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Research Article

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Parabolic oblique derivative problem with discontinuous coefficients in generalized weighted Morrey spaces

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Abstract: We obtain the global weighted Morrey-type regularity of the solution of the regular oblique derivative problem for linear uniformly parabolic operators with VMO coefficients. We show that if the right-hand side of the parabolic equation belongs to certain generalized weighted Morrey space $M^{p,\varphi}(Q,w)$, than the strong solution belongs to the generalized weighted Sobolev- Morrey space $\dot{W}^{p,\varphi}_{2,1}(Q,\omega)$.

Keywords: Generalized weighted Morrey spaces, Uniformly parabolic operators, Regular oblique derivative problem, VMO

MSC: 35K20, 35D35, 35B45, 35R05

1 Introduction

We consider the regular oblique derivative problem in generalized weighted Sobolev- Morrey space $\dot{W}^{p,\varphi}_{2,1}(Q,\omega)$ for linear non-divergence form parabolic equations in a cylinder

$$\begin{cases} u_t - a^{ij}(x)D_{ij}u = f(x) & \text{a.e. in} \quad Q, \\ u(x',0) = 0, & \text{on} \quad \Omega, \\ \partial u/\partial l = l^i(x)D_i u = 0 & \text{on} \quad S. \end{cases}$$

The unique strong solvability of this problem was proved in [38]. In [39] Softova studied the regularity of the solution in the Morrey spaces $L^{p,\lambda}$ with $p \in (1,\infty)$, $\lambda \in (0, n+2)$ and also its Hölder regularity. In [41] Softova extended these studies on generalized Morrey spaces $L^{p,\varphi}$ with a Morrey function φ satisfying the doubling and integral conditions introduced in [27, 31]. The approach associated to the names of Calderón and Zygmund and developed by Chiarenza, Frasca and Longo in [7, 8] consists of obtaining of explicit representation formula for the higher order derivatives of the solution by singular and nonsingular integrals. Further the regularity properties of the solution follows by the continuity properties of these integrals in the corresponding spaces. In [39] and [40] the regularity of the corresponding operators in the Morrey and generalized Morrey spaces is studied, while in [38] we can find the corresponding results obtained in L^p by [9] and [5]. In recent works there have been studied the regularity of the solutions of elliptic and parabolic problems with Dirichlet data on the boundary in generalized Morrey spaces $M^{p,\varphi}$ with a weight φ satisfying (10) with $w \equiv 1$ (cf. [18, 19]). Precisely, a boundedness in $M^{p,\varphi}$ was obtained for

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sub-linear operators generated by singular integrals as the Calderon-Zygmund. More results concerning sub-linear operators in generalized Morrey spaces can be found in [3, 12, 40] see also the references therein.

After studying generalized Morrey spaces in detail, researchers passed to weighted Morrey spaces and generalized weighted Morrey spaces. Recently, Komori and Shirai [23] defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators, such as the Hardy-Littlewood maximal operator or the Calderón-Zygmund operator on these spaces. Also, Guliyev in [13] first introduced the generalized weighted Morrey spaces $M_w^{p,\varphi}$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [15, 17]). Note that, Guliyev [13] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M^{p,\varphi}$ and $L^{p,\kappa}(w)$.

We call *weight* a positive measurable function defined on $\mathbb{R}^n \times \mathbb{R}_+$. In [29] Muckenhoupt shows that the *maximal inequality* holds in *weighted Lebesgue spaces* L_w^q if and only if the weight w satisfies the following integral condition called *parabolic Muckenhoupt condition* or *parabolic A_q-condition*. We say that the measurable, nonnegative function $w : \mathbb{R}^n \to \mathbb{R}_+$ satisfies the parabolic A_q-condition for $q \in (1,\infty)$ if

$$\sup_{\mathcal{I}} \Big(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} w(x,t)\,dxdt\Big)\Big(\frac{1}{|\mathcal{I}|}\int_{\mathcal{I}} w(x,t)^{-\frac{1}{q-1}}\,dxdt\Big)^{q-1} \le A < \infty \tag{1}$$

for all parabolic cylinders $\mathcal{I}$ in $\mathbb{R}^{n+1}$. Then $w(\mathcal{I})$ means the weighted measure of $\mathcal{I}$, that is

$$w(\mathcal{I}) = \int_{\mathcal{I}} w(x,t)\,dxdt\,.$$

This measure satisfies *strong and reverse doubling property.* Precisely, for each $\mathcal{I}$ and each measurable subset $\mathcal{A} \subset \mathcal{I}$, there exist positive constants c_1 and $\tau_1 \in (0,1)$ such that

$$\frac{1}{[w]_q}\left(\frac{|\mathcal{A}|}{|\mathcal{I}|}\right)^q \le \frac{w(\mathcal{A})}{w(\mathcal{I})} \le c_1\left(\frac{|\mathcal{A}|}{|\mathcal{I}|}\right)^{\tau_1}, \tag{2}$$

where c_1 and τ_1 depend on n and q but not on $\mathcal{I}$ and $\mathcal{A}$.

Throughout this paper the following notations are to be used: $x = (x',t) = (x'',x_n,t) \in \mathbb{R}^{n+1}$, $R_+^{n+1} = \{x' \in \mathbb{R}^n, t > 0\}$ and $\mathbb{D}_+^{n+1} = \{x'' \in \mathbb{R}^{n-1}, x_n > 0, t > 0\}$, $D_i u = \partial u/\partial x_i$, $D_{ij}u = \partial^2 u/\partial x_i \partial x_j$, $D_t u = u_t = \partial u/\partial t$ stand for the corresponding derivatives while $Du = (D_1 u,\dots,D_n u)$ and $D^2 u = \{D_{ij}u\}_{i,j=1}^n$ mean the spatial gradient and the Hessian matrix of u. For any measurable function f and $A \subset \mathbb{R}^{n+1}$ we write

$$\|f\|_{p,A} = \Big(\int_A |f(y)|^p dy\Big)^{1/p}, \;\; f_A = \frac{1}{|A|}\int_A f(y)dy$$

where $|A|$ is the Lebesgue measure of A. Through all the paper the standard summation convention on repeated upper and lower indexes is adopted. The letter C is used for various constants and may change from one occurrence to another.

2 Definitions and statement of the problem

Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$ be a bounded $C^{1,1}$-domain, $Q = \Omega \times (0,T)$ be a cylinder in $\mathbb{R}_+^{n+1}$, and $S = \partial\Omega \times (0,T)$ stands for the lateral boundary of Q. We consider the problem

$$\begin{cases} \mathfrak{P}u := u_t - a^{ij}(x)D_{ij}u = f & \text{a.e. in} \quad Q, \\ \mathfrak{I}u := u(x',0) = 0, & \text{on} \quad \Omega, \\ \mathfrak{B}u := \partial u/\partial l = l^i(x)D_i u = 0 & \text{on} \quad S, \end{cases} \tag{3}$$

under the following conditions:

(i) The operator $\mathfrak{B}$ is supposed to be uniformly parabolic, i.e. there exists a constant $\Lambda > 0$ such that for almost all $x \in Q$

$$\begin{cases} \Lambda^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, & \forall \xi \in \mathbb{R}^n, \\ a^{ij}(x) = a^{ji}(x), & i,j = 1,\dots,n. \end{cases} \tag{4}$$

The symmetry of the coefficient matrix $\mathbf{a} = \{a^{ij}\}_{i,j=1}^n$ implies essential boundedness of a^{ij}'s and we set $\|\mathbf{a}\|_{\infty,Q} = \sum_{i,j=1}^n \|a^{ij}\|_{\infty,Q}$.

