}\ I_\delta^{\zeta,\gamma}\{p(t)g(t)\} - I_\delta^{\zeta,\gamma}\{p(t)f(t)\}\ I_\beta^{\eta,\alpha}\{p(t)g(t)\}.\end{aligned}\tag{25}$$

On using (19), the (25) leads to

$$\begin{aligned}&\frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times\\ &\qquad\times p(\tau)p(\rho)\,|\mathcal{H}(\tau,\rho)|\ d\tau d\rho\\ &\le \frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times\\ &\qquad\times p(\tau)p(\rho)\,|\tau-\rho|\left|\int_\tau^\rho \left|f'(y)\right|^r dy\right|^{r^{-1}}\left|\int_\tau^\rho \left|g'(z)\right|^s dz\right|^{s^{-1}} d\tau d\rho.\end{aligned}\tag{26}$$

Making use of the Hölder's inequality, we readily obtain

$$\begin{aligned}&\frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times\\ &\qquad\times p(\tau)p(\rho)\,|\mathcal{H}(\tau,\rho)|\ d\tau d\rho\\ &\le \frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}\,||f'||_r\,||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times\\ &\qquad\times p(\tau)p(\rho)\,|\tau-\rho|\ d\tau d\rho.\end{aligned}\tag{27}$$

Now, one can easily arrive at the left-sided inequality of Theorem 3.2, by taking relations (25) and (27) into account. Further, for $0 \le \tau \le t, \;\; 0 \le \rho \le t$, we have

$$0 \le |\tau - \rho| \le t.$$

Therefore, from (27), we get

$$\frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}p(\tau)p(\rho)\,|\mathcal{H}(\tau,\rho)|\;d\tau d\rho$$

$$\le \frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}\,||f'||_r\,||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^t\int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^\beta-\tau^\beta)^{\alpha-1}(t^\delta-\rho^\delta)^{\gamma-1}\times$$
$$\times p(\tau)p(\rho)\,|\tau-\rho|\;d\tau d\rho.$$

$$= \left|\left|f'\right|\right|_r\;\left|\left|g'\right|\right|_s t\; I_\beta^{\eta,\alpha}\{p(t)\}\, I_\delta^{\zeta,\gamma}\{p(t)\},$$

this leads to the proof of Theorem 3.2. □

Remark 3.3. *For $\beta = \alpha$, Theorem 3.2 immediately reduces to Theorem 3.1.*

4 Special cases

Now, we briefly consider some implications of main results. To this end, if we consider $\beta = 1$ (additionally $\delta = 1$ for Theorem 3.2) and make use of the relation (9), the Theorems 3.1 and 3.2 provide the known fractional integral inequalities involving the Erdélyi-Kober operators, due to Purohit and Raina [29]. Again, if we set $\beta = 1, \eta = 0$ (additionally $\delta = 1$ and $\zeta = 0$ for Theorem 3.2), and make use of the relation (6), the main results recover the known results due to Dahmani *et al.* [28, pp. 39-42, Theorems 3.1 & 3.2].

Further, by setting $\eta = -\frac{1}{2}$, $\beta = 2$ and replacing α by $\alpha + \frac{1}{2}$ (additionally $\zeta = \frac{1}{2}$, $\delta = 2$ and γ replaced by $\gamma + \frac{1}{2}$ for Theorem 3.2), and make use of the relation (11), the main results provide the following integral inequalities involving the Uspensky integral transform:

Corollary 4.1. *Suppose that p is a positive function, f and g are differentiable functions on $[0,\infty)$, $f' \in L_r([0,\infty)), g' \in L_s([0,\infty))$ such that $r^{-1} + s^{-1} = 1$ with $r > 1$. Then for all $t > 0$ and $\alpha > 0$:*

$$\left|P^\alpha\{p(t)\}\,P^\alpha\{p(t)f(t)g(t)\} - \;P^\alpha\{p(t)f(t)\}\;\,P^\alpha\{p(t)g(t)\}\right|$$

$$\le \frac{t^{-4\alpha}\,||f'||_r\,||g'||_s}{\Gamma^2(\alpha+\frac{1}{2})}\int_0^t\int_0^t (t^2-\tau^2)^{\alpha-\frac{1}{2}}(t^2-\rho^2)^{\alpha-\frac{1}{2}}p(\tau)p(\rho)\,|\tau-\rho|\;d\tau d\rho$$
$$\le \left|\left|f'\right|\right|_r\;\left|\left|g'\right|\right|_s t\left(P^\alpha\{p(t)\}\right)^2.$$