(ii) The boundary operator $\mathfrak{B}$ is prescribed in terms of a directional derivative with respect to the unit vector field $l(x) = (l^1(x),\dots,l^n(x))$, $x \in S$. We suppose that $\mathfrak{B}$ is a regular oblique derivative operator, i.e., the field l is never tangential to S:

$$\langle l(x) \cdot \mathbf{n}(x)\rangle = l^i(x)\mathbf{n}_i(x) > 0 \quad \text{on} \quad S,\ l^i \in Lip(\bar S). \tag{5}$$

Here $Lip(\bar S)$ is the class of uniformly Lipschitz continuous functions on $\bar S$ and $\mathbf{n}(x)$ stands for the unit outward normal to $\partial\Omega$.

In the following, besides the parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$ and the defined by it parabolic cylinders

$$\mathcal{I}_r(x) = \{y \in \mathbb{R}^{n+1} : |x'-y'| < r, |t-\tau| < r^2\}, \quad |\mathcal{I}_r| = Cr^{n+2}.$$

we use the equivalent one $\rho(x) = \left(\frac{|x'|^2 + \sqrt{|x'|^4 + 4t^2}}{2}\right)^{\frac12}$ (see [9]). The balls with respect to this metric are ellipsoids

$$\mathcal{E}_r(x) = \left\{y \in \mathbb{R}^{n+1} : \frac{|x'-y'|^2}{r^2} + \frac{|t-\tau|^2}{r^4} < 1\right\}, \quad |\mathcal{E}_r| = Cr^{n+2}.$$

Because of the equivalence of the metrics all estimates obtained over ellipsoids hold true also over parabolic cylinders and in the following we shall use this without explicit references.

Definition 2.1 ([20, 37]). *Let $a \in L^1_{loc}(\mathbb{R}^{n+1})$, denoted by*

$$\eta_a(R) = \sup_{\mathcal{E}_r, r \le R} \frac{1}{|\mathcal{E}_r|}\int_{\mathcal{E}_r} |f(y) - f_{\mathcal{E}_r}|dy, \quad \text{for every} \quad R > 0$$

where $\mathcal{E}_r$ ranges over all ellipsoids in $\mathbb{R}^{n+1}$. The Banach space BMO (bounded mean oscillation) consists of functions for which the following norm is finite

$$\|a\|_* = \sup_{R>0} \eta_a(R) < \infty.$$

A function a belongs to VMO (vanishing mean oscillation) with VMO- modulus $\eta_a(R)$ provided

$$\lim_{R\to 0} \eta_a(R) = 0.$$

For any bounded cylinder Q we define $BMO(Q)$ and $VMO(Q)$ taking $a \in L^1(Q)$ and $Q_r = Q \cap \mathcal{I}_r$ instead of $\mathcal{E}_r$ in the definition above.

According to [1, 21] having a function $a \in BMO/VMO(Q)$ it is possible to extend it in the whole $\mathbb{R}^{n+1}$ preserving its BMO-norm or VMO-modulus, respectively. In the following we use this property without explicit references.

For this goal we recall some well known properties of the BMO functions.

Lemma 2.2 (John-Nirenberg lemma, [20]). *Let $a \in BMO$ and $p \in (1,\infty)$. Then for any ball $\mathcal{B}$ there holds*

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} |a(y) - a_{\mathcal{B}}|^p dy\right)^{\frac1p} \le C(p)\|a\|_*.$$

As an immediate consequence of Lemma 2.2 we get the following property.

Corollary 2.3. *Let $a \in BMO$ then for all $0 < 2r < t$ holds*

$$\left|a_{\mathcal{B}_r^+} - a_{\mathcal{B}_t^+}\right| \le C\|a\|_* \ln\frac{t}{r} \tag{6}$$

where the constant is independent of a, x, t and r.

As mentioned before, we call weight a positive measurable function defined on $\mathbb{R}^n \times \mathbb{R}_+$. Given a weight w and a measurable set $\mathbb{S}$ we denote by

$$w(\mathbb{S}) = \int_{\mathbb{S}} w(x)\,dx$$

the w-measure of $\mathbb{S}$. A weight w belongs to the *Muckenhoupt class* A_p, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_{\mathcal{B}} \left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(x)\,dx\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(x)^{-p'/p}\,dx\right)^{p/p'} < \infty, \tag{7}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for any ball we have (see [11])

$$[w]_{A_p(\mathcal{B})}^{1/p} = |\mathcal{B}|^{-1}\|w\|_{L_1(\mathcal{B})}^{1/p}\,\|w^{-1/p}\|_{L_{p'}(\mathcal{B})} \ge 1. \tag{8}$$

In case $p = 1$, we say that $w \in A_1$ if

$$\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} w(x)\,dx \le A \operatorname*{ess\,inf}_{\mathcal{B}} w(x)$$

and $[w]_{A_1}$ is the smallest A for which the above inequality holds. It is an immediate consequence of (7) that whenever $w \in A_p$ than it satisfies the doubling property, precisely

$$w(2\mathcal{B}_r) \le C(n,p)w(\mathcal{B}_r)\,. \tag{9}$$

The following lemma collects some of the most important properties of the Muckenhoupt weights.

Lemma 2.4 ([11])**.** *We have the following:*

(1) *If $w \in A_p$ for some $1 \le p < \infty$, then for all $\lambda > 1$ we have*

$$w(\lambda\mathcal{B}) \le \lambda^{np}[w]_{A_p} w(\mathcal{B}).$$

(2) *The following equality is valid: $A_\infty = \bigcup_{1 \le p < \infty} A_p$.*

(3) *If $w \in A_\infty$, then for all $\lambda > 1$ we have*

$$w(\lambda\mathcal{B}) \le 2^{\lambda^n}[w]_{A_\infty}^{\lambda^n} w(\mathcal{B}).$$

(4) *If $w \in A_p$ for some $1 \le p \le \infty$, then there exist $C > 0$ and $\delta > 0$ such that for any ball $\mathcal{B}$ and a measurable set $\mathbb{S} \subset \mathcal{B}$,*

$$\frac{1}{[w]_{A_p}}\left(\frac{|\mathbb{S}|}{|\mathcal{B}|}\right) \le \frac{w(\mathbb{S})}{w(\mathcal{B})} \le C\left(\frac{|\mathbb{S}|}{|\mathcal{B}|}\right)^{\delta}.$$