Corollary 4.2. *Suppose that p is a positive function, f and g are differentiable functions on $[0,\infty)$ and $f' \in L_r([0,\infty)), g' \in L_s([0,\infty))$ such that $r > 1$ and $r^{-1} + s^{-1} = 1$. Then for all $t > 0$ the following inequality holds:*

$$\left|P^\alpha\{p(t)\}\;\,P^\gamma\{p(t)f(t)g(t)\} + \;P^\gamma\{p(t)\}\;\,P^\alpha\{p(t)f(t)g(t)\}\right.$$
$$\left. -\;P^\alpha\{p(t)f(t)\}\;\,P^\gamma\{p(t)g(t)\} - P^\gamma\{p(t)f(t)\}\,P^\alpha\{p(t)g(t)\}\right|$$

$$\le \frac{t^{-2\alpha-2\gamma}\,||f'||_r\,||g'||_s}{\Gamma(\alpha+\frac{1}{2})\Gamma(\gamma+\frac{1}{2})}\int_0^t\int_0^t (t^2-\tau^2)^{\alpha-\frac{1}{2}}(t^2-\rho^2)^{\gamma-\frac{1}{2}}p(\tau)p(\rho)\,|\tau-\rho|\;d\tau d\rho$$

$$\le \left|\left|f'\right|\right|_r\;\left|\left|g'\right|\right|_s t\;P^\alpha\{p(t)\}\;\,P^\gamma\{p(t)\}, \qquad \textit{where } \alpha,\gamma > 0.$$

Moreover, by using relations (5) to (11) with suitable values of parameters η, α and β, the results established in this paper can generate some interesting inequalities involving the various type of integral operator.

Now, by suitably choosing the function $p(t)$, we consider some examples of our main results. For example, let us set $p(t) = t^{\lambda} (\lambda \in [0, \infty), t \in (0, \infty))$, then Theorems 3.1 and 3.2 yield the following results:

Example 4.3. *Suppose that f and g are two differentiable functions on $[0, \infty)$ and if $f' \in L_r([0, \infty)), g' \in L_s([0, \infty)), r > 1, r^{-1} + s^{-1} = 1$ then for all $t > 0$, $\lambda \in [0, \infty)$, $\alpha > 0$, $\beta > 0$, $\eta \in \mathbb{R}$ such that $\lambda > -\beta(\eta + 1)$:*

$$2 \left| \frac{\Gamma(1 + \eta + \frac{\lambda}{\beta})}{\Gamma(1 + \alpha + \eta + \frac{\lambda}{\beta})} t^{\lambda} I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} f(t) g(t) \right\} - I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} f(t) \right\} I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} g(t) \right\} \right|$$
$$\leq \frac{\beta^2 \, t^{-2\beta(\eta+\alpha)} \, ||f'||_r \, ||g'||_s}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)+\lambda-1} \rho^{\beta(\eta+1)+\lambda-1} (t^{\beta} - \tau^{\beta})^{\alpha-1} (t^{\beta} - \rho^{\beta})^{\alpha-1} |\tau - \rho| \, d\tau d\rho$$
$$\leq ||f'||_r \; ||g'||_s \frac{\Gamma^2(1 + \eta + \frac{\lambda}{\beta})}{\Gamma^2(1 + \alpha + \eta + \frac{\lambda}{\beta})} t^{1+2\lambda}.$$

Example 4.4. *Let f and g be differentiable functions on $[0, \infty)$ and if $f' \in L_r([0, \infty)), g' \in L_s([0, \infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$ then*

$$\left| \frac{\Gamma(1 + \eta + \frac{\lambda}{\beta})}{\Gamma(1 + \alpha + \eta + \frac{\lambda}{\beta})} t^{\lambda} I_{\delta}^{\zeta,\gamma} \left\{ t^{\lambda} f(t) g(t) \right\} + \frac{\Gamma(1 + \zeta + \frac{\lambda}{\delta})}{\Gamma(1 + \gamma + \zeta + \frac{\lambda}{\delta})} t^{\lambda} I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} f(t) g(t) \right\} \right.$$
$$\left. - I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} f(t) \right\} I_{\delta}^{\zeta,\gamma} \left\{ t^{\lambda} g(t) \right\} - I_{\delta}^{\zeta,\gamma} \left\{ t^{\lambda} f(t) \right\} I_{\beta}^{\eta,\alpha} \left\{ t^{\lambda} g(t) \right\} \right|$$
$$\leq \frac{\beta \, \delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)} \, ||f'||_r \, ||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^t \int_0^t \tau^{\beta(\eta+1)+\lambda-1} \rho^{\delta(\zeta+1)+\lambda-1} (t^{\beta} - \tau^{\beta})^{\alpha-1} (t^{\delta} - \rho^{\delta})^{\gamma-1} \times$$
$$\times |\tau - \rho| \, d\tau d\rho$$
$$\leq ||f'||_r \, ||g'||_s \frac{\Gamma(1 + \eta + \frac{\lambda}{\beta}) \Gamma(1 + \zeta + \frac{\lambda}{\delta})}{\Gamma(1 + \alpha + \eta + \frac{\lambda}{\beta}) \Gamma(1 + \gamma + \zeta + \frac{\lambda}{\delta})} t^{1+2\lambda},$$