Lemma 2.5 ([30, Theorem 5])**.** *Let $w \in A_\infty$. Then the norm of $BMO(w)$ is equivalent to the norm of $BMO(\mathbb{R}^n)$, where*

$$BMO(w) = \{a\colon \|a\|_{*,w} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{w(\mathcal{B}_r(x))}\int_{\mathcal{B}_r(x)} |a(y) - a_{\mathcal{B}_r(x),w}|w(y)\,dy < \infty\}$$

and

$$a_{\mathcal{B}_r(x),w} = \frac{1}{w(\mathcal{B}_r(x))}\int_{\mathcal{B}_r(x)} a(y)w(y)\,dy.$$

Lemma 2.6 (The John-Nirenberg inequality). *Let $a \in BMO$,*

(1) *there exist constants C_1, $C_2 > 0$, such that for all $\beta > 0$*

$$\left|\{x \in \mathcal{B} \colon |a(x) - a_{\mathcal{B}}| > \beta\}\right| \le C_1 |\mathcal{B}| e^{-C_2\beta/\|a\|_*}, \quad \forall\, \mathcal{B} \subset \mathbb{R}^n ;$$

(2) *for all $p \in (1, \infty)$*

$$\|a\|_* = C \sup_{\mathcal{B}} \Big(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |a(y) - a_{\mathcal{B}}|^p \, dy \Big)^{1/p} ;$$

(3) *for all $p \in [1, \infty)$ and $w \in A_\infty$*

$$\|a\|_* = C \sup_{\mathcal{B}} \left(\frac{1}{w(\mathcal{B})} \int_{\mathcal{B}} |a(y) - a_{\mathcal{B}}|^p w(y) \, dy \right)^{1/p} .$$

Definition 2.7. *Let $\varphi(x, r)$ be weight in $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\omega \in A_p$, $p \in [1, \infty)$. The generalized weighted Morrey space $M_{p,\varphi}(\mathbb{R}^n, \omega)$ or $M_{p,\varphi}(\omega)$ consists of all functions $f \in L^{loc}_{p,\omega}(\mathbb{R}^n)$ such that*

$$\|f\|_{p,\varphi;\omega} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} \Big(\omega(\mathcal{E}_r(x))^{-1} \int_{\mathcal{E}_r(x)} |f(y)|^p \omega(y) dy \Big)^{1/p} < \infty.$$

The space $M^{p,\varphi}(Q, w)$ consists of $L^p_\omega(Q)$ functions provided the following norm is finite

$$\|f\|_{p,\varphi,w;Q} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x, r)^{-1} \Big(\omega(Q_r(x))^{-1} \int_{Q_r(x)} |f(y)|^p \omega(y) dy \Big)^{1/p} .$$

The generalized Sobolev-Morrey space $W^{p;\varphi}_{2,1}(Q, \omega)$, $p \in (1, \infty)$ consist of all Sobolev functions $u \in W^p_{2,1}(Q, \omega)$ with distributional derivatives $D^l_t D^s_x u \in M^{p,\varphi}(Q, \omega)$, $0 \le 2l + |s| \le 2$, endowed by the norm

$$\|u\|_{W^{p;\varphi}_{2,1}(Q,\omega)} = \|u_t\|_{p,\varphi,\omega;Q} + \sum_{|s|\le 2} \|D^s u\|_{p,\varphi,\omega;Q}$$

and

$$\dot{W}^{p;\varphi}_{2,1}(Q, \omega) = \{u \in W^{p;\varphi}_{2,1}(Q, \omega) : \; u(x) = 0, \; x \in \partial Q\},$$

$$\|u\|_{\dot{W}^{p;\varphi}_{2,1}(Q,\omega)} = \|u\|_{W^{p;\varphi}_{2,1}(Q,\omega)}$$

where ∂Q means the parabolic boundary $\Omega \cup \{\partial\Omega \times (0, T)\}$.

Theorem 2.8. *(Main result) Let (i) and (ii) hold, $\mathbf{a} \in VMO(Q, \omega)$ and $u \in \dot{W}^p_{2,1}(Q, \omega)$, $p \in (1, \infty)$, $\omega \in A_p$ be a strong solution of (3). If $f \in M^{p,\varphi}(Q, w)$ with $\varphi(x, r)$ being measurable positive function satisfying*

$$\int_r^\infty \Big(1 + \ln \frac{s}{r}\Big) \frac{\operatorname*{ess\,inf}_{s<\varsigma<\infty} \varphi(x, \varsigma)\omega(Q_\varsigma(x))^{\frac{1}{p}}}{\omega(Q_s(x))^{\frac{1}{p}}} \frac{ds}{s} \le C \tag{10}$$

for each $(x, r) \in Q \times \mathbb{R}_+$, then $u \in \dot{W}^{p;\varphi}_{2,1}(Q, \omega)$ and

$$\|u\|_{\dot{W}^{p;\varphi}_{2,1}(Q,\omega)} \le C \|f\|_{p,\varphi,\omega;Q} \tag{11}$$

with $C = C(n, p, [\omega]_{A_p}, \Lambda, \partial\Omega, T, \|\mathbf{a}\|_{\infty;Q}, \eta_{\mathbf{a}})$ and $\eta_{\mathbf{a}} = \sum_{i,j=1}^n \eta_{a^{ij}}$.

If $\varphi(x,r) = r^{(\lambda-n-2)/p}$, then $M^{p,\varphi} \equiv L^{p,\lambda}$ and the condition (10) holds with a constant depending on n, p and λ. If $\varphi(x,r) = \omega(x,r)^{1/p} r^{-(n+2)/p}$ with $\omega : \mathbb{R}^{n+1} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions

$$k_1 \le \frac{\omega(x_0,s)}{\omega(x_0,r)} \le k_2 \quad \forall x_0 \in \mathbb{R}^{n+1},\ r \le s \le 2r$$

$$\int_r^\infty \frac{\omega(x_0,s)}{s} ds \le k_3 \omega(x_0,r) \quad k_i > 0,\ i = 1,2,3$$

than we obtain the spaces $L^{p,\omega}$ studied in [27, 31]. The following results are obtained in [19] and treat continuity in $M^{p,\varphi}(\mathbb{R}^{n+1},\omega)$ of certain singular and nonsingular integrals.

Definition 2.9. *A measurable function $\mathfrak{K}(x;\xi) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is called variable parabolic Calderon-Zygmund kernel ($PCZK$) if:*

i) *$\mathfrak{K}(x;\cdot)$ is a PCZK for a.a. $x \in \mathbb{R}^{n+1}$:*
   a) *$\mathfrak{K}(x;\cdot) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$,*
   b) *$\mathfrak{K}(x;\mu\xi) = \mu^{-(n+2)}\mathfrak{K}(x;\xi)\ \forall \mu > 0$,*
   c) *$\int_{\mathbb{S}^n} \mathfrak{K}(x;\xi)d\sigma_\xi = 0$, $\int_{\mathbb{S}^n} |\mathfrak{K}(x;\xi)|d\sigma_\xi < +\infty$.*

ii) *$\left\| D_\xi^\beta \mathfrak{K} \right\|_{\infty;\mathbb{R}^{n+1}\times\mathbb{S}^n} \le M(\beta) < \infty$ for each multi-index β.*

Consider the singular integrals

$$\mathcal{K}f(x) = P.V. \int_{\mathbb{R}^{n+1}} \mathfrak{K}(x;x-y)f(y)dy,$$

$$\mathfrak{C}[a,f](x) = P.V. \int_{\mathbb{R}^{n+1}} \mathfrak{K}(x;x-y)[a(y)-a(x)]f(y)dy. \tag{12}$$

Theorem 2.10. *For any $f \in M^{p,\varphi}(\mathbb{R}^{n+1},\omega)$ with (p,φ) as in Theorem 2.8 and $a \in BMO$ there exist constants depending on n, p, φ, ω and the kernel such that*

$$\|\mathcal{K}f\|_{p,\varphi,\omega;\mathbb{R}^{n+1}} \le C[\omega]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi,\omega;\mathbb{R}^{n+1}},$$

$$\|\mathfrak{C}[a,f]\|_{p,\varphi,\omega;\mathbb{R}^{n+1}} \le C[\omega]_{A_p}^{\frac{1}{p}} \|a\|_* \|f\|_{p,\varphi,\omega;\mathbb{R}^{n+1}}. \tag{13}$$

Corollary 2.11. *Let Q be a cylinder in $\mathbb{R}_+^{n+1}$, $f \in M^{p,\varphi}(Q,\omega)$, $a \in BMO(Q)$ and $\mathfrak{K}(x,\xi) : Q \times \mathbb{R}_+^{n+1} \setminus \{0\} \to \mathbb{R}$. Then the operators (12) are bounded in $M^{p,\varphi}(Q,\omega)$ with p, φ, and ω as in Theorem 2.10. Then*

$$\|\mathcal{K}f\|_{p,\varphi,\omega;Q} \le C[\omega]_{A_p}^{\frac{1}{p}} \|f\|_{p,\varphi,\omega;Q},$$

$$\|\mathfrak{C}[a,f]\|_{p,\varphi,\omega;Q} \le C[\omega]_{A_p}^{\frac{1}{p}} \|a\|_* \|f\|_{p,\varphi,\omega;Q} \tag{14}$$

with $C = C(n,p,\varphi,[\omega]_{A_p},|\Omega|,\mathcal{K})$.

Corollary 2.12. *Let $a \in VMO$ and (p,φ) be as in Theorem 2.8. Then for any $\varepsilon > 0$ there exists a positive number $r_0 = r_0(\varepsilon,\eta_{\mathbf{a}})$ such that for any $\mathcal{E}_r(x_0)$ with a radius $r \in (0,r_0)$ and all $f \in M^{p,\varphi}(\mathcal{E}_r(x_0),\omega)$*

$$\|\mathfrak{C}[a,f]\|_{p,\varphi,\omega;\mathcal{E}_r(x_0)} \le C\varepsilon \|f\|_{p,\varphi,\omega;\mathcal{E}_r(x_0)} \tag{15}$$

where C is independent of ε, f, r and x_0.

For any $x' \in \mathbb{R}_+^n$ and any fixed $t > 0$ define the generalized reflection

$$\mathcal{T}(x) = (\mathcal{T}'(x),t), \quad \mathcal{T}'(x) = x' - 2x_n \frac{\mathbf{a}^n(x',t)}{a^{nn}(x',t)} \tag{16}$$

where $\mathbf{a}^n(x)$ is the last row of the coefficients matrix $\mathbf{a}(x)$ of (3). The function $\mathcal{T}'(x)$ maps $\mathbb{R}^n_+$ into $\mathbb{R}^n_-$ and the kernel $\mathfrak{K}(x;\mathcal{T}(x)-y)=\mathcal{K}(x;\mathcal{T}'(x)-y',t-\tau)$ is a nonsingular one for any $x,y\in\mathbb{D}^{n+1}_+$. Taking $\tilde{x}=(x'',-x_n,t)$ there exist positive constants k_1 and k_2 such that

$$k_1\rho(\tilde{x}-y)\le\rho(\mathcal{T}(x)-y)\le k_2\rho(\tilde{x}-y) \tag{17}$$

For any $f\in M^{p,\varphi}(\mathbb{D}^{n+1}_+,\omega)$ with a norm

$$\|f\|_{p,\varphi,\omega;\mathbb{D}^{n+1}_+}=\sup_{x\in\mathbb{D}^{n+1}_+,r>0}\varphi(x,r)^{-1}\Big(\omega(\mathcal{E}_r(x))^{-1}\int_{\mathcal{E}_r(x)}|f(y)|^p\omega(y)dy\Big)^{1/p}$$

and $a\in BMO(\mathbb{D}^{n+1}_+,\omega)$ defines the nonsingular integral operators

$$\widetilde{\mathcal{K}}f(x)=\int_{\mathbb{D}^{n+1}_+}\mathfrak{K}(x;\mathcal{T}(x)-y)f(y)dy$$
$$\widetilde{\mathfrak{C}}[a,f](x)=\int_{\mathbb{D}^{n+1}_+}\mathfrak{K}(x;\mathcal{T}(x)-y)[a(x)-a(y)]f(y)dy. \tag{18}$$

Theorem 2.13. *Let $a\in BMO(\mathbb{D}^{n+1}_+)$, $\omega\in A_p$ and $f\in M^{p,\varphi}(\mathbb{D}^{n+1}_+,\omega)$ with (p,φ) as in Theorem 2.8. Then the operators $\widetilde{\mathcal{K}}f$ and $\widetilde{\mathfrak{C}}[a,f]$ are continuous in $M^{p,\varphi}(\mathbb{D}^{n+1}_+,\omega)$ and*

$$\|\widetilde{\mathcal{K}}f\|_{p,\varphi,\omega;\mathbb{D}^{n+1}_+}\le C\|f\|_{p,\varphi,\omega;\mathbb{D}^{n+1}_+},$$
$$\|\widetilde{\mathfrak{C}}[a,f]\|_{p,\varphi,\omega;\mathbb{D}^{n+1}_+}\le C[\omega]^{\frac{1}{p}}_{A_p}\|a\|_*\|f\|_{p,\varphi,\omega;\mathbb{D}^{n+1}_+}. \tag{19}$$

with a constant independend of a and f.

Corollary 2.14. *Let $a\in VMO$, then for any $\varepsilon>0$ there exists a positive number $r_0=r_0(\varepsilon,\eta_{\mathbf{a}})$ such that for any $\mathcal{E}^+_r(x^0)=\mathcal{E}_r(x^0)\cap\mathbb{D}^{n+1}_+$ with a radius $r\in(0,r_0)$ and center $x^0=(x'',0,0)$ and for all $f\in M^{p,\varphi}(\mathcal{E}^+_r(x^0),\omega)$ holds*

$$\|\widetilde{\mathfrak{C}}[a,f]\|_{p,\varphi,\omega;\mathcal{E}^+_r(x^0)}\le C\varepsilon\|f\|_{p,\varphi,\omega;\mathcal{E}^+_r(x^0)}, \tag{20}$$

where C is independent of ε, f, r and x^0.