for all $t > 0$, $\alpha, \beta, \gamma, \delta > 0$, $\eta, \zeta \in \mathbb{R}$, $\lambda \in [0, \infty)$ such that $\lambda > \min\{-\beta(\eta + 1), -\delta(\zeta + 1)\}$.

Further, if we put $\lambda = 0$ in Examples 4.3 and 4.4 (or set $p(t) = 1$ in Theorems 3.1 and 3.2), we obtain the following results:

Example 4.5. *Suppose f and g are differentiable functions on $[0, \infty)$ and if $f' \in L_r([0, \infty)), g' \in L_s([0, \infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$ then for all $t > 0$, $\alpha > 0$, $\beta > 0$, $\eta \in \mathbb{R}$ such that $\eta > -1$:*

$$2 \left| \frac{\Gamma(1 + \eta)}{\Gamma(1 + \alpha + \eta)} I_{\beta}^{\eta,\alpha} \{ f(t) g(t) \} - I_{\beta}^{\eta,\alpha} \{ f(t) \} \, I_{\beta}^{\eta,\alpha} \{ g(t) \} \right|$$
$$\leq \frac{\beta^2 \, t^{-2\beta(\eta+\alpha)} \, ||f'||_r \, ||g'||_s}{\Gamma^2(\alpha)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1} \rho^{\beta(\eta+1)-1} (t^{\beta} - \tau^{\beta})^{\alpha-1} (t^{\beta} - \rho^{\beta})^{\alpha-1} |\tau - \rho| \, d\tau d\rho$$
$$\leq ||f'||_r \; ||g'||_s \; t \frac{\Gamma^2(1 + \eta)}{\Gamma^2(1 + \alpha + \eta)}.$$

Example 4.6. *Let f and g be two differentiable functions on $[0, \infty)$. If $f' \in L_r([0, \infty)), g' \in L_s([0, \infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$, then*

$$\left| \frac{\Gamma(1 + \eta)}{\Gamma(1 + \alpha + \eta)} I_{\delta}^{\zeta,\gamma} \{ f(t) g(t) \} + \frac{\Gamma(1 + \zeta)}{\Gamma(1 + \gamma + \zeta)} I_{\beta}^{\eta,\alpha} \{ f(t) g(t) \} \right.$$

$$- I_{\beta}^{\eta,\alpha}\{f(t)\}\, I_{\delta}^{\zeta,\gamma}\{g(t)\} - I_{\delta}^{\zeta,\gamma}\{f(t)\}\, I_{\beta}^{\eta,\alpha}\{g(t)\}\Big|$$

$$\leq \frac{\beta\,\delta t^{-\beta(\eta+\alpha)-\delta(\zeta+\gamma)}\,||f'||_r\,||g'||_s}{\Gamma(\alpha)\Gamma(\gamma)} \int_0^t \int_0^t \tau^{\beta(\eta+1)-1}\rho^{\delta(\zeta+1)-1}(t^{\beta}-\tau^{\beta})^{\alpha-1}(t^{\delta}-\rho^{\delta})^{\gamma-1}\,|\tau-\rho|\;d\tau d\rho$$

$$\leq \left|\left|f'\right|\right|_r \left|\left|g'\right|\right|_s\, t\, \frac{\Gamma(1+\eta)\Gamma(1+\zeta)}{\Gamma(1+\alpha+\eta)\Gamma(1+\gamma+\zeta)}\,,$$

for all $t > 0$, $\alpha, \beta, \gamma, \delta > 0$, $\eta, \zeta \in \mathbb{R}$, *such that* $\eta > -1$ *and* $\zeta > -1$.

The results established here are giving some contribution to the theory of integral inequalities and fractional calculus, and may find some applications in the theory of fractional differential equations. Moreover, by virtue of the unified nature of the generalized Erdélyi-Kober operator (4) and arbitray function $p(t)$, one can further deduce number of new fractional integral inequalities involving various fractional calculus operators and special functions, from our main results.

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