3 Proof of the main result

As it follows by [39], the problem (3) is uniquely solvable in $\dot{W}^p_{2,1}(Q,\omega)$.

We are going to show that $f\in M^{p,\varphi}(Q,\omega)$ implies $u\in\dot{W}^{p,\varphi}_{2,1}(Q,\omega)$. For this goal we obtain an a priori estimate of u. Following the method used by Chiarenza, Frasca and Longo in [7] and [8], we prove the results considering two steps.

Interior estimate. For any $x_0\in\mathbb{R}^{n+1}_+$ consider the parabolic semi-cylinders $\mathcal{C}_r(x_0)=\mathcal{B}_r(x'_0)\times(t_0-r^2,t_0)$. Let $v\in C^\infty_0(\mathcal{C}_r)$ and suppose that $v(x,t)=0$ for $t\le0$. According to [[5], Theorem 1.4] for any $x\in\operatorname{supp}v$ the following representation formula for the second derivatives of v holds true

$$D_{ij}v(x)=P.V.\int_{\mathbb{R}^{n+1}}\Gamma_{ij}(x;x-y)[a^{hk}(y)-a^{hk}(x)]D_{hk}v(y)dy$$
$$+P.V.\int_{\mathbb{R}^{n+1}}\Gamma_{ij}(x;x-y)\mathfrak{B}v(y)dy+\mathfrak{B}v(x)\int_{\mathbb{S}^n}\Gamma_j(x;y)\nu_i d\sigma_y, \tag{21}$$

where $\nu(\nu_1, \dots, \nu_{n+1})$ is the outward normal to $\mathbb{S}^n$. Here $\Gamma(x;\xi)$ is the fundamental solution of the operator $\mathfrak{B}$ and $\Gamma(x;\xi) = \partial^2\Gamma(x;\xi)/\partial\xi_i\partial\xi_j$.

Because of density arguments the representation formula (21) still holds for any $v \in W^p_{2,1}(\mathcal{C}_r(x_0), \omega)$. The properties of the fundamental solution (cf. [5, 25, 38]) imply Γ_{ij} are Calderon-Zygmund kernels in the sense of Definition 2.9. We denote by $\mathcal{K}_{ij}$ and $\mathfrak{C}_{ij}$ the singular integrals defined in (12) with kernels $\mathfrak{K}(x; x-y) = \Gamma_{ij}(x; x-y)$. Then we can write that

$$D_{ij}v(x) = \mathfrak{C}_{ij}[a^{hk}, D_{hk}v](x) + \mathcal{K}_{ij}(\mathfrak{B}v)(x) + \mathfrak{B}v(x)\int_{\mathbb{S}^n} \Gamma_j(x; y)\nu_i \, d\sigma_y. \tag{22}$$

Because of Corollaries 2.11 and 2.12 and the equivalence of the metrics we get

$$\|D^2v\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)} \le C(\varepsilon\|D^2v\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)} + \|\mathfrak{B}u\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)})$$

for some r small enough. Moving the norm of D^2v on the left-hand side we get

$$\|D^2v\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)} \le C\|\mathfrak{B}v\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)}$$

with a constant depending on n, p, $\eta_{\mathbf{a}}(r)$, $\|\mathbf{a}\|_{\infty,Q}$ and $\|D\Gamma\|_{\infty,Q}$. Define a cut-off function $\phi(x) = \phi_1(x')\phi_2(t)$, with $\phi_1 \in C_0^\infty(\mathcal{B}_r(x_0'))$, $\phi_2 \in C_0^\infty(\mathbb{R})$ such that

$$\phi_1(x') = \begin{cases} 1 & x' \in \mathcal{B}_{\theta r}(x_0') \\ 0 & x' \notin \mathcal{B}_{\theta' r}(x_0'), \end{cases}$$

$$\phi_2(t) = \begin{cases} 1 & t \in (t_0 - (\theta r)^2, t_0] \\ 0 & t < t_0 - (\theta' r)^2 \end{cases}$$

with $\theta \in (0,1)$, $\theta' = \theta(3-\theta)/2 > \theta$ and $|D^s\phi| \le C[\theta(1-\theta)r]^{-s}$, $s = 0,1,2$, $|\phi_t| \sim |D^2\phi|$. For any solution $u \in W^p_{2,1}(Q,\omega)$ of (3) define $v(x) = \phi(x)u(x) \in W^p_{2,1}(\mathcal{C}_r, \omega)$. Then we get

$$\begin{aligned} \|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{\theta r}(x_0)} &\le \|D^2v\|_{p,\varphi,\omega;\mathcal{C}_{\theta',r}(x_0)} \le C\|\mathfrak{B}v\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)} \\ &\le C\Big(\|f\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)} + \frac{\|Du\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)}}{\theta(1-\theta)r} + \frac{\|u\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)}}{[\theta(1-\theta)r]^2}\Big). \end{aligned}$$

By the choice of θ' it holds $\theta(1-\theta) \le 2\theta'(1-\theta')$ which leads to

$$[\theta(1-\theta)r]^2\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{\theta r}(x_0)} \le C(r^2\|f\|_{p,\varphi,\omega;Q} + \theta'(1-\theta')r\|Du\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)} + \|u\|_{p,\varphi,\omega;\mathcal{C}_{\theta' r}(x_0)}\Big).$$

Introducing the semi-norms

$$\Theta_s = \sup_{0<\theta<1} [\theta(1-\theta)r]^s\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{\theta r}(x_0)} \quad s = 0,1,2$$

and taking the supremo with respect to θ and θ' we get

$$\Theta_2 \le C(r^2\|f\|_{p,\varphi,\omega;Q} + \Theta_1 + \Theta_0). \tag{23}$$

The interpolation inequality [26, Lemma 4.2] gives that there exists a positive constant C independent of r such that

$$\Theta_1 \le \varepsilon\Theta_2 + \frac{C}{\varepsilon}\Theta_0 \quad \text{for any } \varepsilon \in (0,2).$$

Thus (23) becomes

$$[\theta(1-\theta)r]^2\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{\theta r}(x_0)} \le \Theta_2 \le C(r^2\|f\|_{p,\varphi,\omega;Q} + \Theta_0)$$

for each $\theta \in (0,1)$. Taking $\theta = 1/2$ we get the Caccioppoli-type estimate

$$\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{r/2}(x_0)} \le C\Big(\|f\|_{p,\varphi,\omega;Q} + \frac{1}{r^2}\|u\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)}\Big).$$

To estimate u_t we exploit the parabolic structure of the equation and the boundedness of the coefficients

$$\|u_t\|_{p,\varphi,\omega;\mathcal{C}_{r/2}(x_0)} \le \|\mathbf{a}\|_{\infty,Q}\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_{r/2}(x_0)} + \|f\|_{p,\varphi,\omega;\mathcal{C}_{r/2}(x_0)}$$
$$\le C\Big(\|f\|_{p,\varphi,\omega;Q} + \frac{1}{r^2}\|u\|_{p,\varphi,\omega;\mathcal{C}_r(x_0)}\Big).$$

Consider cylinders $Q' = \Omega' \times (0,T)$ and $Q'' = \Omega'' \times (0,T)$ with $\Omega' \Subset \Omega'' \Subset \Omega$, by standard covering procedure and partition of the unity we get

$$\|u\|_{W^{p;\varphi}_{2,1}(Q',\omega)} \le C\Big(\|f\|_{p,\varphi,\omega;Q} + \|u\|_{p,\varphi,\omega;Q''}\Big). \tag{24}$$

where C depends on $n,\ p,\ [\omega]^{\frac{1}{2}}_{A_p},\ \Lambda,\ T,\ \|D\Gamma\|_{\infty;Q},\ \eta_{\mathbf{a}}(r),\ \|\mathbf{a}\|_{\infty,Q}$ and $dist(\Omega', \partial\Omega'')$.

Boundary estimates. For any fixed $R > 0$ and $x^0 = (x'', 0, 0)$ define the semi-cylinders

$$\mathcal{C}^+_R(x^0) = \mathcal{C}_R(x^0) \cap \mathbb{D}^{n+1}_+.$$

Without lost of generality we can take $x^0 = (0,0,0)$. Define $\mathcal{B}^+_R = \{|x'| < R, x_n > 0\}$, $S^+_R = \{|x''| < R, x_n = 0, t \in (0,R^2)\}$ and consider the problem

$$\begin{cases} \mathfrak{P}u := u_t - a^{ij}(x)D_{ij}u = f(x) & \text{a.e. in } \ \mathcal{C}^+_R,\\ \mathfrak{I}u := u(x',0) = 0, & \text{on } \ \mathcal{B}^+_R,\\ \mathfrak{B}u := l^i(x)D_iu = 0 & \text{on } \ \mathcal{S}^+_R. \end{cases} \tag{25}$$

Let $u \in W^p_{2,1}(C^+_R,\omega)$ with $u = 0$ for $t \le 0$ and $x_n \le 0$, then the following representation formula holds (see [26, 38])

$$D_{ij}u(x) = I_{ij}(x) - J_{ij}(x) + H_{ij}(x),$$

where

$$I_{ij}(x) = P.V.\int_{\mathcal{C}^+_R} \Gamma_{ij}(x;x-y)F(x;y)dy + f(x)\int_{\mathbb{S}^n}\Gamma_j(x;y)\nu_i\,d\sigma_y, \qquad i,j = 1,\dots,n;$$

$$J_{ij}(x) = \int_{\mathcal{C}^+_R}\Gamma_{ij}(x;\mathcal{T}(x)-y)F(x;y)dy;$$

$$J_{in}(x) = J_{ni}(x) = \int_{\mathcal{C}^+_R}\Gamma_{il}(x;\mathcal{T}(x)-y)\Big(\frac{\partial\mathcal{T}(x)}{\partial x_n}\Big)^l F(x;y)dy, \qquad i,j = 1,\dots,n-1$$

$$J_{nn}(x) = \int_{\mathcal{C}^+_R}\Gamma_{ls}(x;\mathcal{T}(x)-y)\Big(\frac{\partial\mathcal{T}(x)}{\partial x_n}\Big)^l\Big(\frac{\partial\mathcal{T}(x)}{\partial x_n}\Big)^s F(x;y)dy;$$

$$F(x;y)dy = f(y) + [a^{hk}(y) - a^{hk}(x)]D_{hk}u(y),$$

$$H_{ij}(x) = (G_{ij} *_2 g)(x) + g(x'',t)\int_{\mathbb{S}^n} G_j(x;y'',x_n,\tau)n_i\,d\sigma(y'',\tau), \qquad i,j = 1,\dots,n,$$

$$\frac{\partial T(x)}{\partial x_n} = \Big(-2\frac{a^{n1}(x)}{a^{nn}(x)},\dots,-2\frac{a^{nn-1}(x)}{a^{nn}(x)},-1\Big).$$

Here the kernel $G = \Gamma\mathcal{Q}$, is a byproduct of the fundamental solution and a bounded regular function $\mathcal{Q}$. Hence its derivatives G_{ij} behave as Γ_{ij} and the convolution that appears in H_{ij} is defined as follows

$$(G_{ij} *_2 g)(x) = P.V. \int_{S_R^+} G_{ij}(x; x'' - y'', x_n, t - \tau) g(y'', 0, \tau) dy'' d\tau,$$
$$g(x'', 0, t) = [(l^k(0) - l^k(x'', 0, t)) D_k u - l^k(0)(\Gamma_k * F)]\,|_{x_n=0}(x'', 0, t),$$
$$(\Gamma_k * F)(x) = \int_{\mathcal{C}_R^+} \Gamma_k(x; x - y) F(x; y) dy.$$

Here I_{ij} are a sum of singular integrals and bounded surface integrals hence the estimates obtained in Corollaries 2.11 and 2.12 hold true. On the nonsingular integrals J_{ij} we apply the estimates obtained in Theorem 2.13 and Corollary 2.14 that give

$$\|I_{ij}\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \|J_{ij}\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\Big(\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \eta_{\mathbf{a}}(R)\|D^2 u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\Big) \tag{26}$$

for all $i, j = 1, \dots, n$. To estimate the norm of H_{ij} we suppose that the vector field l is extended in $\mathcal{C}_R^+$ preserving its Lipschitz regularity and the norm. This automatically leads to extension of the function g in $\mathcal{C}_R^+$ that is

$$g(x) = \Big(l^k(0) - l^k(x)\Big) D_k u(x) - l^k(0)(\Gamma_k * F)(x). \tag{27}$$

Applying the estimates for the heat potentials [[25], Chapter 4] and the trace theorems in L^p [[2], Theorems 7.48, 7.53] (see also [[38], Theorem 1]) we get

$$\int_{\mathcal{C}_R^+} |(G_{ij} *_2 g)(y)|^p w(y) dy \le C\Big(\int_{\mathcal{C}_R^+} |g(y)|^p w(y) dy + \int_{\mathcal{C}_R^+} |Dg(y)|^p w(y) dy\Big).$$

Taking a parabolic cylinder $\mathcal{I}_r(x)$ centered in some point $x \in \mathcal{C}_R^+$ we have

$$\begin{aligned}\int_{\mathcal{C}_R^+ \cap \mathcal{I}_r(x)} |(G_{ij} *_2 g)(y)|^p w(y) dy &\le C \frac{\omega(\mathcal{I}_r(x))}{\varphi(x,r)^{-p}} \left(\frac{\varphi(x,r)^{-p}}{\omega(\mathcal{I}_r(x))} \int_{\mathcal{C}_R^+ \cap \mathcal{I}_r(x)} |g(y)|^p w(y) dy \right. \\ &\quad \left. + \frac{\varphi(x,r)^{-p}}{\omega(\mathcal{I}_r(x))} \int_{\mathcal{C}_R^+ \cap \mathcal{I}_r(x)} |Dg(y)|^p w(y) dy \right) \\ &\le C \frac{\omega(\mathcal{I}_r(x))}{\varphi(x,r)^{-p}} \left(\|g\|^p_{p,\varphi,\omega;\mathcal{C}_R^+} + \|Dg\|^p_{p,\varphi,\omega;\mathcal{C}_R^+} \right).\end{aligned}$$

Moving $\frac{\varphi(x,r)^{-p}}{\omega(\mathcal{I}_r(x))}$ on the left-hand side and taking the supremo with respect to $(x, r) \in \mathcal{C}_R^+ \times \mathbb{R}_+$ we get

$$\|G_{ij} *_2 g\|^p_{p,\varphi,\omega;\mathcal{C}_R^+} \le C \left(\|g\|^p_{p,\varphi,\omega;\mathcal{C}_R^+} + \|Dg\|^p_{p,\varphi,\omega;\mathcal{C}_R^+} \right).$$

An immediate consequence of (27) is the estimate

$$\begin{aligned}\|g\|_{p,\varphi,\omega;\mathcal{C}_R^+} &\le \|[l^k(0) - l^k(\cdot)] D_k u\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \|l^k(0)(\Gamma_k * F)\|_{p,\varphi,\omega;\mathcal{C}_R^+} \\ &\le CR\|l\|_{Lip(\bar{\mathcal{S}})} \|Du\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \|\Gamma_k * f\|_{p,\varphi,\omega;\mathcal{C}_R^+} \\ &\quad + \|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)] D_{hk} u\|_{p,\varphi,\omega;\mathcal{C}_R^+}.\end{aligned}$$

The convolution $\Gamma_k * f$ is a Riesz potential. On the other hand

$$|(\Gamma_k * f)(x)| \le C \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+1}} dy \le CR \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+2}} dy \le C \int_{\mathcal{C}_R^+} \frac{|f(y)|}{\rho(x-y)^{n+2}} dy$$

with a constant depending on T and $diam\,\Omega$. Apply [[16], Theorem 4.8, [13], Theorem 3.1] that gives

$$\|\Gamma_k * f\|_{p,\varphi,\omega,\mathcal{C}_R^+} \le C\|f\|_{p,\varphi,\omega,\mathcal{C}_R^+}.$$

Analogously

$$|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)]D_{hk}u(\cdot)| \le C \int\limits_{\mathcal{C}_R^+} \frac{|a^{hk}(y) - a^{hk}(x)||D_{hk}u(y)|}{\rho(x-y)^{n+2}}dy$$

with a constant depending on $diam\,\Omega$ and T. The kernel $\rho(x-y)^{-(n+2)}$ is a nonnegative singular one and applying again the results for sub-linear integrals [[13], Theorem 3.7] we get

$$\|\Gamma_k * [a^{hk}(\cdot) - a^{hk}(x)]D_{hk}u\|_{\rho,\omega,\varphi,\mathcal{C}_R^+} \le C\|\mathbf{a}\|_*\|D^2u\|_{\rho,\omega,\varphi,\mathcal{C}_R^+}.$$

Hence

$$\|g\|_{\rho,\omega,\varphi,\mathcal{C}_R^+} \le C\Big(R\|l\|_{Lip(\bar{\mathcal{S}})}\|Du\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} + R\eta_{\mathbf{a}}(R)\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\Big). \tag{28}$$

Further, the Rademacher theorem asserts existence almost everywhere of the derivatives $D_hl^k \in L^\infty$, thus

$$D_hg(x) = -D_hl^k(x)D_ku(x) + [l^k(0) - l^k(x)]D_{kh}u - l^k(0)(\Gamma_{kh} * F)(x).$$

The $M^{p,\varphi}(\omega)$ norm of the last term is estimated as above and

$$\begin{aligned}\|Dg\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\Big(&\|Dl\|_{\infty;\mathcal{S}}\|Du\|_{p,\varphi,\omega;\mathcal{C}_R^+} + R\|l\|_{Lip(\bar{\mathcal{S}})}\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\\ &+ \|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \eta_{\mathbf{a}}(R)\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\Big).\end{aligned} \tag{29}$$

Finally unifying (26), (28) and (29) we get

$$\begin{aligned}\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\Big(&\|f\|_{p,\varphi,\omega;Q} + (1+R)\|Du\|_{p,\varphi,\omega;\mathcal{C}_R^+}\\ &+ (R + \eta_{\mathbf{a}}(R) + R\eta_{\mathbf{a}}(R))\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\Big)\end{aligned}$$

with a constant depending on known quantities and $\|l\|_{Lip(\bar{\mathcal{S}})}$ and $\|Dl\|_{\infty;\mathcal{S}}$. Direct calculations lead to an interpolation inequality in $M^{p,\varphi}(\omega)$ analogous to [[25], Lemma 3.3] (cf. [41])

$$\|Du\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le \delta\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \frac{C}{\delta}\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+},\ \delta \in (0,R).$$

Taking $0 < \delta = \frac{R}{R+1} < R$ we get

$$\begin{aligned}\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\Bigg(&\|f\|_{p,\varphi,\omega;Q} + R\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \frac{C}{R}\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\\ &+ (R + \eta_{\mathbf{a}}(R) + R\eta_{\mathbf{a}}(R))\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\Bigg).\end{aligned}$$

Choosing R small enough and moving the terms containing the norm of D^2u on the left-hand side we get

$$\|D^2u\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\left(\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} + \frac{1}{R}\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\right).$$

Because of the parabolic structure of the equation analogous estimate holds also for u_t. Further the Jensen inequality applied to $u(x) = \int_0^t u_s(x',s)ds$ gives

$$\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le CR^2\|u_t\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\left(R^2\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} + R\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+}\right).$$

Choosing R smaller, if necessary, we get $\|u\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+}$ and therefore

$$\|u\|_{W^{p,\varphi}_{2,1}(\mathcal{C}_R^+,\omega)} \le C\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+} \le C\|f\|_{p,\varphi,\omega;\mathcal{C}_R^+}. \tag{30}$$

Making a covering $\{\mathcal{C}_\alpha^+\}$, $\alpha \in \mathcal{A}$ such that $Q \setminus Q' \subset \bigcup_{\alpha\in\mathcal{A}} \mathcal{C}_\alpha^+$, considering a partition of unity subordinated to that covering and applying (30) for each $\mathcal{C}_\alpha^+$ we get

$$\|u\|_{W_{2,1}^{p,\varphi}(Q\setminus Q',\omega)} \leq C\|f\|_{p,\varphi,\omega;Q} \tag{31}$$

with a constant depending on n, p, $[\omega]_{A_p}^{\frac{1}{2}}$, Λ, T, $diam\Omega$, $\|D\Gamma\|_{\infty;Q}$, $\eta_{\mathbf{a}}$, $\|\mathbf{a}\|_{\infty;Q}$, $\|l\|_{Lip(\bar{\mathcal{S}})}$, and $\|Dl\|_{\infty,\mathcal{S}}$.

The estimate (11) follows from (24) and (31).

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References

[1] Acquistapace P., On BMO regularity for linear elliptic systems, Ann. Mat. Pura Appl., 1992, 161, 231-270

[2] Adams R., Sobolev Spaces. Academic Press, New York, 1975

[3] Akbulut A., Guliyev V.S., Mustafayev R., On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces, Math. Bohem., 2012, 137(1), 27-43

[4] Bramanti M., Commutators of integral operators with positive kernels, Le Matematiche, 1994, 49, 149-168

[5] Bramanti M., Cerutti M.C., $W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, Comm. Partial Diff. Eq., 1993, 18, 1735-1763

[6] Chiarenza F., Frasca M., Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat., 1987, 7, 273-279

[7] Chiarenza, F., Frasca, M., Longo, P., Interior $W^{2,p}$-estimates for nondivergence ellipic equations with discontinuous coefficients, Ricerche Mat., 1991, 40, 149-168

[8] Chiarenza F., Frasca M., Longo P., $W^{2,p}$-solvability of Dirichlet problem for nondivergence ellipic equations with VMO coefficients, Trans. Am. Math. Soc., 1993, 336, 841-853

[9] Fabes E.B., Rivière N., Singular integrals with mixed homogeneity, Studia Math., 1996, 27, 19-38

[10] Gilbarg D., Trudinger N.S., Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer-Verlag, Berlin, 1983

[11] Grafakos L., Classical and Modern Fourier Analysis, Pearson Education, Inc. Upper Saddle River, New Jersey, 2004

[12] Guliyev V.S., Aliyev S.S., Karaman T., Shukurov P., Boundedness of sublinear operators and commutators on generalized Morrey spaces, Int. Eq. Oper. Theory, 2011, 71(3), 327-355

[13] Guliyev V.S., Generalized weighted Morrey spaces and higher order commutators of sublinear operators, Eurasian Math. J., 2012, 3(3), 33-61

[14] Guliyev V.S., Alizadeh F.Ch., Multilinear commutators of Calderon-Zygmund operator on generalized weighted Morrey spaces, Journal of Function Spaces, 2014, http://dx.doi.org/10.1155/2014/710542

[15] Guliyev V.S., Karaman T., Mustafayev R.Ch., Serbetci A., Commutators of sublinear operators generated by Calderón-Zygmund operator on generalized weighted Morrey spaces, Czechoslovak Math. J., 2014, 64(139)(2), 365-386

[16] Guliyev V.S., Karaman T., Serbetci A., Boundedness of sublinear operators generated by Calderon-Zygmund operators on generalized weighted Morrey spaces, Scienti c Annals of "Al.I. Cuza" University of Iasi, 2014, 60(1), 227-244

[17] Guliyev V.S., Omarova M.N., Higher order commutators of vector-valued intrinsic square functions on vector-valued generalized weighted Morrey spaces, Azerb. J. Math., 2014, 4(2), 64-85

[18] Guliyev V.S., Softova L.G., Global regularity in generalized Morrey spaces of solutions to non-divergence elliptic equations with VMO coeficients, Potential Anal., 2013, 38 (3), 843-862

[19] Guliyev V.S., Softova L.G., Generalized Morrey regularity for parabolic equations with discontinuity data, Proc. Edinb. Math. Soc., 2015, (2) 58(1), 199-218

[20] John F., Nirenberg L.,On functions of bounded mean oscillation, Commun. Pure Appl. Math., 1961, 14, 415-426

[21] Jones P.W., Extension theorems for BMO, Indiana Univ. Math. J., 1980, 2(9), 41-66

[22] Kim, D., Parabolic equations with partially BMO coefficients and boundary value problems in Sobolev spaces with mixed norms, Potential Anal., 2010, 33 (1), 17-46

[23] Komori Y., Shirai S., Weighted Morrey spaces and a singular integral operator, Math. Nachr., 2009, 282(2), 219-231

[24] Krylov N.V., Parabolic equations with VMO coefficients in Sobolev spaces with mixed norms, J. Funct. Anal., 2007, 250 (2), 521-558

[25] Ladyzhenskaya O.A., Solonnikov V.A., Ural'tsev N.N., Linear and Quasilinear Equations of Parabolic Type. Transl. Math. Monographs 23, Amer. Math. Soc., Providence, R.I., 1968

[26] Maugeri A., Palagachev D.K., Softova L.G., Elliptic and Parabolic Equations with Discontinuous Coeficients, Wiley-VCH, Berlin, 2000

[27] Mizuhara T., Boundedness of some classical operators on generalized Morrey spaces. Harmonic Anal., Proc. Conf., Sendai/Jap., 1990, ICM-90 Satell. Conf. Proc., 1991, 183-189

[28] Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 1938, 43, 126-166

[29] Muckenhoupt B., Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., 1972, 165, 207-226

[30] Muckenhoupt B., Wheeden R., Weighted bounded mean oscillation and the Hilbert transform, Studia Math., 1976, 54, 221-237

[31] Nakai E., Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, Math. Nachr., 1994, 166, 95-103

[32] Palagachev D.K., Softova L.G., Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's, Potential Anal., 2004, 20, 237-263

[33] Palagachev D.K., Ragusa M.A., Softova L.G., Regular oblique derivative problem in Morrey spaces, Electron. J. Differential Equations, 2000, 39, http://ejde.math.txstate.edu/Volumes/2000/39/palagachev.pdf

[34] Palagachev D.K., Ragusa M.A., Softova L.G., Cauchy-Dirichlet problem in Morrey spaces for parabolic equations with discontinuous coefficients, Boll. Unione Mat. Ital., 2003, 6 B (3), 667-684

[35] Ragusa M.A., Cauchy-Dirichlet problem associated to divergence form parabolic equations, Communications in Contemporary Mathematics, 2004, 6 (3), 377-393

[36] Ragusa M.A., The Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, Mathematical and Computer Modelling, 2005, 42 (11-12), 1245-1254

[37] Sarason D., On functions of vanishes mean oscillation, Trans. Amer. Math. Soc., 1975, 207, 391-405

[38] Softova L.G., Oblique derivative problem for parabolic operators with VMO coeficients, Manuscr. Math., 2002, 103 (2), 203-220

[39] Softova L.G., Parabolic equations with VMO coefficients in Morrey spaces, Electron. J. Differ. Equ., 2001, 51, 1-25

[40] Softova L.G., Singular integrals and commutators in generalized Morrey spaces, Acta Math. Sin., Engl. Ser., 2006, 22, 757-766

[41] Softova L.G., Morrey-type regularity of solutions to parabolic problems with discontinuous data, Manuscr. Math., 2011, 136(3-4), 365-382